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ON SOME DIFFERENCE SEQUENCE SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION

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Abstract: In the this paper, we study köthe-Toeplitz and Null duals of some difference sequence spaces defined by means of a fixed sequence of multiplier and Musielak-Orlicz function $\mathcal{M} = (M_k)$.

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1. Introduction and Preliminaries

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_o be respectively, the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $||x|| = \sup\limits_k |x_k|$.

An orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to define the Orlicz sequence space as follows:

$$l_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}.$$

Note that l_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Also, it was shown in [4] that every Orlicz sequence space l_M contains a subspace

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isomorphic to $l_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M and it satisfies the following properties

(i) $\eta(t) > 0$,

(ii) $\eta(0) = 0$,

(ii) η is right differentiable for $t \geq 0$,

(iv) η is non-decreasing, and

(v) $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see (see [5], [6]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_M(cx) < \infty, \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \ \ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for a sequence space E, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \big\{x = (x_k) \in w : (\lambda_k x_k) \in E\big\}.$$

Goes and Goes [2] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and k respectively. Kizmaz [3] defined the sequence spaces

$$X(\Delta) = \Big\{ x = (x_k) : (\Delta x_k) \in X \Big\}$$

for $X = l_\infty$, c or c_o, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$. For more details about sequence spaces see (see [7], [8], [9]).

Let $\Lambda = (\lambda_k)$ be a non-zero sequence of scalars and $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. Then we define the following sequence spaces :

$$c_0(M_k, \Lambda, \Delta) = \big\{ x = (x_k) : \lim_k M_k\big(\frac{|\Delta\lambda_k x_k|}{\rho}\big) = 0 \text{ for some } \rho > 0 \big\},$$

$$c(M_k, \Lambda, \Delta) = \big\{ x = (x_k) : \lim_k M_k\big(\frac{|\Delta\lambda_k x_k - L|}{\rho}\big) = 0 \text{ for some } L \text{ and } \rho > 0 \big\}$$

and

$$l_\infty(M_k, \Lambda, \Delta) = \big\{ x = (x_k) : \sup_k M_k\big(\frac{|\Delta\lambda_k x_k|}{\rho}\big) < \infty \text{ for some } \rho > 0 \big\},$$

where $\Delta\lambda_k x_k = \lambda_k x_k - \lambda_{k+1} x_{k+1}$, for all $k \in N$. It is obvious that

$$c_0(M_k, \Lambda, \Delta) \subset c(M_k, \Lambda, \Delta) \subset l_\infty(M_k, \Lambda, \Delta).$$

For the sequence spaces c_0, c and l_∞ we denote it by Z throughout the paper. The sequence spaces $Z(M_k)$ are Banach spaces normed by

$$||x||_\Delta = |\lambda_1 x_1| + \inf\big\{\rho > 0 : \sup_k M_k\big(\frac{|\Delta\lambda_k x_k|}{\rho}\big) \le 1 \big\}.$$

We will write $\Delta^{-1} x_k = x_k - x_{k-1}$, for all $k \in N$. It is obvious that $(\Delta\lambda_k x_k) \in Z(M_k)$ if and only if $(\Delta^{-1}\lambda_k x_k) \in Z(M_k, \Lambda, \Delta)$. For $x \in Z(M_k, \Lambda, \Delta^{-1})$, we define

$$||x||_{\Delta^{-1}} = \inf\big\{\rho > 0 : \sup_k M_k\big(\frac{|\Delta^{-1}\lambda_k x_k|}{\rho}\big) \le 1 \big\}.$$

It can be shown that $Z(M_k, \Lambda, \Delta)$ is a BK-space under the norm $||x||_\Delta$. Also $||x||_{\Delta^{-1}}$ and the norm $||x||_\Delta$ and $||x||_{\Delta^{-1}}$ are equivalent. Obviously $\Delta^{-1} : Z(M_k, \Lambda, \Delta) \to Z(M_k)$, defined by $\Delta^{-1} x = y = (\Delta^{-1}\lambda_k x_k)$, is isometric isomorphism. Hence $c_0(M_k, \Lambda, \Delta^{-1})$, $c(M_k, \Lambda, \Delta^{-1})$ and $l_\infty(M_k, \Lambda, \Delta^{-1})$ are isometrically isomorphic to $c_0(M_k)$, $c(M_k)$ and $l_\infty(M_k)$ respectively.

