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ON SOME PROJECTIVELY FLAT (α, β)-METRICS

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Abstract: In this article, we consider some (α, β)-metric, and discuss the condition for a Finsler space with (α, β)-metric to be projectively flat on the basis of Matsumoto's results where α is a Riemannian metric and β is a differential 1-form.

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1. Introduction

The (α, β)-metric is an important class of Finsler metrics. The Finsler space $F^n = (M^n, L)$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_i(x)y^i$. A change $L \to \bar{L}$ of a Finsler metric on a same underlying manifold M^n is called projective, if any geodesic on (M^n, L) remains to be a geodesic in $(M^n, \bar{L})$ and vice versa. A Finsler space is called projectively flat, or with rectillinear geodesic, if the space is covered by coordinate neighborhoods in which the geodesics can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called rectillinear. The condition for a Finsler space to be projectively flat was studied by L. Berwald[5]. The condition for a Finsler space with (α, β)-metric to be projectively flat was studied by M. Matsumoto[7]. Later on many authors worked on projective flatness of (α, β)-metric[1][2][3][4]. M. Hashiguchi and Y. Ichijyo[6] have worked on some special (α, β)-metric, In our paper we concentrated on the metrics $\frac{(\alpha+\beta)^2}{\alpha}$ and $\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$(second approximate Matsumoto metric).

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2. Preliminaries

We consider a Finsler space with an (α,β)-metric $L(\alpha,\beta)$. The space $R^n=(M^n,\alpha)$ is called the associated Riemannian space. Let $\gamma_j{}^i{}_k(x)$ be the Christoffel symbols constructed from α and we denote the covariant differentiation with respect to $\gamma_j{}^i{}_k(x)$ by $(/)$. From the differential 1-form $\beta(x,y)=b_i(x)y^i$ we define

$$2r_{ij}=b_{i/j}+b_{j/i},\quad 2s_{ij}=b_{i/j}-b_{j/i},$$
$$s^i_j=a^{ih}s_{hj},\quad s_j=b_is^i_j,\quad b^i=a^{ih}b_h,\quad b^2=b^ib_i.$$

We use the well known Matsumoto's theorem.

Theorem 1. *A Finsler space (M,L) with an (α,β)-metric $L(\alpha,\beta)$ is projectively flat if and only if for any point of space M there exist local coordinate neighborhoods containing the point such that $\gamma_j{}^i{}_k$ satisfies:*

$$(\gamma_0{}^i{}_0-\gamma_{000}y^i/\alpha^2)/2+(\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C+\alpha r_{00}/2\beta)(\alpha^2b^i/\beta-y^i)=0, \quad (2.1)$$

where C is given by

$$C+(\alpha^2L_\beta/\beta L_\alpha)s_0+(\alpha L_{\alpha\alpha}/\beta^2L_\alpha)(\alpha^2b^2-\beta^2)(C+\alpha r_{00}/2\beta)=0. \quad (2.2)$$

Equation (2.2) can be written as

$$(C+\alpha r_{00}/2\beta)\{1+(\alpha L_{\alpha\alpha/\beta^2L_\alpha})(\alpha^2b^2-\beta^2)\}-(\alpha/2\beta)\{r_{00}-(2\alpha L_\beta/L_\alpha)s_0\}=0,$$

that is,

$$C+\alpha r_{00}/2\beta=\frac{\alpha\beta(r_{00}L_\alpha-2\alpha L_\beta s_0)}{2\{\beta^2L_\alpha+\alpha L_{\alpha\alpha}(\alpha^2b^2-\beta^2)\}}. \quad (2.3)$$

3. Finsler Space with the Metric $\frac{(\alpha+\beta)^2}{\alpha}$

Let F^n be a Finsler space with an (α,β)-metric given by

$$L=\frac{(\alpha+\beta)^2}{\alpha}. \quad (3.1)$$

The partial derivatives with respect to α and β of a metric (3.1) are given by

$$L_\alpha=\frac{\alpha^2-\beta^2}{\alpha^2},\quad L_\beta=\frac{2(\alpha+\beta)}{\alpha},\quad L_{\alpha\alpha}=\frac{2\beta^2}{\alpha^3}. \quad (3.2)$$

