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MULTIPLE FIXED POINT THEOREMS OF OPERATOR TYPE

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Abstract: This paper presents multiple fixed point theorems of operator type applying the compression-expansion fixed point theorem of operator type. An application and example to a boundary value problem are provided to illustrate the key arguments.

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1. Introduction

The Leggett-Williams triple fixed point theorem [11] and its generalization, the five functionals fixed point theorem [2], have been the primary tools used to prove the existence of at least three positive solutions to boundary value problems over the past decade. There have been many different types of fixed point theorems involving functionals and norms that have extended and further generalized these results in recent years. See Kwong [10], Avery-Anderson-Henderson [4], and Sun-Zhang [16]

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to mention a few. Mavridis [12] published the first extension to the work of Leggett-Williams [11] which replaced the arguments involving functionals with arguments involving operators. Invariance arguments were key components in the paper by Mavridis. Avery-Anderson-Henderson-Liu in [5] removed the invariance condition in the spirit of the original work done by Leggett-Williams by working in a function space to prove the existence of at least one fixed point for an operator.

The difficulty in replacing the arguments involving functionals in [2] and other similar generalizations of the work of Leggett-Williams with arguments involving operators lies in the ability to compare the output of an operator to a function using the comparison generated by an underlying cone P. That is, for an operator R and a specified function x_R, one needs to be able to say, for any $y \in P$, that either $Ry < x_R$ or $x_R \leq Ry$. In [12] this issue was dealt away with invariance like conditions. However, in [5], by considering a cone P of a real Banach space E which is a subset of $F(K)$ (the set of real valued functions defined on a set K), the spirit of the original work of Leggett-Williams [11] and the extensions to the outer boundary by Avery-Anderson-Henderson [4] are maintained by avoiding any invariance-like conditions in the arguments. This manuscript extends the existence result of Avery-Anderson-Henderson-Liu in [5] to an existence of at least two and at least three fixed points with a nontrivial application and example to a boundary value problem.

2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if, for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$, and if $x, -x \in P$ then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$, and we say that $x < y$ whenever $x \leq y$ and $x \neq y$. Let K be a subset of real numbers and $F(K)$ the set of all real valued functions defined over K. If $J \subset K$ and $x, y \in F(K)$ we will say that:

$$x <_J y \text{ if and only if } x(t) < y(t) \text{ for all } t \in J,$$

and

$$x \leq_J y \text{ if and only if } x(t) \leq y(t) \text{ for all } t \in J.$$

Furthermore, we will say that

$$x \leqq_J y \text{ if and only if } x \leq_J y \text{ and there exists a } t_0 \in J \text{ such that } x(t_0) = y(t_0).$$

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 3. Let P be a cone in a real Banach space E. Then we say that $A : P \to P$ is a continuous concave operator on P if $A : P \to P$ is continuous and

$$tA(x) + (1-t)A(y) \leq A(tx + (1-t)y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say that $B : P \to P$ is a continuous convex operator on P if $B : P \to P$ is continuous and

$$B(tx + (1-t)y) \leq tB(x) + (1-t)B(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let R and S be operators on a cone P of a real Banach space E which is a subset of $F(K)$, the set of real valued functions defined on a set K. For $J_R, J_S \subset K$ and $x_R, x_S \in E$ we define the sets,

$$P_{J_R}(R, x_R) = \{y \in P : R(y) <_{J_R} x_R\}$$

and

$$P(R, S, x_R, x_S, J_R, J_S) = P_{J_S}(S, x_S) - \overline{P_{J_R}(R, x_R)}.$$

Definition 4. Let R be an operator on a cone P of a real Banach space E which is a subset of $F(K)$, the set of real valued functions defined on a set K. For $J_R \subset K$ and $x_R \in E$, we say that R is comparable to x_R on P relative to J_R if, given any $y \in P$, either

$$R(y) <_{J_R} x_R \text{ or } x_R \leq_{J_R} R(y).$$

Definition 5. Let D be a subset of a real Banach space E. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a *retract* of E, and the map r is a *retraction*. The *convex hull* of a subset D of a real Banach space X is given by

$$conv(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \right\}.$$

The following theorem is due to Dugundji and its proof can be found in [13, *p* 22].

Theorem 6. *Let E and X be Banach spaces and let $f : C \to K$ be a continuous mapping, where C is closed in E and K is convex in X. Then there exists a continuous mapping $\tilde{f} : E \to K$ such that $\tilde{f}(u) = f(u),\ u \in C$.*

Yet in establishing our main results, we will use the following form of Dugundji's theorem [6, *p* 44].

