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RELATION BETWEEN THE SMALLEST AND GREATEST PARTS OF THE OVERPARTITIONS OF n

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Abstract: In this paper the formulae for the number of smallest parts of overpartitions of and relations between smallest parts and greatest parts are obtained.

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1. Introduction

We adopt the common notation on partitions as used in [1], [2] and [7]. Throughout this article λ stands for an *overpartition* of n; $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r$.

Let $\overline{\xi}(n)$ denote the set of all *overpartitions* of n, and $\overline{p}(n)$ the cardinality of $\overline{\xi}(n)$ for $n \in N$. If $1 \leq r \leq n$ we write $\overline{p_r}(n)$ for the number of *overpartitions* of n each consisting of exactly r parts, that is, r-*overpartitions* of n. If $r \leq 0$ or $r \geq n$ we write $\overline{p_r}(n) = 0$ and $\overline{\xi^*}(n)$ denotes the set of all conjugate *overpartitions* of n.

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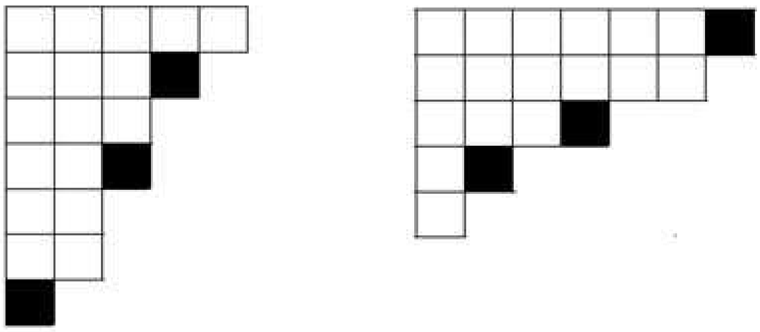

$\lambda = (5, \bar{4}, 3, \bar{3}, 2, 2, \bar{1})$ $\lambda^* = (\bar{7}, 6, \bar{4}, \bar{2}, 1)$

Figure 1: Illustration of the Conjugate of an Overpartition

Figure 1 helps us to formulate the definition of the conjugate λ^* of an *overpartition* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$. The conjugate $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_r^*)$ is a *s-overpartition* of n for some $n \in N$.

Let $\overline{spt}\,(n)$ denote the number of smallest parts including repititions in all *overpartitions* of n. For $i \geq 1$, we adopt the folllowing notation:

$$\overline{spt}\,(n) = \sum_{\lambda\,\in\,\bar{\xi}(n)} \overline{n_s}\,(\lambda)\,.$$

where $\overline{n_s}\,(\lambda)$ is the number of the smallest parts of λ.

If $\mu_1, \mu_2, ...\mu_k$ are the distinct parts in λ where $\mu_1 > \mu_2 > ... > \mu_k$, then $\mu_i = g_i = ith$ greatest part of λ, $\mu_{k-i+1} = ith$ smallest part of λ and $\overline{n_{s_i}}\,(\lambda)$ is the number of the *ith* smallest parts in λ, that is, the number of μ_{k-i+1}'s in λ. We write $s_i\,(\lambda) = \mu_{k-i+1}$ and $t_i = s_i - s_{i-1}$ if both s_i, s_{i-1} exist, $t_i = s_i$ if s_i exists but s_{i-1} does not exist and $t_i = 0$ if s_i does not exist but s_{i-1} exists. Let $\overline{spt_i}\,(n)$ denote the number of the *ith* smallest parts in all *partitions* of n, and

$$sum\,(s_i) = \sum_{\lambda\,\in\,\xi,(n)} \mu_{k-i+1}.$$

Dually we define the greatest part of λ, $\overline{gpt}\,(\lambda)\,, \overline{n_g}\,(\lambda)\,, ith$ greatest part $g_i\,(\lambda)$ of λ and $sum\,(g_i)\,.$

A list of *overpartitions* of 4 with their corresponding

$$s_1\,(\lambda)\,,\quad s_2\,(\lambda)\,,\quad n_{s_i}\,(\lambda)\,,\quad n_{s_2}\,(\lambda)\,,\quad g_1\,(\lambda)\,,\quad g_2\,(\lambda)\,,\quad n_{g_1}\,(\lambda)\quad \text{and}\quad n_{g_2}\,(\lambda)$$

is provided in table 1. We see that $\overline{spt_1}\,(4) = 26$, $\overline{spt_2}\,(4) = 8$, $\overline{gpt_1}\,(4) = 22$, $\overline{gpt_2}\,(4) = 12$, $gpt_2\,(n) = 14$, $sum\,(s_2 - s_1) = 12$ and $sum\,(g_1 - g_2) = 26$.