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Now the spaces $\tilde{c}_0(M_k,\Lambda,\Delta)$, $\tilde{c}(M_k,\Lambda,\Delta)$ and $\tilde{l}_\infty(M_k,\Lambda,\Delta)$ which is the subspaces of $c_0(M_k,\Lambda,\Delta)$, $c(M_k,\Lambda,\Delta)$ and $l_\infty(M_k,\Lambda,\Delta)$ respectively are define as follows :

$$\tilde{c}_0(M_k,\Lambda,\Delta)=\Big\{x=(x_k):\lim_k M_k\big(\frac{|\Delta\lambda_k x_k|}{d}\big)=0 \text{ for each } d>0\Big\},$$

$$\tilde{c}(M_k,\Lambda,\Delta)=\Big\{x=(x_k):\lim_k M_k\big(\frac{|\Delta\lambda_k x_k-L|}{d}\big)=0 \text{ for some } L \text{ and } d>0\Big\}$$

and

$$\tilde{l}_\infty(M_k,\Lambda,\Delta)=\Big\{x=(x_k):\sup_k M_k\big(\frac{|\Delta\lambda_k x_k|}{d}\big)<\infty \text{ for each } d>0\Big\}.$$

It is obvious that $\tilde{c}_0(M_k,\Lambda,\Delta)\subset\tilde{c}(M_k,\Lambda,\Delta)\subset\tilde{l}_\infty(M_k,\Lambda,\Delta)$.

Hence $\tilde{c}_0(M_k,\Lambda,\Delta)$, $\tilde{c}(M_K,\Lambda,\Delta)$ and $\tilde{l}_\infty(M_k,\Lambda,\Delta)$ are isometrically isomorphic to $\tilde{c}_0(M_k)$, $\tilde{c}(M_k)$ and $\tilde{l}_\infty(M_k)$ respectively. Moreover $Z(M_k,\Lambda)\subset Z(M_k,\Lambda,\Delta)$ and $\tilde{Z}(M_k,\Delta)\subset\tilde{Z}(M_k,\Lambda,\Delta)$ which can be shown by using the following inequality :

$$M_k\big(\frac{|\Delta\lambda_k x_k|}{2\rho}\big)\le\frac{1}{2}M_k\big(\frac{|\lambda_k x_k|}{\rho}\big)+\frac{1}{2}M_k\big(\frac{|\lambda_{k+1}x_{k+1}|}{\rho}\big).$$

Let E and F be two sequence spaces. Then the F dual of E is defined as

$$E^F=\big\{x=(x_k)\in w:(x_k y_k)\in F \text{ for all } (y_k)\in E\big\}.$$

For $F=l_1$ and c_0, the duals are termed as α (or Köthe-Toeplitz) dual and N (or Null) dual of E and denoted by E^α and E^N respectively. If $X\subset Y$, then $Y^z\subset X^z$ for $z=\alpha,\ N$.

In this paper we study some relation on the above sequence spaces.

2. Main Results

Lemma 2.1. *For $x\in l_\infty(M_k,\Lambda,\Delta)$ implies that*

$$\sup_k M_k\big(\frac{|k^{-1}\lambda_k x_k|}{\rho}\big)<\infty \quad \textit{for some} \quad \rho>0.$$

Proof. Let $x\in l_\infty(M_k,\Lambda,\Delta)$, then

$$\sup_k M_k\big(\frac{|\lambda_k x_k-\lambda_{k+1}x_{k+1}|}{\rho}\big)<\infty \text{ for some } \rho>0.$$

Then there exists a $S > 0$ such that

$$M_k\Big(\frac{|\lambda_k x_k - \lambda_{k+1}x_{k+1}|}{\rho}\Big) < S \text{ for all } k \in \mathbb{N}.$$

Taking $\sigma = k\rho$, for an arbitrary fixed positive integer k, by the subadditivity of modulus, the monotonicity and convexity of M_k

$$M_k\Big(\frac{|\lambda_1 x_1 - \lambda_{k+1}x_{k+1}|}{\sigma}\Big) < \frac{1}{k}\sum_{l=1}^{k} M_k\Big(\frac{|\lambda_l x_l - \lambda_{l+1}x_{l+1}|}{\rho}\Big) < S.$$