Corollary 7. *For Banach spaces X and Y, let $D \subset X$ be closed and let $F : D \to Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \to Y$ such that $\tilde{F}(X) \subset \overline{conv(F(D))}$.*

Corollary 8. *Every closed convex set in a Banach space is a retract of the Banach space.*

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [7, *pp* 82-86]; an elementary proof can be found in [6, *pp* 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 9. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

$(G1)$ *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*

$(G2)$ *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*

$(G3)$ *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$;*

$(G4)$ *Solution: If $i(A, U, X) \neq 0$, then A has at least one fixed point in U.*

Moreover, $i(A, U, X)$ is uniquely defined.

3. Main Results

The following two lemmas were the key components in the proof of the expansion-compression fixed point theorem of operator type and their proofs can be found in [5].

Lemma 10. *Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, J_A and J_B be subsets of K with J_B being compact, and P be a cone of non-negative functions in a real Banach space E which is a subset of $F(K)$. Suppose that A is a concave operator on P, B is a continuous convex operator on P, and $T : P \to P$ is a completely continuous operator. Suppose there exist $x_A, x_B \in E$ such that*

$(B0)$ *A is comparable to x_A on P relative to J_A;*

$(B1)$ $\{y \in P \ : x_A <_{J_A} A(y) \text{ and } B(y) <_{J_B} x_B\} \neq \emptyset$;

$(B2)$ *if* $y \in P$ *with* $B(y) \leqq_{J_B} x_B$ *and* $x_A \leq_{J_A} A(y)$*, then* $B(Ty) <_{J_B} x_B$*;*

$(B3)$ *if* $y \in P$ *with* $B(y) \leqq_{J_B} x_B$ *and* $A(Ty) <_{J_A} x_A$*, then* $B(Ty) <_{J_B} x_B$*.*

If $\overline{P_{J_B}(B, x_B)}$ *is bounded, then* $i(T, P_{J_B}(B, x_B), P) = 1$*.*

Lemma 11. *Let* $F(K)$ *be the set of real valued functions defined on* $K \subset \mathbb{R}$*,* J_C *and* J_D *be subsets of* K *with* J_C *being compact, and* P *be a cone of non-negative functions in a real Banach space* E *which is a subset of* $F(K)$*. Suppose that* C *is a continuous concave operator on* P*,* D *is a convex operator on* P*, and* $T : P \to P$ *is a completely continuous operator. Suppose there exist* $x_C, x_D \in E$ *such that*

$(A0)$ D *is comparable to* x_D *on* P *relative to* J_D*;*

$(A1)$ $\{y \in P\ : x_C <_{J_C} C(y) \text{ and } D(y) <_{J_D} x_D\} \neq \emptyset$*;*

$(A2)$ *if* $y \in P$ *with* $C(y) \leqq_{J_C} x_C$ *and* $D(y) \leq_{J_D} x_D$*, then* $x_C <_{J_C} C(Ty)$*;*

$(A3)$ *if* $y \in P$ *with* $C(y) \leqq_{J_C} x_C$ *and* $x_D <_{J_D} D(Ty)$*, then* $x_C <_{J_C} C(Ty)$*.*

If $\overline{P_{J_C}(C, x_C)}$ *is bounded, then* $i(T, P_{J_C}(C, x_C), P) = 0$*.*

To simplify the statements of the multiple fixed point theorems of operator type, in the following definition we define what it means to be LW-inward and LW-outward (LW for Leggett-Williams). Note that the operator is void of invariance-like conditions on the underlying sets in the spirit of the original Leggett-Williams arguments [11] for the outward conditions and in the spirit of the Avery-Anderson-Henderson arguments [4] for the inward conditions.

Definition 12. Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, J_A, J_B, J_C and J_D be subsets of K, with J_B and J_C being compact, and P be a cone of nonnegative functions in a real Banach space E which is a subset of $F(K)$. If A and C are concave operators on P, B and D are convex operators on P, with B and C being continuous, $x_A, x_B, x_C, x_D \in E$, and $T : P \to P$ is a completely continuous operator then we say that:

(i) T is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$ if the conditions $(B0)$, $(B1)$, $(B2)$, and $(B3)$ of Lemma 10 are satisfied,

and

(ii) T is LW-outward with respect to $O(C, D, x_C, x_D, J_C, J_D)$ if the conditions $(A0), (A1), (A2)$, and $(A3)$ of Lemma 11 are satisfied.