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$\lambda \in \xi(n)$	$s_1(\lambda)$	$s_2(\lambda)$	$s_2 - s_1$	$n_{s_1}(\lambda)$	$n_{s_2}(\lambda)$	$g_1(\lambda)$	$g_2(\lambda)$	$g_1 - g_2$	$n_{g_1}(\lambda)$	$n_{g_2}(\lambda)$
(4)	4	$--$	0	1	0	4	$--$	4	1	0
$(\overline{4})$	$\overline{4}$	$--$	0	1	0	$\overline{4}$	$--$	4	1	0
$(3,1)$	1	3	2	1	1	3	1	2	1	1
$(\overline{3},1)$	1	$\overline{3}$	2	1	1	$\overline{3}$	1	2	1	1
$(3,\overline{1})$	$\overline{1}$	3	2	1	1	3	$\overline{1}$	2	1	1
$(\overline{3},\overline{1})$	$\overline{1}$	$\overline{3}$	2	1	1	$\overline{3}$	$\overline{1}$	2	1	1
$(2,2)$	2	$--$	0	2	0	2	$--$	2	2	0
$(\overline{2},2)$	2	$--$	0	2	0	$\overline{2}$	$--$	2	2	0
$(2,1,1)$	1	2	1	2	1	2	1	1	1	2
$(\overline{2},1,1)$	1	$\overline{2}$	1	2	1	$\overline{2}$	1	1	1	2
$(2,\overline{1},1)$	1	2	1	2	1	2	$\overline{1}$	1	1	2
$(\overline{2},\overline{1},1)$	1	$\overline{2}$	1	2	1	$\overline{2}$	1	1	1	2
$(1,1,1,1)$	1	$--$	0	4	0	1	$--$	1	4	0
$(\overline{1},1,1,1)$	1	$--$	0	4	0	$\overline{1}$	$--$	1	4	0

Table 1: List of overpartitions of 4

Let $\overline{p}(k,n)$ represent the number of *overpartitions of* n using natural numbers at least as large as k only, and $\overline{G}(s,n)$ denote the number of *overpartitions* of n having the greatest part s.

We now obtain a relation between the ith smallest parts and the ith greatest parts of the *overpartitions* of the positive integer n. In particular, we show that

$$\overline{spt}(n) = \sum_{\lambda \in \xi(n)} g_1(\lambda) - \sum_{\lambda \in \xi(n)} g_n(\lambda).$$

Theorem 1. $\overline{\xi}(n) = \overline{\xi^*}(n)$.

Proof. The Ferrer diagram for λ consists of r rows of dots, the ith row having λ_i dots. This clearly the columns also have dots in decreasing numbers. Hence the rows in λ^* have dots in decreasing numbers. Since λ, λ^* have the same dots, $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_s^*)$ is an *overpartition* of n. Further $\lambda^{**} = \lambda$. Hence $\overline{\xi}(n) = \overline{\xi^*}(n)$. □

Theorem 2. For $1 \leq i \leq n$ and $\lambda \in \overline{\xi}(n)$, let g_i be the ith greatest part of λ. Then

$$\sum_{\lambda \in \overline{\xi}(n)} g_i = \sum_{\lambda \in \overline{\xi}(n)} g_i^*.$$

Proof. From Theorem 1, we have

$$\overline{\xi}(n) = \overline{\xi^*}(n) \Rightarrow \sum_{\lambda \in \overline{\xi}(n)} g_i = \sum_{\lambda \in \overline{\xi}(n)} g_i^*.$$

□

Theorem 3. For each i, if

(i) g_i^* is the ith greatest part of λ^*, then $\overline{n_{s_i}}(\lambda) = g_i^* - g_{i+1}^*$

(ii) the $(i+1)\,th$ greatest part does not exist, then $\overline{n_{s_i}}(\lambda) = g_i^*$

(iii) the ith greatest part does not exist, then $\overline{n_{s_i}}(\lambda) = 0$.

