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LAPLACE AND FOURIER TRANSFORMS ON ARITHMETIC ASIAN OPTIONS

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Abstract: The solution of the Black-Scholes PDE for arithmetic Asian options is the most difficult problem in the financial Mathematics; as a result, there are varieties ways in the literature have been appeared to address the problem. In this paper we follow the PDE approach; we present an efficient method for pricing continuous arithmetic Asian options. Using Laplace transforms and changing some of variables we reduce the PDE of the Asian options to the less complicated one. Firstly, We transform the PDE in three variables from the second order to a non-homogeneous parabolic equation; which have been attempted by many researchers so far. Next, we transform the parabolic equation to ODE in two variables from the first order with an initial condition using Fourier transform then, we provide its final analytical solution. This method can be applicable to all types of arithmetic Asian options.

Key Words: partial differential equations, arithmetic Asian option, Laplace transform, Fourier transform

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1. Introduction

Asian options are path dependent options whose payoff functions depend on the average of stock price over a specific period of time named life of the option. There are two types of Asian options according to the way of computing the average, arithmetic and geometric, and each one includes two types, call options which give the holder of the option the right to buy, and put options give the right to sell. For the case of geometric Asian options there is a closed-form solution to value these options (see Barucci et. al. [1]). However the most difficult task is arithmetic type, because the arithmetic average of a set of lognormal random variables is not lognormally distributed. Since till now there is no closed-form solution to value these types of option different ways in the literature have been developed, some of them, Geman and Yor [5] used Laplace transform in time of the Asian option price. However, this transform is only applicable in some cases. Rogers and Shi [6] transform the problem to the problem of solving a parabolic PDE in two variables from the second order. But there is no analytical solution for this PDE, and it is difficult to use numerical methods on this PDE. Also they derive lower-bound formulas for Asian options by computing the expectation based on some zero-mean Gaussian variable. Zhang [9] present a theory of continuously-sampled Asian option pricing, and he solves the PDE with perturbation approach. Vecer's approach [7] is based on Asian option as option on a traded account; he provides a one-dimensional PDE for Asian options. Dubois and Lelievre [4] derive accurate and fast numerical methods to solve Rogers and Shi PDE. Chen and Lyuu [2] develop the lower-bound pricing formulas of Rogers and Shi PDE [6] to include general maturities instead one year. Dewynne and Shaw [3] provide a simplified means of pricing arithmetic Asian options by PDE approach, they derive an analytical formula for the Laplace transform in time of the Asian option, and they obtained asymptotic solutions for Black-Scholes PDE for Asian options for low-volatility limit which is the big problem on using Laplace transform. Yang et. al. [8] derive quasi analytical expressions for price and hedge arithmetic Asian call option.

In this paper, we provide a new direct solution for arithmetic Asian options by using Laplace and Fourier transforms. Firstly, we make some change of variables of the PDE, as suggested by Rogers and Shi [6]. also we perform another change of variables. Then, we apply Laplace transform in time to get ODE in two variables, and we use Fourier transform to have an analytical solution for the arithmetic Asian options. Hence, in this paper we solved the problem of computing the value of the arithmetic Asian options.

The Black-Scholes PDE for the arithmetic Asian options is:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}+\frac{1}{t}\left(S-A\right)\frac{\partial C}{\partial A}-rC=0, t\geq 0, S>0 \tag{1}$$

$$C\left(T,S,A\right)=g(S,A)$$

With a boundary condition, S is a stock price, r is the interest rate, σ is the asset volatility, T the expiration date, and $A = \frac{1}{T}\int_0^T S(t)\,dt$ is the arithmetic average of the stock price.

There are four different types of arithmetic Asian option according to the payoff function $g(S,A)$

1. Fixed strike call $g(S,A) = (A-k)^+$
2. Fixed strike put $g(S,A) = (k-A)^+$
3. Floating strike call $g(S,A) = (S-A)^+$
4. Floating strike put $g(S,A) = (A-S)^+$

Where k is the exercise price.

