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RANKS AND SUBDEGREES OF THE SYMMETRIC GROUP S_n ACTING ON UNORDERED r-ELEMENT SUBSETS

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Abstract: The main aim of this paper is to determine the ranks and subdegrees of the symmetric group S_n acting on unordered r-element subsets of $X = \{1, 2, 3, ..., n\}$. These areas have not received much attention, in fact most of the research has been focused on the action of S_n on unordered pairs. In this paper it has been shown that the action of S_n on $X^{(r)}$ is transitive. The ranks and suborbits of S_n acting on $X^{(4)}$ and $X^{(5)}$ are determined, after which it is proved that the rank of S_n acting on $X^{(r)}$ is $r + 1$ if $n \geq 2r$. It has been shown that the suborbits of S_n acting on $X^{(r)}$ are self paired. It is also proved that the subdegrees of S_n acting on $X^{(r)}$ are $1, r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{r}{3}\binom{n-r}{r-3}, \ldots, \binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r}$.

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1. Introduction

In 1970, Higman [3] gave a characterization of families of rank 3 permutation groups

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by the subdegrees. He proved that the symmetric group S_n on $X = \{1, 2, \ldots, n\}$, $n \geq 4$ acts as a rank 3 group on the set $X^{(2)}$, 2-element subsets of X, with subdegrees 1, $2(n-2)$, $\binom{n-2}{2}$.

In this paper we generalize Higman's work to S_n acting on r-element subsets; $X^{(r)}$.

2. Preliminary Definitions and Results

A *permutation* of X is a one-to-one mapping of X onto itself. The *symmetric group* of degree n and denoted by S_n is the group of all permutations of X under the binary operation of composition of maps. The order of S_n is $n!$.

Let X be a nonempty set and G be a group. We say that G *acts on the left* on X if for each $x \in X$ and $g \in G$ there corresponds a unique element $gx \in X$ such that, for all $x \in X$ and $g_1, g_2 \in G$

1. $(g_1 g_2)x = g_1(g_2)x$;
2. $1.x = x$, where 1 is the identity in G.

The action of G from the right can be written in a similar way.

Let G act on a set X. Then X is partitioned into disjoint equivalence classes (with respect to an equivalence relation) called *orbits or transitivity classes* of the action. For each $x \in X$, the orbit containing x is denoted by $Orb_G(x)$. Thus,

$$Orb_G(x) = \{gx | g \in G\}.$$

The action of a group G on X is said to be *transitive* if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit.

Let G act on X and let $x \in X$. The *stabilizer* of x in G, denoted by $Stab_G(x)$ or G_x is given by

$$Stab_G(x) = \{g \in G | gx = x\}.$$

Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. Suppose $\Delta_0 = \{x\}, \Delta_1, \Delta_2, ..., \Delta_{r-1}$ are the orbits of G_x on X, the *rank* of G is then r. The sizes $n_i = |\Delta_i|, (i = 1, 2, ..., r-1)$, are known as the *subdegrees* of G_x. The orbits of G_x on X are also called the suborbits of G.

Theorem 2.1. (Orbit-Stabilizer Theorem, see [5], p. 72) *Let G be a group acting on a finite set X and $x \in X$. Then*

$$|Orb_G(x)| = |G : Stab_G(x)|. \tag{1}$$

Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g, denoted by $Fix(g)$. Thus

$$Fix(g) = \{x \in X | gx = x\}.$$

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of of length 2,..., α_n cycles of length n, we say that σ and hence g has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$

Theorem 2.2. (see [4], p. 68) *Two permutations in S_n are conjugate if and only if they have the same cycle type; and if $g \in G$ has cycle type $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then the number of permutations in S_n conjugate to g is*

$$\frac{n!}{\Pi_{i=1}^{n} \alpha_i! i^{\alpha_i}} \tag{2}$$

Theorem 2.3. (see [1], 1969, p. 98) *Let G be a finite group acting on a set X. The number of orbits of G is*

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)|. \tag{3}$$

This theorem is referred to as Cauchy-Frobenius Lemma.

Let Δ be an orbit of G_x on X. Define $\Delta^* = \{gx | g \in G, x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit paired with Δ. If $\Delta^* = \Delta$, then Δ is called a self-paired orbit of G_x.

3. Main Results

3.1. Order of $Stab_G\{1, 2, 3, ..., r\}$

Let $G = S_n$ act on $X = \{1, 2, ..., n\}$. Then the action of G on X induces an action of G on $X^{(r)}$; the set of all unordered r-element subsets of X defined by $g\{a_1, g_2, \dots, a_r\} = \{g(a_1), g(a_2), ..., g(a_r)\}$. If $Stab_G\{1, 2, 3, ..., r\}$ is the stabilizer of $\{1, 2, ..., r\}$, the order of $Stab_G\{1, 2, 3, ..., r\}$ is determined in this section. The result is given by the following Theorem.