Proof. Let $\lambda \in (\lambda_1, \lambda_2, ..., \lambda_r) \in \overline{\xi}(n)$ and $\lambda^* \in (\lambda_1^*, \lambda_2^*, ..., \lambda_r^*) \in \overline{\xi^*}(n)$.

The Ferrer diagram of λ and λ^* can be partitioned into rows having equal number of blocks which can be put in matrix form as shown below. We observe that the number of rows in the ith matrix from bottom to top of the diagram of λ is equal to the number for the ith smallest part of λ, and that the number of columns in the ith matrix from top to bottom of the diagram for λ^* is equal to the ith greatest part of λ^*. We also observe that

(i) If both the ith and the $(i+1)\,th$ greatest parts of λ^* exist, then the difference between the ith and the $(i+1)\,th$ greatest parts of λ^* is equal to the number of the ith smallest parts of λ. (See Figure 2).

(ii) If the ith greatest part of λ^* exists and the $(i+1)\,th$ greatest part of λ^* does not exist, then the value of the ith greatest part of λ^* is equal to the number of the ith smallest part of λ.

(iii) If the ith greatest part of λ^* does not exist and $(i+1)\,th$ greatest part of λ^* exists, then the difference between the ith and the $(i+1)\,th$ greatest part of λ^* is zero.

Hence it follows that $n_{s_i}(\lambda) = g_i^* - g_{i+1}^*$ if both g_i^* and g_{i+1}^* exist, $n_{s_i}(\lambda) = g_i^*$ if g_{i+1}^* does not exist, and $n_{s_i}(\lambda) = 0$ if g_i^* does not exist. □

Theorem 4. $\overline{spt_i}(n) = \sum_{\lambda \in \overline{\xi}(n)} g_i(\lambda) - \sum_{\lambda \in \overline{\xi}(n)} g_{i+1}(\lambda)$

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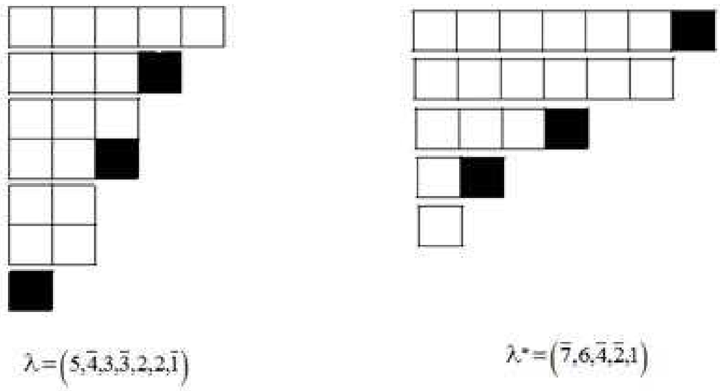

Figure 2: Illustration of the Conjugate of an Overpartition

Proof. Since $\overline{spt_i}(n) = \sum_{\lambda \in \overline{\xi}(n)} \overline{n_{s_i}}(\lambda)$, from Theorem (3), it follows that

$$\overline{n_{s_i}}(\lambda) = g_i^*(\lambda) - g_{i+1}^*(\lambda),$$

or

$$\overline{n_{s_i}}(\lambda) = g_i^*(\lambda).$$

$$\Rightarrow \sum_{\lambda \in \overline{\xi}(n)} \overline{n_{s_i}}(\lambda) = \sum_{\lambda \in \overline{\xi}(n)} \left[g_i^*(\lambda) - g_{i+1}^*(\lambda)\right],$$

which means that from theorem (2)

$$\Rightarrow \sum_{\lambda \in \overline{\xi}(n)} \overline{n_{s_i}}(\lambda) = \sum_{\lambda \in \overline{\xi}(n)} \left[g_i(\lambda) - g_{i+1}(\lambda)\right]$$

$$\Rightarrow \overline{spt_i}(n) = \sum_{\lambda \in \overline{\xi}(n)} g_i(\lambda) - \sum_{\lambda \in \overline{\xi}(n)} g_{i+1}(\lambda).$$

□

As a consequence of Theorem 4, we have the following.