Equatio (1) is not easy to solve since the parabolic operator is degenerated in the A-variable. However, it is possible to reduce Eq. (1) to easier equation using change of variables and Laplace transforms. Make the first following change of variables on Eq. (1)

$$C(t,S,A) = Sf(t,\xi), \xi = \frac{k - \frac{tA}{T}}{S}, \xi \geq 0$$

Eq. (1) is reduced to

$$\frac{\partial f}{\partial t} + \left(\frac{1}{T} + r\xi\right)\frac{\partial f}{\partial \xi} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 f}{\partial \xi^2} = 0 \tag{2}$$

$$f(T,\xi) = \varphi(\xi)$$

For the case of fixed strike call option $\varphi(\xi) = \max(-\xi,0) = \xi^-$, fixed strike put option $\varphi(\xi) = \max(\xi,0) = \xi^+$, floating strike call $\varphi(\xi) = \max(\xi,0) = (1+\xi)^-$, and floating strike put $\varphi(\xi) = \max(\xi,0) = (1+\xi)^+$. Also Eq. (2) is difficult to solve since there is no the variable ξ in the second term with $1/T$, so we will try to make another changes of variables.

Let $\tau = T - t$

$$\frac{\partial f}{\partial \tau} = \left(\frac{1}{T} + r\xi\right)\frac{\partial f}{\partial \xi} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 f}{\partial \xi^2} \tag{3}$$

$$f(0,\xi) = \varphi(\xi)$$

Under the following changes Eq. (3) is reduced to

$$h(\tau,x) = f(\tau, \xi = x + \frac{\tau}{T})$$

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$$\frac{\partial h}{\partial \tau} = r\left(x+\frac{\tau}{T}\right)\frac{\partial h}{\partial x}+\frac{1}{2}\sigma^2\left(x+\frac{\tau}{T}\right)^2\frac{\partial^2 h}{\partial x^2} \tag{4}$$

$$h\left(0,x\right)=\varphi(x)$$

To get rid from the derivation respect to the time and have ODE instead PDE we will apply Laplace transform in time.

The Laplace transform in τ is defined as:

$$L\left\{h\left(\tau\right)\right\}=h^*\left(p\right)=\int_0^\infty h(\tau)e^{-\tau p}d\tau$$

$$L^{-1}\left\{h^*\left(p\right)\right\}=h\left(\tau\right)=\frac{1}{2\pi i}\int_0^\infty h^*\left(p\right)e^{\tau p}dp$$

$$L\{\frac{\delta h}{\delta \tau}\}=ph^*(p)-h(0)$$

And

$$L\left\{\tau\right\}=\int_0^\infty \tau e^{-\tau p}d\tau=\frac{1}{p^2}$$

After using the above transforms Eq. (4) becomes

$$ph^*-\varphi\left(x\right)=r\left(x+\frac{1}{p^2T}\right)\frac{\partial h^*}{\partial x}+\frac{1}{2}\sigma^2\left(x+\frac{1}{p^2T}\right)^2\frac{\partial^2 h^*}{\partial x^2} \tag{5}$$

Assume $h^*\left(p,x\right)=h^*\left(p,k\right), k=x+\frac{1}{p^2T}$

$$ph^*-\varphi\left(k-\frac{1}{p^2T}\right)=rk\frac{\partial h^*}{\partial k}+\frac{1}{2}\sigma^2k^2\frac{\partial^2 h^*}{\partial k^2} \tag{6}$$

Applying inverse Laplace in p

$$L^{-1}\left\{ph^*(p)\right\}=\frac{\partial h\left(\tau\right)}{\partial \tau}$$

$$\frac{\partial h}{\partial \tau}-\varphi\left(k\right)=rk\frac{\partial h}{\partial k}+\frac{1}{2}\sigma^2k^2\frac{\partial^2 h}{\partial k^2} \tag{7}$$

$$h\left(0,k\right)=\varphi(k)$$

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In the above equation the variable coefficients with the derivation term respect to x are similar, they only differ in the power so it's not difficult to make it constant using the following changes.