Theorem 3.1.

$$|Stab_G\{1, 2, 3, ..., r\}| = (n - r)!r! \tag{4}$$

Proof. Clearly, the stabilizer of unordered r-element subset $\{1, 2, 3, ..., r\}$ is isomorphic to $S_r \times S_{n-r}$ whose order is $r!(n - r)!$. Thus

$$|Stab_G\{1, 2, 3, ..., r\}| = (n - r)!r!$$

We next give some results which will be used extensively in this section.

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Lemma 3.2. *Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then the number of permutations in S_n fixing $\{1, 2, 3\} \in X^{(3)}$ and having the same cycle type as g is given by*

$$\frac{(n-3)!}{(\alpha_1-3)!1^{(\alpha_1-3)}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{3(n-3)!}{(\alpha_1-1)!1^{(\alpha_1-1)}(\alpha_2-1)!2^{\alpha_2-1}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{2(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}}. \quad (5)$$

Proof. Consider a permutation $g \in S_n$ that fixes $\{1, 2, 3\} \in X^{(3)}$. Then g fixes $S = \{1, 2, 3\}$ if either each member of S comes from a 1-cycle of g or if one element of S comes from a 1-cycle and the other two elements come from a 2-cycle in g or if all the three elements of S come from a 3-cycle of g. We consider the three cases as follows:

1. When each of the members of S comes from a 1-cycle, we apply Theorem 2.2 to a permutation of S_{n-3} with cycle type $(\alpha_1 - 3, \alpha_2, \alpha_3, ..., \alpha_n)$ to get

$$\frac{(n-3)!}{(\alpha_1-3)!1^{(\alpha_1-3)}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}}$$

permutations.

2. If one of members of S comes from a 1-cycle, and the other two from a 2-cycle, the number of elements coming from the 1-cycle can be chosen in three ways. We apply Theorem 2.2 to a permutation of S_{n-3} with cycle type $(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, ..., \alpha_n)$ and considering the three possible ways to get

$$\frac{3(n-3)!}{(\alpha_1-1)!1^{(\alpha_1-1)}(\alpha_2-1)!2^{\alpha_2-1}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}}$$

permutations.

3. Finally, if all the elements of S come from a 3-cycle, there are two different permutations from a cycle of length three. Applying Theorem 2.2 to a permutation of S_{n-3} with cycle type $(\alpha_1, \alpha_2, \alpha_3 - 1, ..., \alpha_n)$ and considering the two cases, we get

$$\frac{2(n-3)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}}$$

permutations. Summing up all the three cases (a), (b), and (c) gives the required result.

Lemma 3.3. *Let $g \in S_n$ be a permutation with a cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$ Then the number of permutations in S_n fixing $\{1, 2, 3, 4\} \in X^{(4)}$ and having the same cycle type as g is given by*

$$\frac{(n-4)!}{(\alpha_1-4)!\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{6(n-4)!}{(\alpha_1-2)!(\alpha_2-1)!2^{\alpha_2-1}\Pi_{i=3}^{n}\alpha_1!i^{\alpha_i}} + \frac{8(n-4)!}{(\alpha_1-1)!\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{\alpha_3-1}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}} + \frac{3(n-4)!}{\alpha_1!(\alpha_2-2)!2^{\alpha_2-2}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}} + \frac{6(n-4)!}{\alpha_1!\alpha_2!2^{\alpha_2}\alpha_3!3^{\alpha_3}(\alpha_4-1)!4^{\alpha_4-1}\Pi_{i=5}^{n}\alpha_i!i^{\alpha_i}} \quad (6)$$

Proof. The idea of proof of this Lemma is given. We consider the five cases in which g fixes $S = \{1, 2, 3, 4\}$. The first case is when all of the four elements of S come from 1-cycles of g. Second case is when two of the members of S come from 1-cycles while the other two come from a 2-cycle. Case number three is when one of the members come from a 1-cycle while the other three come from a 3-cycle. Case number four is when the members of S come from two 2-cycles. Lastly we consider the case where all the members come from a 4-cycle. The proof can be completed by proceeding as in Lemma 3.2.