Then the above inequality reduces to

$$\frac{|\lambda_{k+1}x_{k+1}|}{(k+1)\rho} \leq \frac{1}{k+1}\Big(\frac{|\lambda_1 x_1|}{\rho}\Big) + k\frac{|\lambda_1 x_1 - \lambda_{k+1}x_{k+1}|}{k\rho}\Big)$$

and the convexity of M_k implies that

$$M_k\Big(\frac{|\lambda_{k+1}x_{k+1}|}{(k+1)\rho}\Big) \leq \frac{1}{k+1}\Big(M_k\Big(\frac{|\lambda_1 x_1|}{\rho}\Big) + kM_k\Big(\frac{|\lambda_1 x_1 - \lambda_{k+1}x_{k+1}|}{k\rho}\Big)\Big)$$

$$\leq \max\Big\{M_k\Big(\frac{|\lambda_1 x_1|}{\rho}, S\Big\} < \infty.$$

Hence the required result. □

Corollary 2.2. *For $x \in \tilde{l}_\infty(M_k, \Lambda, \Delta)$ implies that*

$$\sup_k M_k\Big(\frac{|k^{-1}\lambda_k x_k|}{d}\Big) < \infty \text{ for each } d > 0.$$

Lemma 2.3. *For $x \in l_\infty(M_k, \Lambda, \Delta)$ implies that*

$$\sup_k k^{-1}|\lambda_k x_k| < \infty.$$

Proof. It is obvious in view of Lemma 2.1. □

For the next theorem , let

$$D_1 = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k|\lambda_k^{-1} a_k| < \infty\Big\},$$

$$D_2 = \Big\{b = (b_k) : \sup_k k^{-1}|\lambda_k b_k| < \infty\Big\}.$$

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then:*

(i) $[c(M_k,\Lambda,\Delta)]^\alpha = [l_\infty(M_k,\Lambda,\Delta)]^\alpha = D_1$,

(ii) $[\tilde{c}(M_k,\Lambda,\Delta)]^\alpha = [\tilde{l}_\infty(M_k,\Lambda,\Delta)]^\alpha = D_2$,

(iii) $D_1^\alpha = D_2$.

Proof. (i) Let $y \in D_1$. Then $\sum_{k=1}^{\infty} |k\lambda_k^{-1} y_k| < \infty$. Now for any $x \in l_\infty(M_k,\Lambda,\Delta)$, we have $\sup_k |k^{-1}\lambda_k x_k| < \infty$. Thus we have

$$\sum_{k=1}^{\infty} |y_k x_k| \leq \sup_k |k^{-1}\lambda_k x_k| \sum_{k=1}^{\infty} |k\lambda_k^{-1} y_k| < \infty.$$

Hence $y \in [l_\infty(M_k,\Lambda,\Delta)]^\alpha$ and so

$$D_1 \subseteq [l_\infty(M_k,\Lambda,\Delta)]^\alpha. \tag{1}$$

Moreover we know that

$$[l_\infty(M_k,\Lambda,\Delta)]^\alpha \subseteq [c(M_k,\Lambda,\Delta)]^\alpha \subseteq [c_0(M_k,\Lambda,\Delta)]^\alpha. \tag{2}$$

Again suppose that $y \in [c(M_k,\Lambda,\Delta)]^\alpha$. Then $\sum_{k=1}^{\infty} |y_k x_k| < \infty$ for each $x \in c(M_k,\Lambda,\Delta)$. If we take

$$x_k = k\lambda_k^{-1}, \quad k \geq 1, \quad \text{then}$$

$$\sum_{k=1}^{\infty} |k\lambda_k^{-1} y_k| = \sum_{k=1}^{\infty} |y_k x_k| < \infty.$$

This implies that $y \in D_1$. Thus

$$[c(M_k,\Lambda,\Delta)]^\alpha \subseteq D_1. \tag{3}$$

On combining equation (1), (2) and (3), we get

$$[c(M_k,\Lambda,\Delta)]^\alpha = [l_\infty(M_k,\Lambda,\Delta)]^\alpha = D_1.$$

This completes the proof of (i). The proof of (ii) is similar to that of part (i). We omit the details.

Finally, we prove (iii). The proof of $D_1^\alpha \supseteq D_2$ is similar to that of $D_1 \subseteq [l_\infty(M_k,\Lambda,\Delta)]^\alpha$.