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Substituting (3.2) in (2.3), we obtain

$$C+\alpha r_{00}/2\beta=\frac{\alpha\{r_{00}(\alpha^2-\beta^2)-4\alpha^2(\alpha+\beta)s_0\}}{2\beta\{(1+2b^2)\alpha^2-3\beta^2\}}. \tag{3.3}$$

Plugging (3.2) and (3.3) in (2.1) we obtain

$$\begin{aligned}&(\alpha^2-\beta^2)(\alpha^2(1+2b^2)-3\beta^2)(\alpha^2\gamma_0{}^i{}_0-\gamma_{000}y^i)\\&+4\alpha^4(\alpha+\beta)(\alpha^2(1+2b^2)-3\beta^2)s^i_0+2\alpha^2\{(\alpha^2-\beta^2)r_{00}-4\alpha^2(\alpha+\beta)s_0\}(\alpha^2b^i-\beta y^i)\\&=0.\end{aligned} \tag{3.4}$$

Only the term $-3\beta^4\gamma_{000}y^i$ of (3.4) seemingly does not contain α^2, hence we must have hp(6)v^i_6 satisfying $-3\beta^4\gamma_{000}y^i=\alpha^2v^i_6$. For the sake of brevity we suppose $\alpha^2\not\equiv 0(mod\beta)$. Then the above is written as

$$\gamma_{000}=v_0\alpha^2, \tag{3.5}$$

where v_0 is hp(1). Substituting (3.5) in (3.4), we obtain

$$\begin{aligned}&(\alpha^2-\beta^2)(\alpha^2(1+2b^2)-3\beta^2)(\gamma_0{}^i{}_0-v_0y^i)+4\alpha^2(\alpha+\beta)(\alpha^2(1+2b^2)-3\beta^2)s^i_0\\&+2\{(\alpha^2-\beta^2)r_{00}-4\alpha^2(\alpha+\beta)s_0\}(\alpha^2b^i-\beta y^i)=0.\end{aligned} \tag{3.6}$$

The terms $\beta^3\{3\beta(\gamma_0{}^i{}_0-v_0y^i)+2r_{00}y^i\}$ of (3.6) seemingly does not contain α^2. Consequently we must have hp(1)u^i_0 such that the above is equal to $\alpha^2\beta^3u^i_0$, that is

$$3\beta(\gamma_0{}^i{}_0-v_0y^i)+2r_{00}y^i=\alpha^2u^i_0. \tag{3.7}$$

We contract (3.7) by $a_{ir}y^r$, which yields

$$2r_{00}=u^i_0y_i. \tag{3.8}$$

Now from (3.7) and (3.8), we obtain

$$\gamma_0{}^i{}_0=v_0y_i, \tag{3.9}$$

which implies

$$2\gamma_j{}^i{}_k=v_k\delta^i_j+v_j\delta^i_k. \tag{3.10}$$

which shows that the associated Riemannian space is projectively flat.

Substituting (3.9) in (3.6), we obtain

$$\begin{aligned}&4\alpha^2(\alpha+\beta)(\alpha^2(1+2b^2)-3\beta^2)s^i_0+\\&2\{(\alpha^2-\beta^2)r_{00}-4\alpha^2(\alpha+\beta)s_0\}(\alpha^2b^i-\beta y^i)=0.\end{aligned} \tag{3.11}$$