Lemma 3.4. *Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then the number of permutations in S_n fixing $\{1, 2, 3, 4, 5\} \in X^{(5)}$ and having the same cycle type as g is given by*

$$\frac{(n-5)!}{(\alpha_1-5)!1^{(\alpha_1-5)}\Pi_{i=2}^{n}\alpha_i!i^{\alpha_i}} + \frac{10(n-5)!}{(\alpha_1-3)!1^{(\alpha_1-3)}(\alpha_2-1)!2^{(\alpha_2-1)}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}} + \frac{20(n-5)!}{(\alpha_1-2)!1^{\alpha_1-2}\alpha_2!2^{\alpha_2}(\alpha_3-1)!3^{(\alpha_3-1)}\Pi_{i=4}^{n}\alpha_i!i^{\alpha_i}} + \frac{15(n-5)!}{(\alpha_1-1)!1^{(\alpha_1-1)}(\alpha_2-2)!2^{(\alpha_2-2)}\Pi_{i=3}^{n}\alpha_i!i^{\alpha_i}} + \frac{30(n-5)!}{(\alpha_1-1)!1^{(\alpha_1-1)}(\alpha_4-1)!4^{(\alpha_4-1)}\Pi_{i=2}^{3}\alpha_i i^{\alpha_i}\Pi_{i=5}^{n}\alpha_i!i^{\alpha_i}} + \frac{20(n-5)!}{\alpha_1!1^{\alpha_1}(\alpha_2-1)!2^{(\alpha_2-1)}(\alpha_3-1)!3^{(\alpha_3-1)}\Pi_{i=4}^{5}\alpha_i!i^{\alpha_i}} + \frac{24(n-5)!}{(\alpha_5-1)!5^{(\alpha_5-1)}\Pi_{i=1}^{4}\alpha_i!i^{\alpha_i}\Pi_{i=6}^{n}\alpha_i!i^{\alpha_i}}. \quad (7)$$

The proof for this Lemma is similar to that of Lemma 3.2 and Lemma 3.3.

Remark 3.5. Any of the summands in Lemmas 3.2, 3.3, and 3.4 yields a zero whenever $(\alpha_i - j) < 0$. This is because $(\alpha_i - j)!$ does not exist when $(\alpha_i - j) < 0$, consequently

$$\frac{b}{(\alpha_i - j)!} = 0$$

Permutation type	No. fixing $\{1,2,3\}$	Permutation type	No. fixing $\{1,2,3\}$
I	1	(a b)(c d)(e f)(g h)	0
(a b)	13	(a b)(c d)	45
(a b c)	22	(a b)(c d e)	100
(a b c d)	30	(a b)(c d)(e f)	45
(a b c d e)	24	(a b)(c d)(e f g)	90
(a b c d e f)	0	(a b c d)(e f g h)	0
(a b c d e f g)	0	(a b c)(d e f)	40
(a b c d e f g h)	0	(a b c)(d e f g)	60
(a b)(c d e f g h)	0	(a b c)(d e f g h)	48
(a b)(c d e f g)	72	(a b c)(d e f)(g h)	40
(a b)(c d e f)	90	*Total*	*720*

Table 1: Number of permutations fixing {1,2,3}

Some examples on the order of $Stab_G\{1,2,3,...,r\}$ are now given.

Example 3.6. Consider $G = S_8$ acting on $X^{(3)}$.

Then by Theorem 3.1

$$|Stab_G\{1,2,3\}| = 5!3! = 720.$$

Alternatively, the problem may solved by using Lemma 3.2 and coming up with Table 1. The second and fourth columns give the number of permutations in S_8 fixing $\{1,2,3\}$ and having the same cycle type, which are obtained by using Lemma 3.2.

From Table 1, the order of $Stab_G\{1,2,3\}$ is 720.

Example 3.7. Let $G = S_9$ acting on $X^{(4)}$.

Then by Theorem 3.1

$$|Stab_G\{1,2,3,4\}| = 5!4! = 2880.$$

Table 2 may also be used to find $|Stab_G\{1,2,3,4\}|$ where the second and fourth columns are obtained by using Lemma 3.3.

The total sum entries in the second and fourth columns of Table 2 gives the order of $Stab_G\{1,2,3,4\}$ as equal to 2880.

3.2. Transitivity of S_n

We next show that S_n acts transitively on $X^{(r)}$.

Theorem 3.8. *S_n acts transitively on $X^{(r)}$*

Permutation type	No. fixing $\{1,2,3,4\}$	Permutation type	No. fixing $\{1,2,3,4\}$
I	1	(a b)	16
(a b c)	28	(a b c e)	36
(a b c d e)	24	(a b c d e f)	0
(a b c d e f g)	0	(a b c d e f g h)	0
(a b c d e f g h i)	0	(a b)(c d e f g h i)	0
(a b)(c d e f g h)	0	(a b)(c d e f g)	144
(a b)(c d e f)	240	(a b)(c d e)	220
(a b)(c d)	78	(a b)(c d)(e f)(g h)	120
(a b)(c d)(e f)	45	(a b)(c d e)(f g h)	60
(a b)(c d)(e f g h i)	300	(a b)(c d)(e f g h)	180
(a b)(c d)(e f g)	72	(a b)(c d e)(f g h i)	0
(a b)(c d)(e f)(g h i)	160	(a b c)(d e f)(g h i)	360
(a b c)(d e f)	192	(a b c)(d e f g)	0
(a b c)(d e f g h)	180	(a b c)(d e f g h i)	144
(a b c d)(e f g h)	120	(a b c d)(e f g h i)	160
		Total	*2880*