For the converse part suppose $y \in D_1^\alpha$ and $y \notin D_2$. Then we have

$$\sup_k |k^{-1}\lambda_k y_k| = \infty.$$

Hence we can find a strictly increasing sequence (k_j) of positive integers k_j such that

$$|k_j^{-1}\lambda_{k_j}y_{k_j}| > j^2 \text{ for all } j \geq 1.$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} |y_{k_j}^{-1}|, & \text{if } k = k_j \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in D_1$, because

$$\sum_{k=1}^{\infty} |k\lambda_k^{-1}x_k| = \sum_{j=1}^{\infty} |k_j\lambda_{k_j}^{-1}y_{k_j}^{-1}| \leq \sum_{j=1}^{\infty} j^{-2} < \infty.$$

Thus $x \in D_1$ but $\sum_{k=1}^{\infty} |y_kx_k| = \sum_{j=1}^{\infty} |a_{k_j}x_{k_j}| = \infty$. This is a contradiction to $y \in D_1^{\alpha}$. Hence $y \in D_2$. □

Corollary 2.5. *For $Z = c$ and l_∞:*

(i) $[Z(M_k, \Delta)]^{\alpha} = [\tilde{Z}(M_k, \Delta)]^{\alpha} = H_1$,

(ii) $H_1^{\alpha} = H_2$, *where*

$$H_1 = \{a = (a_k) : \sum_{k=1}^{\infty} |ka_k| < \infty\} \text{ and}$$

$$H_2 = \{b = (b_k) : \sup_k |k^{-1}b_k| < \infty\}.$$

Proof. The proof follows by taking $\lambda_k = 1$, for all $k \in \mathbb{N}$ and proceeding as in theorem 2.4 so we omit it. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then:*

(i) $[c(M_k, \Lambda, \Delta)]^N = [l_\infty(M_k, \Lambda, \Delta)]^N = G_1$,

(ii) $[\tilde{c}(M_k, \Lambda, \Delta)]^N = [\tilde{l}_\infty(M_k, \Lambda, \Delta)]^N = G_1$, *where* $G_1 = \{a = (a_k) : \lim_k k\lambda_k^{-1}a_k = 0\}$.

Proof. Once again proof is on similar lines as the proof of Theorem 2.4 so we omit it. □

Corollary 2.7. *For $Z = c$ and l_∞,*

(i) $[Z(M_k, \Delta)]^N = [\tilde{Z}(M_k, \Delta)]^N = L_1$, *where* $L_1 = \{a = (a_k) : \lim_k ka_k = 0\}$.

Theorem 2.8. *If M_k satisfies the Δ_2-condition, then we have $Z(M_k, \Lambda, \Delta) = \tilde{Z}(M_k, \Lambda, \Delta)$, for every $Z = c_0$, c and l_∞.*

Proof. We prove for the case $Z = l_\infty$ and the other will be prove by applying similar arguments.

To prove the theorem, it is enough to show that $l_\infty(M_k, \Lambda, \Delta)$ is a subspace of $\tilde{l}_\infty(M_k, \Lambda, \Delta)$.

Let $x \in l_\infty(M_k, \Lambda, \delta)$, then for some $\rho > 0$,

$$\sup_k M_k\Big(\frac{|\Delta\lambda_k x_k|}{\rho}\Big) < \infty.$$

Therefore,

$$M_k\Big(\frac{|\Delta\lambda_k x_k|}{\rho}\Big) < \infty \text{ for every } k \in \mathbb{N}.$$

Choose an arbitrary $\sigma > 0$. If $\rho \leq \sigma$ then $M_k\big(\frac{|\Delta\lambda_k x_k|}{\sigma}\big) < \infty$ for every $k \in \mathbb{N}$. Let $\sigma < \rho$ and put $l = \frac{\rho}{\sigma} > 1$.

Since M_k satisfies the Δ_2-condition, there exists a constant K such that

$$M_k\Big(\frac{|\Delta\lambda_k x_k|}{\sigma}\Big) \leq K\Big(\frac{\rho}{\sigma}\Big)^{\log_2 k} M_k\Big(\frac{|\Delta\lambda_k x_k|}{\rho}\Big) < \infty \text{ for every } k \in \mathbb{N}.$$

Now let us denote

$$U = \sup_k M_k\Big(\frac{|\Delta\lambda_k x_k|}{\rho}\Big) < \infty \text{ for fixed } \rho > 0.$$

Then it follows that for every $\sigma > 0$, we have

$$\sup_k M_k\Big(\frac{|\Delta\lambda_k x_k|}{\sigma}\Big) \leq K\Big(\frac{\rho}{\sigma}\Big)^{\log_2 k} U < \infty.$$

This completes the proof. □

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