The following theorem is the operator type expansion-compression fixed point theorem of Avery-Anderson-Henderson-Liu [5].

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Theorem 13. *Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, J_A, J_B, J_C and J_D be subsets of K, with J_B and J_C being compact, and P be a cone of nonnegative functions in a real Banach space E which is a subset of $F(K)$. Suppose that A and C are concave operators on P, B and D are convex operators on P, with B and C being continuous, and $T : P \to P$ is a completely continuous operator. Suppose there exist $x_A, x_B, x_C, x_D \in E$ such that*

$(D1)$ *T is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$;*

$(D2)$ *T is LW-outward with respect to $O(C, D, x_C, x_D, J_C, J_D)$.*

If

$(H1)$ *$\overline{P_{J_B}(B, x_B)} \subsetneq P_{J_C}(C, x_C)$ and $P_{J_C}(C, x_C)$ is bounded, then T has a fixed point $y \in P(B, C, x_B, x_C, J_B, J_C)$,*

whereas, if

$(H2)$ *$\overline{P_{J_C}(C, x_C)} \subsetneq P_{J_B}(B, x_B)$ and $P_{J_B}(B, x_B)$ is bounded, then T has a fixed point $y \in P(C, B, x_C, x_B, J_C, J_B)$.*

The following theorem is the double fixed point theorem of operator type.

Theorem 14. *Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, J_A, J_B, J_{C_l}, J_{D_l}, J_{C_o}, J_{D_o} be subsets of K with J_B, J_{C_l} and J_{C_o} being compact and P be a cone of nonnegative functions in a real Banach space E which is a subset of $F(K)$. Suppose that A, C_l and C_o are concave operators on P, B, D_l and D_o are convex operators on P with B, C_l and C_o being continuous, and $T : P \to P$ is a completely continuous operator. If there exist $x_A, x_B, x_{C_l}, x_{D_l}, x_{C_o}, x_{D_o} \in E$ such that*

$(D1)$ *T is LW-outward with respect to $O(C_l, D_l, x_{C_l}, x_{D_l}, J_{C_l}, x_{D_l})$;*

$(D2)$ *T is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$;*

$(D3)$ *T is LW-outward with respect to $O(C_o, D_o, x_{C_o}, x_{D_o}, J_{C_o}, x_{D_o})$;*

and if $\overline{P_{J_{C_l}}(C_l, x_{C_l})} \subset P_{J_B}(B, x_B)$ and $\overline{P_{J_B}(B, x_B)} \subset P_{J_{C_o}}(C_o, x_{C_o})$ with $P_{J_{C_o}}(C_o, x_{C_o})$ being bounded, then T has at least two fixed points x^* and x^{**} with

$$x^* \in P(C_l, B, x_{C_l}, x_B, J_{C_l}, J_B) \text{ and } x^{**} \in P(B, C_o, x_B, x_{C_o}, J_B, J_{C_o}).$$

Proof. The operator T is LW-outward with respect to $O(C_l, D_l, x_{C_l}, x_{D_l}, J_{C_l}, x_{D_l})$ and is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$ with $\overline{P_{J_{C_l}}(C_l, x_{C_l})} \subset P_{J_B}(B, x_B)$. Thus by (H2) of Theorem 13, T has a fixed point $x^* \in P(C_l, B, x_{C_l}, x_B, J_{C_l}, J_B)$. Similarly, the operator T is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$ and LW-outward with respect to $O(C_o, D_o, x_{C_o}, x_{D_o}, J_{C_o}, x_{D_o})$ with $\overline{P_{J_B}(B, x_B)} \subset P_{J_{C_o}}(C_o, x_{C_o})$. Thus by (H1) of Theorem 13, T has a fixed point $x^{**} \in P(B, C_o, x_B, x_{C_o}, J_B, J_{C_o})$.

□

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The following theorem is the triple fixed point theorem of operator type.