Table 2: Number of permutations fixing $\{1,2,3,4\}$

Proof. It suffices to show that $|Orb_G\{1,2,3,...,r\}| = |X^{(r)}| = nC_r$. By using Orbit-Stabilizer Theorem (Theorem 2.1) and Theorem 3.1,

$$\begin{aligned} |Orb_G\{1,2,3,...,r\}| &= |G : Stab_G\{1,2,3,...,r\}| \\ &= \frac{|G|}{|Stab_G\{1,2,3,...,r\}|} \\ &= \frac{n!}{(n-r)!r!} \\ &= nC_r. \end{aligned} \tag{8}$$

3.3. Number of Fixed Points

Derivation of some formulas for finding the number of elements of $X^{(r)}$ fixed by a permutation $g \in S_n$ is given in this section. The formulas are given in the following results.

Lemma 3.9. *Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|Fix(g)|$ in $X^{(3)}$ is given by the formula*

$$|Fix(g)| = \binom{\alpha_1}{3} + \alpha_1\alpha_2 + \alpha_3. \tag{9}$$

Proof. Let $g \in S_n$ have cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$ and let $S \in X^{(3)}$. S is fixed by g if each member of S comes from a 1-cycle in g or one of members of S comes from a 1-cycle in g and the other two from a 2-cycle in g or if all members of S come from a 3-cycle in g. From the first case, the number of unordered triples fixed by g is $\binom{\alpha_1}{3}$, while in the second case the number of unordered triples fixed by g is $\alpha_1\alpha_2$ and in the third case the number of unordered triples fixed by g is α_3. Adding up the results from the three cases gives the required result.

Lemma 3.10. *Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|Fix(g)|$ in $X^{(4)}$ is given by the formula*

$$|Fix(g)| = \binom{\alpha_1}{4} + \alpha_2\binom{\alpha_1}{2} + \alpha_1\alpha_3 + \binom{\alpha_2}{2} + \alpha_4. \tag{10}$$

Proof. Let $g \in S_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$ and let $S \in X^{(4)}$. S is fixed by g if each member of S comes from a 1-cycle in g; if two members of S come from 1-cycles in g and the other two members from a 2-cycle; if one element of S comes from a 1-cycle in g and the other three from a 3-cycle; if four elements of S come from two 2-cycles in g; if all the elements of S come from a 4-cycle in g. From each of the cases, the number of unordered quadruples fixed by g is $\binom{\alpha_1}{4}$, $\alpha_2\binom{\alpha_1}{2}$, $\alpha_1\alpha_3$, $\binom{\alpha_2}{2}$, and α_4 respectively.

Lemma 3.11. *Let the cycle type of $g \in S_n$ be $(\alpha_1, \alpha_2, ..., \alpha_n)$. Then $|Fix(g)|$ in $X^{(5)}$ is given by the formula*

$$|Fix(g)| = \binom{\alpha_1}{5} + \alpha_2\binom{\alpha_1}{3} + \alpha_3\binom{\alpha_1}{2} + \alpha_1\alpha_4 + \alpha_1\binom{\alpha_2}{2} + \alpha_2\alpha_3 + \alpha_5. \tag{11}$$

The proof of this Lemma is similar to that of Lemmas 3.9 and 3.10.

3.4. Ranks and Suborbits of S_n

The results on the ranks of S_n acting on $X^{(r)}$ are proved in this section.

Lemma 3.12. *Let $G = S_8$ acting on $X^{(4)}$. The number of orbits of $G_{\{1,2,3,4\}}$ acting on $X^{(4)}$ is 5.*

Proof. Lemma 3.3 and Lemma 3.10 are applied to get the values in columns two and three in Table 3 respectively.