Corollary 5. If $\lambda \in \overline{\xi}(n)$, then

$$\overline{spt}(n) = \sum_{\lambda \in \overline{\xi}(n)} g_1(\lambda) - \sum_{\lambda \in \overline{\xi}(n)} g_2(\lambda).$$

Theorem 6. If the *overpartition* $\lambda \in \overline{\xi}(n)$ has k distinct parts, then $g_i = s_{k-i+1}$ for $1 \leq i \leq k$.

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Proof. By theorem, the ith matrix from top to bottom in the Ferrer diagram is the same as the $(k-i+1)\,th$ matrix from bottom to top. Hence $g_i = s_{k-i+1}$ for $1 \le i \le k$. □

Theorem 7. $\overline{spt_i}\,(n) = \sum_{\lambda \,\in\, \overline{\xi}(n)} t_i\,(\lambda)$.

Proof. From Theorem 4,

$$\overline{spt_i}\,(n) = \sum_{\lambda \,\in\, \overline{\xi}(n)} \left[g_i\,(\lambda) - g_{i+1}\,(\lambda)\right],$$

and from Theorem 6,

$$\overline{gpt}_{k-i+1}\,(n) = \sum_{\lambda \,\in\, \overline{\xi}(n)} \left[s_{k-i+1}\,(\lambda) - s_{k-i+1+1}\,(\lambda)\right],$$

which gives

$$\overline{gpt}_t\,(n) = \sum_{\lambda \,\in\, \overline{\xi}(n)} \left[s_t\,(\lambda) - s_{t+1}\,(\lambda)\right], \quad \text{where } t = k-i+1$$

$$\Rightarrow \overline{gpt}_t\,(n) = \sum_{\lambda \,\in\, \overline{\xi}(n)} t_i\,(\lambda)\,.$$

□

As a consequence of Theorem 7, we have the following

Corollary 8. If $\lambda \in \xi\,(n)$, then

$$\overline{gpt}\,(n) = \sum_{\lambda \,\in\, \overline{\xi}(n)} \left[s_1\,(\lambda) - s_2\,(\lambda)\right].$$

Note. $\sum_{\lambda \,\in\, \overline{\xi}(n)} \left[s_i\,(\lambda) - s_{t+1}\,(\lambda)\right] \neq \sum_{\lambda \,\in\, \overline{\xi}(n)} s_i\,(\lambda) - \sum_{\lambda \,\in\, \overline{\xi}(n)} s_{i+1}\,(\lambda)$ because when $s_{i+1}\,(\lambda)$ does not exist, then $s_i\,(\lambda) - s_{t+1}\,(\lambda) = 0$.

Theorem 9.

$$\sum_{\lambda \,\in\, \overline{\xi}(n)} g_i\,(\lambda)$$

$$= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_{i-1}=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1}\cdots\sum_{g_i=1}^{g_{i-1}-1} 2g_i\overline{p}_{g_i}\,(n - k_1g_1 - k_2g_2 - \ldots - k_{i-1}g_{i-1})\,.$$

Proof. From [5], for every g we have

$$G(g,n) = p_g(n).$$

Similarly we have for every g

$$\overline{G}(g,n) = \overline{p}_g(n).$$

Hence

$$\sum_{\lambda \in \overline{\xi}(n)} g_1(\lambda) = \sum_{g_1=1}^{n} g_1 \cdot p_{g_1}(n).$$

If the greatest part g_1 appears k_1 times (without the overline) followed by its successor g_2, then

$$\overline{G}(g_2, n-k_1g_1) = \overline{p}_{g_2}(n-k_1g_1),$$

and if the greatest part g_1 appears k_1 times (with the overline in the first part) followed by its successor g_2, then

$$\overline{G}(g_2, n-k_1g_1) = \overline{p}_{g_2}(n-k_1g_1).$$

Therefore if the greatest part g_1 appears k_1 times (irrespective of the overline) followed by its successor g_2, then

$$\overline{G}(g_2, n-k_1g_1) = 2\overline{p}_{g_2}(n-k_1g_1).$$

The sum of these second greatest parts taken over all partitions is

$$\sum_{\lambda \in \overline{\xi}(n)} 2g_2\overline{p}_{g_2}(n-k_1g_1).$$

Hence

$$\sum_{\lambda \in \overline{\xi}(n)} g_2(\lambda) = \sum_{k_1=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_{1-1}} 2g_2\overline{p}_{g_2}(n-k_1g_1).$$