Theorem 15. *Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, J_{A_l}, J_{B_l}, J_{A_o}, J_{B_o}, J_C, J_D be subsets of K with J_{B_l}, J_{B_o} and J_C being compact and P be a cone of nonnegative functions in a real Banach space E which is a subset of $F(K)$. Suppose that A_l, A_o and C are concave operators on P, B_l, B_o, and D are convex operators on P with B_l, B_o, and C being continuous, and $T : P \to P$ is a completely continuous operator. If there exist $x_{A_l}, x_{B_l}, x_{A_o}, x_{B_o}, x_C, x_D \in E$ such that*

$(D1)$ *T is LW-inward with respect to $I(A_l, B_l, x_{A_l}, x_{B_l}, J_{A_l}, x_{B_l})$;*

$(D2)$ *T is LW-outward with respect to $O(C, D, x_C, x_D, J_C, J_D)$;*

$(D3)$ *T is LW-inward with respect to $I(A_o, B_o, x_{A_o}, x_{B_o}, J_{A_o}, x_{B_o})$;*

and if $\overline{P_{J_{B_l}}(B_l, x_{B_l})} \subset P_{J_C}(C, x_C)$ and $\overline{P_{J_C}(C, x_C)} \subset P_{J_{B_o}}(B_o, x_{B_o})$ with $P_{J_{B_o}}(B_o, x_{B_o})$ being bounded, then T has at least three fixed points x^*, x^{**} and x^{***} with

$$x^* \in P_{J_{B_l}}(B_l, x_{B_l}), x^{**} \in P(B_l, C, x_{B_l}, x_C, J_{B_l}, J_C), \text{ and } x^{***} \in P(C, B_o, x_C, x_{B_o}, J_C, J_{B_o}).$$

Proof. The operator T is LW-inward with respect to $I(A_l, B_l, x_{A_l}, x_{B_l}, J_{A_l}, x_{B_l})$. Thus by Lemma 10, $i(T, P_{J_{B_l}}(B_l, x_{B_l}), P) = 1$. Hence by the solution property of the fixed point index, T has a fixed point $x^* \in P_{J_{B_l}}(B_l, x_{B_l})$. The operator T is LW-outward with respect to $O(C, D, x_C, x_D, J_C, J_D)$ and is LW-inward with respect to $I(A_l, B_l, x_{A_l}, x_{B_l}, J_{A_l}, x_{B_l})$ with $\overline{P_{J_{B_l}}(B_l, x_{B_l})} \subset P_{J_C}(C, x_C)$. Thus by (H1) of Theorem 13, T has a fixed point $x^{**} \in P(B_l, C, x_{B_l}, x_C, J_{B_l}, J_C)$. Similarly, the operator T is LW-inward with respect to $I(A_o, B_o, x_{A_o}, x_{B_o}, J_{A_o}, x_{B_o})$ and LW-outward with respect to $O(C, D, x_C, x_D, J_C, J_D)$ with $\overline{P_{J_C}(C, x_C)} \subset P_{J_{B_o}}(B_o, x_{B_o})$. Thus by (H2) of Theorem 13, T has a fixed point $x^{***} \in P(C, B_o, x_C, x_{B_o}, J_C, J_{B_o})$. □

4. Applications

Two-point right focal boundary value problems for differential equations have received substantial interest by boundary value problem researchers for a long time, see, [1, 3, 4, 5, 8, 9, 11, 14], and [15], to mention a few. In this section, we will apply the double fixed point theorem of operator type to the second-order nonlinear right focal boundary value problem,

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \tag{1}$$

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$$x(0) = 0 = x'(1), \tag{2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous.

It is well known that the Green's function for $-x'' = 0$ with the boundary conditions (2) is given by

$$G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1].$$

For any fixed $s \in [0, 1]$, $G(t, s)$ is nondecreasing in t and satisfies $G(t, s) \geq tG(1, s)$ for $t \in [0, 1]$.

Let the Banach space in this section be $E = C[0, 1]$ with the maximum norm. Define the cone $P \subset E$ by

$$P := \{x \in E \mid x \text{ is nonnegative, nondecreasing, concave, and } x(0) = 0\}.$$

For any $x \in P$, from its concavity, we have $x(t_1) \geq \frac{t_1}{t_2} x(t_2)$ for $0 \leq t_1 \leq t_2 \leq 1$ with $t_2 \neq 0$.

It is also well known that a fixed point of the operator $T : E \to E$ defined by

$$Tx(t) := \int_0^1 G(t, s) f(x(s)) ds$$

is a solution of the boundary value problem (1), (2). Also, by properties of the Green's function, we have that T maps P to P. It is a standard exercise to show $T : P \to P$ is a completely continuous operator by applying the Arzela-Ascoli Theorem.

In the following theorem, we demonstrate how to apply the double fixed point theorem (Theorem 14) to prove the existence of at least two positive solutions to (1), (2).