Applying Cauchy-Frobenius Lemma (Theorem 2.3), we get: Number of orbits of $G_{\{1,2,3,4\}}$ acting on $X^{(4)}$ is given by:

$$\frac{1}{|G_{\{1,2,3,4\}}|}\sum_{g\in G_{\{1,2,3,4\}}} |\text{Fix}(g)| = \frac{1}{576}[1 \times 70 + 12 \times 30 + 42 \times 14 + 16 \times 10 +$$
$$12 \times 2 + 36 \times 6 + 96 \times 6 + 72 \times 2 + 9 \times 6 +$$

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Permutations in $G_{\{1,2,3,4\}}$	No. of permutations	$\lvert\mathrm{Fix}(g)\rvert$ in $X^{(4)}$
(1)(2)(3)(4)(5)(6)(7)(8)	1	70
(1)(2)(3)(4)(5)(6)(7 8)	12	30
(1)(2)(3)(4)(5 6)(7 8)	42	14
(1)(2)(3)(4)(5)(6 7 8)	16	10
(1)(2)(3)(4)(5 6 7 8)	12	2
(1)(2)(3 4)(5 6)(7 8)	36	6
(1)(2)(3 4)(5)(6 7 8)	96	6
(1)(2)(3 4)(5 6 7 8)	72	2
(1 2)(3 4)(5 6)(7 8)	9	6
(1 2)(3 4)(5)(6 7 8)	48	2
(1 2)(3 4)(5 6 7 8)	36	2
(1)(2 3 4)(5)(6 7 8)	64	4
(1)(2 3 4)(5 6 7 8)	96	2
(1 2 3 4)(5 6 7 8)	36	2
Total	576	

Table 3: Permutations in $G_{\{1,2,3,4\}}$ and the number of fixed points

$$
\begin{aligned}
& 48 \times 2 + 36 \times 2 + 64 \times 4 + 96 \times 2 + 36 \times 2] \\
= & \frac{1}{576}[70 + 360 + 588 + 160 + 24 + 216 + 576 \\
& +144 + 54 + 96 + 72 + 256 + 192 + 72] \\
= & \frac{2880}{576} \\
= & 5.
\end{aligned}
$$

The five orbits of $G_{\{1,2,3,4\}}$ are:

1. $G_{\{1,2,3,4\}}\{1,2,3,4\} = \Delta_0$, the trivial orbit.

2. $G_{\{1,2,3,4\}}$\{1, 5, 6, 7\}=\{\{1, 5, 6, 7\}, \{1, 5, 6, 8\}, \{1, 5, 7, 8\}, \{1, 6, 7, 8\}, \{2, 5, 6, 7\}, \{2, 5, 6, 8\}, \{2, 5, 7, 8\}, \{2, 6, 7, 8\}, \{3, 5, 6, 7\}, \{3, 5, 6, 8\}, \{3, 5, 7, 8\}, \{3, 6, 7, 8\}, \{4, 5, 6, 7\}, \{4, 5, 6, 8\}, \{4, 5, 7, 8\}, \{4, 6, 7, 8\}\}=Δ_1, the orbit containing exactly one of 1, 2, 3, 4.

3. $G_{\{1,2,3,4\}}$\{1, 2, 5, 6\}=\{\{1, 2, 5, 6\}, \{1, 2, 5, 7\}, \{1, 2, 5, 8\}, \{1, 2, 6, 7\}, \{1, 2, 6, 8\}, \{1, 2, 7, 8\}, \{1, 3, 5, 6\}, \{1, 3, 5, 7\}, \{1, 3, 5, 8\}, \{1, 3, 6, 7\}, \{1, 3, 6, 8\}, \{1, 3, 7, 8\}, \{1, 4, 5, 6\}, \{1, 4, 5, 7\}, \{1, 4, 5, 8\}, \{1, 4, 6, 7\}, \{1, 4, 6, 8\}, \{1, 4, 7, 8\}, \{2, 3, 5, 6\}, \{2, 3, 5, 7\}, \{2, 3, 5, 8\}, \{2, 3, 6, 7\}, \{2, 3, 6, 8\}, \{2, 3, 7, 8\}, \{2, 4, 5, 6\}, \{2, 4, 5, 7\}, \{2, 4, 5, 8\}, \{2, 4, 6, 7\}, \{2, 4, 6, 8\}, \{2, 4, 7, 8\}, \{3, 4, 5, 6\}, \{3, 4, 5, 7\}, \{3, 4, 5, 8\}, \{3, 4, 6, 7\}, \{3, 4, 6, 8\}, \{3, 4, 7, 8\}\}=Δ_2, the orbit containing exactly two of 1, 2, 3, 4.

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4. $G_{\{1,2,3,4\}}$\{1, 2, 3, 5\}=\{\{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 3, 7\}, \{1, 2, 3, 8\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 4, 7\}, \{1, 2, 4, 8\}, \{2, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 3, 4, 7\}, \{2, 3, 4, 8\}, \{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 4, 7\}, \{1, 3, 4, 8\}\}=Δ_3, the orbit containing exactly three of 1, 2, 3, 4.