The theorem follows by repeating the process:

$$\sum_{\lambda \in \overline{\xi}(n)} g_i(\lambda)$$
$$= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_{i-1}=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1}\cdots\sum_{g_i=1}^{g_{i-1}-1} 2g_i\overline{p}_{g_i}(n-k_1g_1-k_2g_2-\ldots-k_{i-1}g_{i-1}).$$

□

In general, Theorem 5 leads to the following:

Theorem 10.

$$\overline{spt_i}(n) = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}...\sum_{k_{i-1}=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1}...\sum_{g_i=1}^{g_{i-1}-1} 2g_i\overline{p}_{g_i}(n-k_1g_1-k_2g_2-...-k_{i-1}g_{i-1})$$
$$-\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}...\sum_{k_i=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1}...\sum_{g_{i+1}=1}^{g_i-1} 2g_{i+1}\overline{p}_{g_{i+1}}(n-k_1g_1-k_2g_2-...-k_ig_i).$$

Proof. From Theorem 5,

$$\overline{spt_i}(n) = \sum_{\lambda\in\overline{\xi}(n)} g_i(\lambda) - \sum_{\lambda\in\overline{\xi}(n)} g_{i+1}(\lambda)$$

$$\Rightarrow \overline{spt_i}(n) = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}...\sum_{k_{i-1}=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1}...\sum_{g_i=1}^{g_{i-1}-1} 2g_i\overline{p}_{g_i}(n-k_1g_1-k_2g_2-...-k_{i-1}g_{i-1})$$
$$-\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}...\sum_{k_i=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1}...\sum_{g_{i+1}=1}^{g_i-1} 2g_{i+1}\overline{p}_{g_{i+1}}(n-k_1g_1-k_2g_2-...-k_ig_i).$$

□

Corollary 11. $\overline{spt}(n) = \sum_{g_i=1}^{\infty} 2g_1\overline{p}_{g_1}(n) - \sum_{k_1=1}^{\infty}\sum_{g_1=1}^{\infty}\sum_{g_2=1}^{g_1-1} 2g_2\overline{p}_{g_2}(n-k_1g_1)$.

Proof. Put $i=1$ in Corollary 10. □

Theorem 12. The number of *r-overpartitions of* n having k as a non-overlined smallest part is

$$j + \sum_{i=0}^{\infty}\overline{p}_{r-1-i}\left[n-(k-1)r-1-i\right].$$

where $j=1$ if r divides n, otherwise $j=0$.

Proof. From [4], the number of *r-partitions* of n with the smallest part k is given by

$$p_{r-1}\left[n-(k-1)r-1\right].$$

So we easily observe that the number of *r-overpartitions* of n with non-overlined smallest part k is

$$\overline{p}_{r-1}\left[n-(k-1)r-1\right].$$

We fix $k\in\{1,2,...,n\}$. For $1\le i\le r$, the number of *r-overpartitions* of n with $(r-i)$ non-overlined smallest parts each being k is the number of *i-overpartitions* of $n-(r-i)\,k$. Summing over $i=1$ to r we get the total number of *r-overpartitions* of n with non-overlined smallest parts k. This number is

$$j+\sum_{i=0}^{\infty}\overline{p}_{r-1-i}\left[n-(k-1)\,r-1-i\right],$$

where $j=1$ if r divides n, otherwise $j=0$. □

As k varies from 1 to n, we have the following:

Corollary 13. The total number of *r-overpartitions* of n having k as a non-overlined smallest part is

$$\sum_{k\,=\,1}^{\infty}\left[j+\sum_{i\,=\,0}^{\infty}\overline{p}_{r-1-i}\left[n-(k-1)\,r-1-i\right]\right],$$

where $j=1$ if r divides n, otherwise $j=0$.