Theorem 16. *Suppose that $0 < \tau_1, \eta, \tau_2 < 1$ and there are positive numbers c_1, d_1, b, c_2, d_2 with $\frac{c_1}{\tau_1} < d_1 \leq b \leq c_2\eta$, $c_2 < d_2\tau_2$, and $f : \mathbb{R} \to [0, \infty)$ is continuous such that*

(a) $f(x) > \frac{6x}{\tau_1(3-\tau_1^2)}$ *for* $0 \leq x \leq c_1$, *and* $f(x) > \frac{6c_1}{\tau_1(3-\tau_1^2)}$ *for* $c_1 \leq x < d_1$,

(b) $f(x) < \frac{2b}{\eta^2} - \frac{2}{\eta}\int_\eta^1 f(\frac{b}{\eta}s)ds$ *for* $0 \leq x \leq b$, *and* $f(x)$ *is nondecreasing for* $x \in [b, \frac{b}{\eta}]$,

(c) $f(x) > \frac{2c_2}{\tau_2(1-\tau_2^2)}$ *for* $c_2 \leq x < d_2$.

Then the focal problem (1), (2) has at least two positive solutions x^* and x^{**} such that $x^*(\eta) < b$ and $x^*(t) > \frac{c_1}{\tau_1}t$ for $t \in [\tau_1, t_0]$ and some $t_0 \in [\tau_1, 1]$, $x^{**}(t) < \frac{c_2}{\tau_2}t$ for $t \in [\tau_2, 1]$ and $x^{**}(\eta) > b$.

Proof. Let $A = B = C_l = D_l = C_o = D_o = I$ be the identity operator on P. All are linear continuous operators mapping P onto P and hence concave or convex operators as well. Let $J_A = J_{D_l} = J_{D_o} = \{1\}$, $J_B = \{\eta\}$, $J_{C_l} = [\tau_1, 1]$, $J_{C_o} = [\tau_2, 1]$, which are all compact intervals. Let $x_A(t) = x_B(t) = b$, $x_{C_l} = \frac{c_1}{\tau_1}t$, $x_{D_l} = d_1 t$, $x_{C_o} = \frac{c_2}{\tau_2}t$, $x_{D_o} = d_2 t$, for $t \in [0, 1]$, which are all in the Banach space E.

Claim 1. *T is LW-outward with respect to $O(C_l, D_l, x_{C_l}, x_{D_l}, J_{C_l}, J_{D_l})$.*

D_l is comparable to x_{D_l} on P relative to J_{D_l} since J_{D_l} is a degenerate interval. Also, let $x_0(t) := \alpha t$ on $[0, 1]$ with $\alpha \in (\frac{c_1}{\tau_1}, d_1)$. Then, $x_0(t) > \frac{c_1}{\tau_1}t$ for $t \in [\tau_1, 1]$ and $y(1) < d_1$. So, $x_0 \in \{y \in P \ : x_{C_l} <_{J_{C_l}} C_l(y)$ and $D_l(y) <_{J_{D_l}} x_{D_l}\}$ which implies that the set is nonempty.

Subclaim 1.1. *If $y \in P$ with $C_l(y) \leqq_{J_{C_l}} x_{C_l}$ and $D_l(y) \le_{J_{D_l}} x_{D_l}$, then $x_{C_l} <_{J_{C_l}} C_l(Ty)$.*

Suppose $y \in P$ with $C_l(y) \leqq_{J_{C_l}} x_{C_l}$ and $D_l(y) \le_{J_{D_l}} x_{D_l}$. From $C_l(y) \leqq_{J_{C_l}} x_{C_l}$, we have that $y(t) \le \frac{c_1}{\tau_1}t$ for $t \in [\tau_1, 1]$, and there is some $t_0 \in [\tau_1, 1]$ such that $y(t_0) = \frac{c_1}{\tau_1}t_0$. Hence $y(\tau_1) \ge \frac{\tau_1}{t_0}y(t_0) = c_1$. From $y(t) \le \frac{c_1}{\tau_1}t$ for $t \in [\tau_1, 1]$, we have $y(\tau_1) \le c_1$. Therefore, $y(\tau_1) = c_1$.