Transvecting (3.11) by b_i, we get

$$2\alpha^2(\alpha+\beta)s_0 + (\alpha^2 b^2 - \beta^2)r_{00} = 0. \tag{3.12}$$

Since $\alpha^2 b^2 - \beta^2$ of (3.11) does not contain α^2 as a factor, we must have a function $k(x)$ such that

$$r_{00} = k(x)\alpha^2. \tag{3.13}$$

Substituting (3.13) in (3.12), we obtain $2(\alpha+\beta)s_0 + k(x)(\alpha^2 b^2 - \beta^2) = 0$, leads to $k(x) = 0$ because $(\alpha^2 b^2 - \beta^2)$ does not vanish. Hence we have

$$r_{00} = 0;\ \ r_{ij} = 0 \ \ \textit{and} \ \ s_0 = 0;\ \ s_i = 0. \tag{3.14}$$

Plugging (3.14) in (3.11), we get $s^i_0 = 0$ that is $s_{ij} = 0$.

Since $r_{ij} = 0$ and $s_{ij} = 0$, $b_{i/j} = 0$. Conversely, if $b_{i/j} = 0$, then we get $r_{00} = 0$, $s^i_0 = 0$ and $s_0 = 0$. So (3.6) follows from (3.9).

Hence we conclude that

Theorem 2. *A Finsler space F^n with an (α,β)-metric given by (3.1) is projectively flat, if and only if we have $b_{i/j} = 0$ and the associated Riemannian space (M^n,α) is projectively flat.*

4. Finsler Space with the Metric $\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$

The Matsumoto metric $L = \alpha^2/(\alpha - \beta)$ is expressed as the form

$$L = \alpha \left\{ \sum_{k=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \right\} \tag{4.1}$$

for $|\beta| < |\alpha|$. We regard $b_i(x)$ as very small numerically. If we neglect all the power ≥ 2 of $b_i(x)$ in (4.1), then $L = \alpha + \beta$, that is a Randers metric. If we neglect all the power ≥ 3 of $b_i(x)$ in (4.1), then L is the first approximate Matsumoto metric. Hereafter we neglect all the power ≥ 4 of $b_i(x)$ in (4.1), then (α,β)-metric

$$L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} \tag{4.2}$$

is an approximate metric to the Matsumoto metric. We shall call the (α,β)-metric (4.2) the second approximate Matsumoto metric.

The partial derivatives with respect to α and β of a metric (4.2) are given by

$$L_\alpha = \frac{\alpha^3 - \alpha\beta^2 - 2\beta^3}{\alpha^3}, \quad L_\beta = \frac{\alpha^2 + 2\alpha\beta + 3\beta^2}{\alpha^2}, \quad L_{\alpha\alpha} = \frac{2\alpha\beta^2 + 6\beta^3}{\alpha^4}. \tag{4.3}$$

Substituting (4.3) in (2.3), we obtain

$$C + \alpha r_{00}/2\beta = \frac{\alpha\{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)s_0\}}{2\beta\{(1 + 2b^2)\alpha^3 - 3\alpha\beta^2 + 6b^2\alpha^2\beta - 8\beta^3\}}. \tag{4.4}$$

Plugging (4.3) and (4.4) in (2.1) we obtain

$$\begin{aligned}&(\alpha^3 - \alpha\beta^2 - 2\beta^3)\{\alpha^3(1 + 2b^2) - 3\alpha\beta^2 + 6b^2\alpha^2\beta - 8\beta^3\}(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)\\ &+2\alpha^4(\alpha^2 + 2\alpha\beta + 3\beta^2)\{\alpha^3(1 + 2b^2) - 3\alpha\beta^2 + 6b^2\alpha^2\beta - 8\beta^3\}s^i_0\\ &+2\alpha^2(\alpha + 3\beta)\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)r_{00}\\ &-2\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)s_0\}(\alpha^2b^i - \beta y^i) = 0.\end{aligned} \tag{4.5}$$