5. $G_{\{1,2,3,4\}}\{1,2,3,4\} = \{5,6,7,8\} = \Delta_4$, the orbit containing none of 1, 2, 3, 4.

Lemma 3.13. *Let $G = S_{10}$ acting on $X^{(5)}$. The number of orbits of $G_{\{1,2,3,4,5\}}$ acting on $X^{(5)}$ is 6.*

Proof. Lemma 3.4 and Lemma 3.11 are applied to get the values in columns two and three in Table 4 respectively.

Permutations in $G_{\{1,2,3,4,5\}}$	No. of permutations	$\|\text{Fix}(g)\|$ in $X^{(5)}$
(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)	1	252
(1)(2)(3)(4)(5)(6)(7)(8)(9 10)	20	112
(1)(2)(3)(4)(5)(6)(7 8)(9 10)	130	52
(1)(2)(3)(4)(5)(6)(7)(8 9 10)	40	42
(1)(2)(3)(4)(5)(6)(7 8 9 10)	60	12
(1)(2)(3)(4)(5)(6 7)(8 9 10)	440	22
(1)(2)(3)(4)(5)(6 7 8 9 10)	48	2
(1)(2)(3)(4 5)(6)(7 8)(9 10)	300	24
(1)(2)(3)(4 5)(6)(7 8 9 10)	600	8
(1)(2)(3)(4 5)(6 7)(8 9 10)	1000	10
(1)(2)(3)(4 5)(6 7 8 9 10)	480	2
(1)(2 3)(4 5)(6)(7 8)(9 10)	225	12
(1)(2 3)(4 5)(6)(7 8 9 10)	900	4
(1)(2 3)(4 5)(6 7)(8 9 10)	600	6
(1)(2 3)(4 5)(6 7 8 9 10)	720	2
(1)(2)(3 4 5)(6)(7)(8 9 10)	400	12
(1)(2)(3 4 5)(6)(7 8 9 10)	1200	6
(1)(2)(3 4 5)(6 7)(8 9 10)	800	4
(1)(2)(3 4 5)(6 7 8 9 10)	960	2
(1)(2 3 4 5)(6)(7 8 9 10)	900	4
(1)(2 3 4 5)(6 7)(8 9 10)	1200	2
(1)(2 3 4 5)(6 7 8 9 10)	1440	2
(1 2)(3 4 5)(6 7)(8 9 10)	400	4
(1 2)(3 4 5)(6 7 8 9 10)	960	2
(1 2 3 4 5)(6 7 8 9 10)	576	2
Total	14400	

Table 4: Permutations in $G_{\{1,2,3,4,5\}}$ and the number of fixed points

Applying Theorem 2.3, the number of orbits of $G_{\{1,2,3,4,5\}}$ acting on $X^{(5)}$ is given

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by:

$$\frac{1}{|G_{\{1,2,3,4,5\}}|}\sum_{g\in G_{\{1,2,3,4,5\}}}|\mathrm{Fix}(g)| = \frac{1}{14400}[1\times 252 + 20\times 112 + 130\times 52 + 40\times 42$$
$$+ 60\times 12 + 440\times 22 + 48\times 2 + 300\times 24$$
$$+ 600\times 8 + 1000\times 10 + 480\times 2 + 900\times 4$$
$$+ 225\times 12 + 600\times 6 + 720\times 2 + 400\times 12$$
$$+ 1200\times 6 + 800\times 4 + 960\times 2 + 900\times 4$$
$$+ 1200\times 2 + 1440\times 2 + 400\times 4 + 960\times 2 + 576\times 2]$$
$$= \frac{1}{14400}[252 + 2240 + 6760 + 1680 + 720 + 9680 + 96$$
$$+ 7200 + 4800 + 10000 + 960 + 3600 + 2700 + 3600$$
$$+ 1440 + 4800 + 7200 + 3200 + 1920 + 3600 + 2400$$
$$+ 2880 + 1600 + 1920 + 1152]$$
$$= \frac{86400}{14400}$$
$$= 6$$

The six orbits of $G_{\{1,2,3,4,5\}}$ are:

1. $G_{\{1,2,3,4,5\}}\{1,2,3,4,5\} = \Delta_0$, the trivial orbit.