For $t \in [0, \tau_1]$, we have that $c_1 = y(\tau_1) \ge y(t) \ge \frac{t}{\tau_1}y(\tau_1) \ge \frac{c_1}{\tau_1}t \ge 0$. For $t \in [\tau_1, 1]$, we have $c_1 = y(\tau_1) \le y(t) \le \frac{c_1}{\tau_1}t < d_1 t \le d_1$. Hence for $t \in [\tau_1, 1]$, from condition (a) it follows that

$$
\begin{aligned}
(C_l Ty)(t) &= (Ty)(t) = \int_0^1 G(t,s) f(y(s))ds \\
&= \int_0^{\tau_1} s f(y(s))ds + \int_{\tau_1}^t s f(y(s))ds + \int_t^1 t f(y(s))ds \\
&> \frac{6}{\tau_1(3-\tau_1^2)}\left[\int_0^{\tau_1} s y(s)ds + \int_{\tau_1}^t s c_1 ds + \int_t^1 t c_1 ds\right] \\
&\ge \frac{6c_1}{\tau_1(3-\tau_1^2)}\left[\int_0^{\tau_1} \frac{s^2}{\tau_1}ds + \int_{\tau_1}^t s ds + \int_t^1 t ds\right] \\
&= \frac{c_1}{\tau_1(3-\tau_1^2)}(-3t^2 + 6t - \tau_1^2) \ge \frac{c_1}{\tau_1}t,
\end{aligned}
$$

where the last inequality is true for $t = \tau_1$ and 1, and so is for $t \in [\tau_1, 1]$. Hence, $x_{C_l} <_{J_{C_l}} C_l(Ty)$.

Subclaim 1.2. *If $y \in P$ with $C_l(y) \leqq_{J_{C_l}} x_{C_l}$ and $x_{D_l} <_{J_{D_l}} D_l(Ty)$, then $x_{C_l} <_{J_{C_l}} C_l(Ty)$.*

Suppose $y \in P$ with $C_l(y) \leqq_{J_{C_l}} x_{C_l}$ and $x_{D_l} <_{J_{D_l}} D_l(Ty)$. From $x_{D_l} <_{J_{D_l}} D_l(Ty)$, we have that $(Ty)(1) > d_1$. By the concavity of Ty, we get $(Ty)(t) \ge t(Ty)(1) > td_1 > \frac{c_1}{\tau_1}t$ for $t \in [\tau_1, 1]$, i.e., $x_{C_l} <_{J_{C_l}} C_l(Ty)$.

It is easy to see that $\overline{P_{J_{C_l}}(C_l, x_{C_l})}$ is bounded. Therefore, we have verified that T is LW-outward with respect to $O(C_l, D_l, x_{C_l}, x_{D_l}, J_{C_l}, J_{D_l})$.

Claim 2. *T is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$.*

The operator A is comparable to x_A on P relative to J_A since J_A is degenerate. Let $y_0(t) := \beta t$ with $b < \beta < \frac{b}{\eta}$ defined on $[0,1]$. Obviously $y_0 \in \{y \in P : x_A <_{J_A} A(y)$ and $B(y) <_{J_B} x_B\}$ which means the set is nonempty. Also, if $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $A(Ty) <_{J_A} x_A$, then $(Ty)(1) < b$. By the nondecreasing property of Ty, we have that $(Ty)(\eta) \le (Ty)(1) < b$, i.e., $B(Ty) <_{J_B} x_B$.

Subclaim 2.1. *If $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $x_A \le_{J_A} A(y)$, then $B(Ty) <_{J_B} x_B$.*

Suppose $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $x_A \le_{J_A} A(y)$. Then we have $y(\eta) = b$ and $b \le y(1)$. By the concavity of y, we have that for $t \in [0, \eta]$, $0 \le \frac{b}{\eta}t = \frac{t}{\eta}y(\eta) \le y(t) \le b$, and for $t \in [\eta, 1]$, $b = y(\eta) \le y(t) \le \frac{t}{\eta}y(\eta) = \frac{b}{\eta}t \le \frac{b}{\eta}$. By condition (b), we have that

$$\begin{aligned}
(BTy)(\eta) &= (Ty)(\eta) = \int_0^1 G(\eta, s) f(y(s))ds \\
&= \int_0^\eta s f(y(s))ds + \int_\eta^1 \eta f(y(s))ds \\
&< \left(\frac{2b}{\eta^2} - \frac{2}{\eta}\int_\eta^1 f\left(\frac{b}{\eta}s\right)ds\right)\int_0^\eta s ds + \int_\eta^1 \eta f\left(\frac{b}{\eta}s\right)ds \\
&= b.
\end{aligned}$$

Obviously, $\overline{P_{J_B}(B, x_B)}$ is bounded. Therefore, we have verified that T is LW-inward with respect to $I(A, B, x_A, x_B, J_A, J_B)$.