Only the term $-16\beta^6\gamma_{000}y^i$ of (4.5) seemingly does not contain α^2, hence we must have hp(8)v^i_8 satisfying $-16\beta^6\gamma_{000}y^i = \alpha^2v^i_8$. For the sake of brevity we suppose $\alpha^2 \not\equiv 0(mod\beta)$. Then the above is written as

$$\gamma_{000} = v_0\alpha^2, \tag{4.6}$$

where v_0 is hp(1). Substituting (4.6) in (4.5), we obtain

$$\begin{aligned}&(\alpha^3 - \alpha\beta^2 - 2\beta^3)\{\alpha^3(1 + 2b^2) - 3\alpha\beta^2 + 6b^2\alpha^2\beta - 8\beta^3\}(\gamma_0{}^i{}_0 - v_0y^i)\\ &+2\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)\{\alpha^3(1 + 2b^2) - 3\alpha\beta^2 + 6b^2\alpha^2\beta - 8\beta^3\}s^i_0\\ &+2(\alpha + 3\beta)\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)r_{00}\\ &-2\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)s_0\}(\alpha^2b^i - \beta y^i) = 0.\end{aligned} \tag{4.7}$$

The terms $4\beta^5\{4\beta(\gamma_0{}^i{}_0 - v_0y^i) + 3r_{00}y^i\}$ of (4.7) seemingly does not contain α^2. Consequently we must have hp(1)u^i_0 such that the above is equal to $\alpha^2\beta^5u^i_0$, that is

$$4\beta(\gamma_0{}^i{}_0 - v_0y^i) + 3r_{00}y^i = \alpha^2u^i_0. \tag{4.8}$$

We contract (4.8) by $a_{ir}y^r$, which yields

$$3r_{00} = u^i_0y_i. \tag{4.9}$$

Now from (4.8) and (4.9), we obtain

$$\gamma_0{}^i{}_0 = v_0y_i, \tag{4.10}$$

which implies

$$2\gamma_j{}^i{}_k = v_k\delta^i_j + v_j\delta^i_k. \tag{4.11}$$

which shows that the associated Riemannian space is projectively flat.

Substituting (4.10) in (4.7), we obtain

$$2\alpha^2(\alpha^2 + 2\alpha\beta + 3\beta^2)\{\alpha^3(1 + 2b^2) - 3\alpha\beta^2 + 6b^2\alpha^2\beta - 8\beta^3\}s^i_0$$

$$+2(\alpha+3\beta)\{(\alpha^3-\alpha\beta^2-2\beta^3)r_{00}$$
$$-2\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)s_0\}(\alpha^2 b^i-\beta y^i)=0. \tag{4.12}$$

Transvecting (4.12) by b_i, we get

$$\alpha^2(\alpha^2+2\alpha\beta+3\beta^2)s_0+(\alpha+3\beta)(\alpha^2 b^2-\beta^2)r_{00}=0. \tag{4.13}$$

Since $\alpha^2 b^2-\beta^2$ of (4.13) does not contain α^2 as a factor, we must have a function $k(x)$ such that

$$r_{00}=k(x)\alpha^2. \tag{4.14}$$

Plugging (4.14) in (4.13), we obtain $(\alpha^2+2\alpha\beta+3\beta^2)s_0+(\alpha+3\beta)(\alpha^2b^2-\beta^2)k(x)=0$, leads to $k(x)=0$ because $(\alpha+3\beta)(\alpha^2b^2-\beta^2)$ does not vanish. Hence we have

$$r_{00}=0;\ \ r_{ij}=0 \ \ and\ \ s_0=0;\ \ s_i=0. \tag{4.15}$$

Substituting (4.15) in (4.12), we get $s^i_0=0$ that is $s_{ij}=0$.

Since $r_{ij}=0$ and $s_{ij}=0$, $b_{i/j}=0$. Conversely, if $b_{i/j}=0$, then we get $r_{00}=0$, $s^i_0=0$ and $s_0=0$. Consequently (4.7) follows from (4.10).

Hence we conclude that

Theorem 3. *A Finsler space F^n with an (α,β)-metric given by (4.2) is projectively flat, if and only if we have $b_{i/j}=0$ and the associated Riemannian space (M^n,α) is projectively flat.*

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