Corollary 14. By taking the sum as r varies, we ger the number of non-overlined smallest parts

$$\sum_{r\,=\,1}^{\infty}\sum_{k\,=\,1}^{\infty}\left[j+\sum_{i=0}^{\infty}\overline{p}_{r-1-i}\left[n-(k-1)\,r-1-i\right]\right],$$

where $j=1$ if r divides n, otherwise $j=0$.

We now derive independently another formula for the number of non-overlined smallest parts of all *overpartitions* of a positive integer n.

Theorem 15. The number of non-overlined smallest parts of all overpartitions of a positive integer n is

$$\sum_{t_1\,=\,1}^{\infty}\sum_{s_{i_1}\,=\,1}^{\infty}p\left(s_{i_1},n-t_1s_{i_1}\right)+d\left(n\right),$$

where $d\left(n\right)$ is the number of positive divisors of n.

Proof. Any *overpartition* in $\overline{\xi}\left(n\right)$ has a smallest part which possibly repeats.

(i) If the non-overlined smallest part d is a divisor of n, then the number of *overpartitions* with d as a non-overlined smallest part is

$$1+\sum_{t\,=\,1}^{\left[\frac{n}{d}\right]-1}\overline{p}\left(d,n-td\right).$$

Here 1 corresponds to $(d,d,...d)$ where d repeats $\dfrac{n}{d}$ times.

□

If d is not a divisor of n, there is no *overpartition* with equal parts. In this case, the total number of *overpartition* with d as a non-overlined smallest part is

$$\sum_{t\,=\,1}^{\left[\frac{n}{d}\right]-1} \overline{p}\left(d, n-td\right).$$

Therefore the number of overlined smallest parts of all *overpartitions* of n is

$$\sum_{t\,=\,1}^{\left[\frac{n}{d}\right]-1} \overline{p}\left(d, n-td\right) + \sum_{d/n} 1 = \sum_{t\,=\,1}^{\left[\frac{n}{d}\right]-1} \overline{p}\left(d, n-td\right) + d\left(n\right) = RHS.$$

As a consequence of Theorem 4.4.15, we have

$$d^1\left(n-ts\right) = \text{ number of divisors of } n-ts \text{ that are greater than } s.$$

The number of non-overlined second smallest parts of all *overpartitions* of a positive integer n is therefore

$$\sum_{s_1\,=\,1}^{\infty} \sum_{t_1\,=\,1}^{\infty} \left[\sum_{s_2=s_1+1}^{\infty} \sum_{t_2=1}^{\infty} p\left(s_1, n-t_1 s_1 - t_2 s_2\right) + d\left(n - t_1 s_1\right) \right]$$

More generally, if

$$d^1\left(n - t_1 s_1 - t_2 s_2 - \ldots - t_{i-1} s_{i-1}\right)$$

is the number of divisors of $n^1 - t_1 s_1 - t_2 s_2 - \ldots - t_{i-1} s_{i-1}$ that are greater than s_{i-1}, the number of non-overlined *ith* smallest parts of all *overpartitions* of a positive integer n is therefore

$$\sum_{s_1\,=\,1}^{\infty} \sum_{s_2=s_1+1}^{\infty} \ldots \sum_{s_i=s_{i-i}+1}^{\infty} \sum_{t_1\,=\,1}^{\infty} \sum_{t_2\,=\,1}^{\infty} \ldots \sum_{t_{i-1}\,=\,1}^{\infty} \sum_{t_i\,=\,1}^{\infty} p\left(s_1, n - t_1 s_1 - t_2 s_2 - \ldots - t_i s_i\right)$$
$$+ \sum_{s_1\,=\,1}^{\infty} \sum_{s_2=s_1+1}^{\infty} \ldots \sum_{s_{i-1}=s_{i-2}+1}^{\infty} \sum_{t_1\,=\,1}^{\infty} \sum_{t_2\,=\,1}^{\infty} \ldots \sum_{t_{i-2}\,=\,1}^{\infty} \sum_{t_{i-1}\,=\,1}^{\infty} d\left(n - t_1 s_1 - t_2 s_2 - \ldots - t_{i-1} s_{i-1}\right).$$

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