Claim 3. *T is LW-outward with respect to $O(C_o, D_o, x_{C_o}, x_{D_o}, J_{C_o}, J_{D_o})$.*

D_o is comparable to x_{D_o} on P relative to J_{D_o} since J_{D_o} is degenerate. Also, let $z_0(t) := \gamma t$ on $[0,1]$ with $\gamma \in (\frac{c_2}{\tau_2}, d_2)$. Then, $z_0 \in \{y \in P : x_{C_o} <_{J_{C_o}} C_o(y)$ and $D_o(y) <_{J_{D_o}} x_{D_o}\}$, which is nonempty. Similarly as in Subclaim 1.2, we have that, if $y \in P$ with $C_o(y) \leqq_{J_{C_o}} x_{C_o}$ and $x_{D_o} <_{J_{D_o}} D_o(Ty)$, then $x_{C_o} <_{J_{C_o}} C_o(Ty)$.

Subclaim 3.1. *If $y \in P$ with $C_o(y) \leqq_{J_{C_o}} x_{C_o}$ and $D_o(y) \le_{J_{D_o}} x_{D_o}$, then $x_{C_o} <_{J_{C_o}} C_o(Ty)$.*

Suppose $y \in P$ with $C_o(y) \leqq_{J_{C_o}} x_{C_o}$ and $D_o(y) \le_{J_{D_o}} x_{D_o}$. Similarly as in Subclaim 1.1, we have that for $t \in [0, \tau_2]$, $0 \le y(t) \le c_2$, and for $t \in [\tau_2, 1]$, $c_2 \le y(t) < d_2$. Hence for $t \in [\tau_2, 1]$, by condition (c), we have

$$(C_oTy)(t) = (Ty)(t) = \int_0^1 G(t,s) f(y(s))ds$$

$$\begin{aligned} &= \int_0^{\tau_2} sf(y(s))ds + \int_{\tau_2}^{t} sf(y(s))ds + \int_t^1 tf(y(s))ds \\ &\geq \int_{\tau_2}^{t} sf(y(s))ds + \int_t^1 tf(y(s))ds \\ &> \frac{2c_2}{\tau_2(1-\tau_2^2)}\left[\int_{\tau_2}^{t} sds + \int_t^1 tds\right] \\ &= \frac{c_2}{\tau_2(1-\tau_2^2)}(-t^2+2t-\tau_2^2) \geq \frac{c_2}{\tau_2}t, \end{aligned}$$

where the last inequality is true for $t=\tau_2$ and $t=1$, and so is true for $t\in[\tau_2,1]$.

It is easy to see that $\overline{P_{J_{C_o}}(C_o,x_{C_o})}$ is bounded. Therefore we have verified that T is LW-outward with respect to $O(C_o,D_o,x_{C_o},x_{D_o},J_{C_o},J_{D_o})$.

Claim 4. $\overline{P_{J_{C_l}}(C_l,x_{C_l})} \subsetneq P_{J_B}(B,x_B)$ and $\overline{P_{J_B}(B,x_B)} \subsetneq P_{J_{C_o}}(C_o,x_{C_o})$.

Suppose $y\in\overline{P_{J_{C_l}}(C_l,x_{C_l})}$. Then, $y\in P$ and $y(t)\leq \frac{c_1}{\tau_1}t$ for $t\in[\tau_1,1]$. If $\tau_1\leq\eta$, then $y(\eta)\leq\frac{c_1}{\tau_1}\eta<d_1\eta<b$, since $d_1\leq b$. If $\eta<\tau_1$, then $y(\eta)\leq y(\tau_1)\leq c_1<d_1\tau_1<b$. Hence $B(y)<_{J_B} x_B$, i.e., $y\in P_{J_B}(B,x_B)$ and so $\overline{P_{J_{C_l}}(C_l,x_{C_l})}\subset P_{J_B}(B,x_B)$. Define $h_0(t):=d_1t$ on $[0,1]$. Then, $h_0(t)=d_1t>\frac{c_1}{\tau_1}t$ for $t\in[\tau_1,1]$, and $h_0(\eta)=d_1\eta<b$. So, $h_0\in P_{J_B}(B,x_B)-\overline{P_{J_{C_l}}(C_l,x_{C_l})}$, i.e., $\overline{P_{J_{C_l}}(C_l,x_{C_l})}\subsetneq P_{J_B}(B,x_B)$.