2. $G_{\{1,2,3,4,5\}}$\{1, 6, 7, 8, 9\}=\{\{1, 6, 7, 8, 9\}, \{1, 6, 7, 8, 10\}, \{1, 6, 7, 9, 10\}, \{1, 6, 8, 9, 10\},
\{1, 7, 8, 9, 10\}, \{2, 6, 7, 8, 9\}, \{2, 6, 7, 8, 10\}, \{2, 6, 7, 9, 10\}, \{2, 6, 8, 9, 10\}, \{2, 7, 8, 9, 10\}, \{3, 6, 7, 8, 9\}, \{3, 6, 7, 8, 10\}, \{3, 6, 7, 9, 10\}, \{3, 6, 8, 9, 10\}, \{3, 7, 8, 9, 10\},
\{4, 6, 7, 8, 9\}, \{4, 6, 7, 8, 10\}, \{4, 6, 7, 9, 10\}, \{4, 6, 8, 9, 10\}, \{4, 7, 8, 9, 10\}, \{5, 6, 7, 8, 9\},
\{5, 6, 7, 8, 10\}, \{5, 6, 7, 9, 10\}, \{5, 6, 8, 9, 10\}, \{5, 7, 8, 9, 10\}\}=Δ_1, the orbit containing exactly one of 1, 2, 3, 4, 5.

3. $G_{\{1,2,3,4,5\}}$\{1, 2, 6, 7, 8\}=\{\{1, 2, 6, 7, 8\}, \{1, 2, 6, 7, 9\}, \{1, 2, 6, 7, 10\}, \{1, 2, 6, 8, 9\}, \{1, 2, 6, 8, 10\}, \{1, 2, 6, 9, 10\}, \{1, 2, 7, 8, 9\}, \{1, 2, 7, 8, 10\}, \{1, 2, 7, 9, 10\}, \{1, 2, 8, 9, 10\}, \{1, 3, 6, 7, 8\}, \{1, 3, 6, 7, 9\}, \{1, 3, 6, 7, 10\}, \{1, 3, 6, 8, 9\}, \{1, 3, 6, 8, 10\}, \{1, 3, 6, 9, 10\}, \{1, 3, 7, 8, 9\}, \{1, 3, 7, 8, 10\}, \{1, 3, 7, 9, 10\}, \{1, 3, 8, 9, 10\}, \{1, 4, 6, 7, 8\}, \{1, 4, 6, 7, 9\}, \{1, 4, 6, 7, 10\}, \{1, 4, 6, 8, 9\}, \{1, 4, 6, 8, 10\}, \{1, 4, 6, 9, 10\}, \{1, 4, 7, 8, 9\}, \{1, 4, 7, 8, 10\}, \{1, 4, 7, 9, 10\}, \{1, 4, 8, 9, 10\}, \{1, 5, 6, 7, 8\}, \{1, 5, 6, 7, 9\}, \{1, 5, 6, 7, 10\}, \{1, 5, 6, 8, 9\}, \{1, 5, 6, 8, 10\}, \{1, 5, 6, 9, 10\}, \{1, 5, 7, 8, 9\}, \{1, 5, 7, 8, 10\}, \{1, 5, 7, 9, 10\}, \{1, 5, 8, 9, 10\}, \{2, 3, 6, 7, 8\}, \{2, 3, 6, 7, 9\}, \{2, 3, 6, 7, 10\}, \{2, 3,

6, 8, 9}, {2, 3, 6, 8, 10}, {2, 3, 6, 9, 10}, {2, 3, 7, 8, 9}, {2, 3, 7, 8, 10}, {2, 3, 7, 9, 10}, {2, 3, 8, 9, 10}, {2, 4, 6, 7, 8}, {2, 4, 6, 7, 9}, {2, 4, 6, 7, 10}, {2, 4, 6, 8, 9}, {2, 4, 6, 8, 10}, {2, 4, 6, 9, 10}, {2, 4, 7, 8, 9}, {2, 4, 7, 8, 10}, {2, 4, 7, 9, 10}, {2, 4, 8, 9, 10}, {2, 5, 6, 7, 8}, {2, 5, 6, 7, 9}, {2, 5, 6, 7, 10}, {2, 5, 6, 8, 9}, {2, 5, 6, 8, 10}, {2, 5, 6, 9, 10}, {2, 5, 7, 8, 9}, {2, 5, 7, 8, 10}, {2, 5, 7, 9, 10}, {2, 5, 8, 9, 10}, {3, 4, 6, 7, 8}, {3, 4, 6, 7, 9}, {3, 4, 6, 7, 10}, {3, 4, 6, 8, 9}, {3, 4, 6, 8, 10}, {3, 4, 6, 9, 10}, {3, 4, 7, 8, 9}, {3, 4, 7, 8, 10}, {3, 4, 7, 9, 10}, {3, 4, 8, 9, 10}, {3, 5, 6, 7, 8}, {3, 5, 6, 7, 9}, {3, 5, 6, 7, 10}, {3, 5, 6, 8, 9}, {3, 5, 6, 8, 10}, {3, 5, 6, 9, 10}, {3, 5, 7, 8, 9}, {3, 5, 7, 8, 10}, {3, 5, 7, 9, 10}, {3, 5, 8, 9, 10}, {4, 5, 6, 7, 8}, {4, 5, 6, 7, 9}, {4, 5, 6, 7, 10}, {4, 5, 6, 8, 9}, {4, 5, 6, 8, 10}, {4, 5, 6, 9, 10}, {4, 5, 7, 8, 9}, {4, 5, 7, 8, 10}, {4, 5, 7, 9, 10}, {4, 5, 8, 9, 10}}=Δ_2, the orbit containing exactly two of 1, 2, 3, 4, 5.