We have $k > 0$, so we can assume $z = lnk$

$$\frac{\partial h}{\partial \tau} - \varphi\left(e^{z}\right) = \left(r - \frac{1}{2}\sigma^{2}\right)\frac{\partial h}{\partial z} + \frac{1}{2}\sigma^{2}\frac{\partial^{2} h}{\partial z^{2}} \tag{8}$$

$$\frac{\partial h}{\partial \tau} - \varphi\left(e^{z}\right) = \rho\frac{\partial h}{\partial z} + \frac{1}{2}\sigma^{2}\frac{\partial^{2} h}{\partial z^{2}}, \rho = r - \frac{1}{2}\sigma^{2} \tag{9}$$

$$h\left(0, z\right) = \varphi(e^{z})$$

The above equation is inhomogeneous PDE with constant coefficient. So it is easy to get red from some terms by changing some of variables.

Make the following changes

$$g\left(\tau, \xi\right) = h\left(\tau, z = -\left(\xi + \rho\tau\right)\right)$$

$$\frac{\partial g}{\partial \tau} - \varphi\left(e^{-\xi}\right) = \frac{1}{2}\sigma^{2}\frac{\partial^{2} g}{\partial \xi^{2}} \tag{10}$$

$$g\left(0, \xi\right) = \varphi(e^{-\xi})$$

The inhomogeneous parabolic equation (10) can be solved straightforward way using Fourier transforms in ξ.

Fourier transform for a function $g\left(\xi\right)$ is defined by

$$F\left\{g(\xi)\right\} = g^{*}\left(\omega\right) = \int_{-\infty}^{\infty} g(\xi)e^{-i\omega\xi}d\xi$$

And

$$F\left\{\frac{\partial^{n} g}{\partial \xi^{n}}\right\} = (i\omega)^{n} g^{*}\left(\omega\right)$$

$$\frac{\partial g^{*}}{\partial \tau} - \varphi\left(\omega\right) = -\frac{1}{2}\sigma^{2}\omega^{2} g^{*} \tag{11}$$

$$\frac{\partial g^{*}}{\partial \tau} + \frac{1}{2}\sigma^{2}\omega^{2} g^{*} = \varphi\left(\omega\right) \tag{12}$$

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$$g^*(0,\omega) = \varphi\left(\frac{1}{1+i\omega}\right)$$

This is ordinary differential equation from the first order, so its solution is

$$g^*(\tau,\omega) = e^{-\frac{1}{2}\sigma^2\omega^2\tau}\left(-\int_0^T \varphi(\frac{1}{1+i\omega})e^{\frac{1}{2}\sigma^2\omega^2\tau}d\tau + c\right) \tag{13}$$

$$g^*(0,\omega) = c = \varphi\left(\frac{1}{1+i\omega}\right)$$

$$g^*(\tau,\omega) = \varphi\left(\frac{1}{1+i\omega}\right)e^{-\frac{1}{2}\sigma^2\omega^2\tau}\left(-\int_0^T e^{\frac{1}{2}\sigma^2\omega^2\tau}d\tau + 1\right) \tag{14}$$

$$g^*(\tau,\omega) = \varphi\left(\frac{1}{1+i\omega}\right)e^{-\frac{1}{2}\sigma^2\omega^2\tau}\left(\left(\frac{1-e^{\frac{1}{2}\sigma^2\omega^2T}}{\frac{1}{2}\sigma^2\omega^2}\right) + 1\right) \tag{15}$$

The function $\varphi(\frac{1}{1+i\omega})$ depends on the type of the options as we explain in the beginning.

2. Conclusion

Since the numerical methods for solving arithmetic Asian option PDE is not very accurate because the low volatility level or short maturity, so we tried to solve it analytically by means of partial differential equations. We have shown that, the PDE of the arithmetic Asian options in three variables from the second order could be transformed to ODE in two variables from the first order with an initial condition. We have given the analytical solution for all types of arithmetic Asian options.

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