Suppose $y\in\overline{P_{J_B}(B,x_B)}$. Then, $y\in P$ and $y(\eta)\leq b$. If $\eta\leq\tau_2$, then $y(t)\leq\frac{t}{\eta}y(\eta)\leq\frac{b}{\eta}t\leq c_2t<\frac{c_2}{\tau_2}t$ for $t\in[\tau_2,1]$. If $\tau_2<\eta$, then $y(t)\leq y(\eta)\leq b\leq c_2\eta<c_2\leq\frac{c_2}{\tau_2}t$ for $t\in[\tau_2,\eta]$ and $y(t)\leq\frac{t}{\eta}y(\eta)\leq\frac{b}{\eta}t\leq c_2t<\frac{c_2}{\tau_2}t$ for $t\in[\eta,1]$. Hence, $y(t)<\frac{c_2}{\tau_2}t$ for $t\in[\tau_2,1]$, i.e., $C_o(y)<_{J_{C_o}} x_{C_o}$, which means $\overline{P_{J_B}(B,x_B)}\subset P_{J_{C_o}}(C_o,x_{C_o})$. Let $g_0(t):=\delta t$ for $t\in[0,1]$ with $\delta\in(c_2,\frac{c_2}{\tau_2})$. Then, $g_0(\eta)>c_2\eta\geq b$ and $g_0(t)<\frac{c_2}{\tau_2}t$ for $t\in[\tau_2,1]$. Hence, $g_0\in P_{J_{C_o}}(C_o,x_{C_o})-\overline{P_{J_B}(B,x_B)}$, i.e., $\overline{P_{J_B}(B,x_B)}\subsetneq P_{J_{C_o}}(C_o,x_{C_o})$.

By Theorem 14, we conclude that the operator T has at least two fixed points x^* and x^{**} with

$$x^*\in P(C_l,B,x_{C_l},x_B,J_{C_l},J_B) \text{ and } x^{**}\in P(B,C_o,x_B,x_{C_o},J_B,J_{C_o}).$$

For $x^*\in P(C_l,B,x_{C_l},x_B,J_{C_l},J_B)$, we have that $x^*\in P_{J_B}(B,x_B)-\overline{P_{C_l}(C_l,x_{C_l})}$. From $x^*\in P_{J_B}(B,x_B)$, we have that $x^*(\eta)<b$. From $x^*\notin\overline{P_{C_l}(C_l,x_{C_l})}$, it follows that there is some $t_0\in[\tau_1,1]$ such that $x^*(t_0)>\frac{c_1}{\tau_1}t_0$, which by the concavity of x^* implies $x^*(t)\geq\frac{t}{t_0}x^*(t_0)>\frac{c_1}{\tau_1}t$ for $t\in[\tau_1,t_0]$. From $x^{**}\in P(B,C_o,x_B,x_{C_o},J_B,J_{C_o})$, we get that $x^{**}\in P_{J_{C_o}}(C_o,x_{C_o})-\overline{P_{J_B}(B,x_B)}$, i.e., $x^{**}(t)<\frac{c_2}{\tau_2}t$ for $t\in[\tau_2,1]$ and $x^{**}(\eta)>b$. □

Example. Consider the right focal boundary value problem,

$$x''(t)+|e^x-x^e|=0,\quad t\in(0,1),$$

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$$x(0) = x'(1) = 0.$$

We notice the function $f(x) = |e^x - x^e|$ has a unique local maximal value $e - 1$ at $x = 1$ and a unique local minimal value 0 at $x = e$ on $[0, \infty)$. Choose $\tau_1 = 0.2$, $\eta = 0.8$, $\tau_2 = 0.5$, $c_1 = 0.1$, $d_1 = 1$, $b = e$, $c_2 = 4.4$, $d_2 = 9$. Then it is easy to verify that all conditions in Theorem 16 are satisfied and hence the problem has at least two positive solutions x^* and x^{**} on $[0, 1]$ with $x^*(0.8) < e$, $x^*(t) > 0.5t$, for $t \in [0.2, t_0]$ and some $t_0 \in [0.2, 1]$, and $x^{**}(t) < 8.8t$ for $t \in [0.5, 1]$, $x^{**}(0.8) > e$.

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