4. $G_{\{1,2,3,4,5\}}$\{1, 2, 3, 6, 7}={{1, 2, 3, 6, 7}, {1, 2, 3, 6, 8}, {1, 2, 3, 6, 9}, {1, 2, 3, 6, 10}, {1, 2, 3, 7, 8}, {1, 2, 3, 7, 9}, {1, 2, 3, 7, 10}, {1, 2, 3, 8, 9}, {1, 2, 3, 8, 10}, {1, 2, 3, 9, 10}, {1, 2, 4, 6, 7}, {1, 2, 4, 6, 8}, {1, 2, 4, 6, 9}, {1, 2, 4, 6, 10}, {1, 2, 4, 7, 8}, {1, 2, 4, 7, 9}, {1, 2, 4, 7, 10}, {1, 2, 4, 8, 9}, {1, 2, 4, 8, 10}, {1, 2, 4, 9, 10}, {1, 2, 5, 6, 7}, {1, 2, 5, 6, 8}, {1, 2, 5, 6, 9}, {1, 2, 5, 6, 10}, {1, 2, 5, 7, 8}, {1, 2, 5, 7, 9}, {1, 2, 5, 7, 10}, {1, 2, 5, 8, 9}, {1, 2, 5, 8, 10}, {1, 2, 5, 9, 10}, {1, 3, 4, 6, 7}, {1, 3, 4, 6, 8}, {1, 3, 4, 6, 9}, {1, 3, 4, 6, 10}, {1, 3, 4, 7, 8}, {1, 3, 4, 7, 9}, {1, 3, 4, 7, 10}, {1, 3, 4, 8, 9}, {1, 3, 4, 8, 10}, {1, 3, 4, 9, 10}, {1, 3, 5, 6, 7}, {1, 3, 5, 6, 8}, {1, 3, 5, 6, 9}, {1, 3, 5, 6, 10}, {1, 3, 5, 7, 8}, {1, 3, 5, 7, 9}, {1, 3, 5, 7, 10}, {1, 3, 5, 8, 9}, {1, 3, 5, 8, 10}, {1, 3, 5, 9, 10}, {1, 4, 5, 6, 7}, {1, 4, 5, 6, 8}, {1, 4, 5, 6, 9}, {1, 4, 5, 6, 10}, {1, 4, 5, 7, 8}, {1, 4, 5, 7, 9}, {1, 4, 5, 7, 10}, {1, 4, 5, 8, 9}, {1, 4, 5, 8, 10}, {1, 4, 5, 9, 10}, {2, 3, 4, 6, 7}, {2, 3, 4, 6, 8}, {2, 3, 4, 6, 9}, {2, 3, 4, 6, 10}, {2, 3, 4, 7, 8}, {2, 3, 4, 7, 9}, {2, 3, 4, 7, 10}, {2, 3, 4, 8, 9}, {2, 3, 4, 8, 10}, {2, 3, 4, 9, 10}, {2, 3, 5, 6, 7}, {2, 3, 5, 6, 8}, {2, 3, 5, 6, 9}, {2, 3, 5, 6, 10}, {2, 3, 5, 7, 8}, {2, 3, 5, 7, 9}, {2, 3, 5, 7, 10}, {2, 3, 5, 8, 9}, {2, 3, 5, 8, 10}, {2, 3, 5, 9, 10}, {2, 4, 5, 6, 7}, {2, 4, 5, 6, 8}, {2, 4, 5, 6, 9}, {2, 4, 5, 6, 10}, {2, 4, 5, 7, 8}, {2, 4, 5, 7, 9}, {2, 4, 5, 7, 10}, {2, 4, 5, 8, 9}, {2, 4, 5, 8, 10}, {2, 4, 5, 9, 10}, {3, 4, 5, 6, 7}, {3, 4, 5, 6, 8}, {3, 4, 5, 6, 9}, {3, 4, 5, 6, 10}, {3, 4, 5, 7, 8}, {3, 4, 5, 7, 9}, {3, 4, 5, 7, 10}, {3, 4, 5, 8, 9}, {3, 4, 5, 8, 10}, {3, 4, 5, 9, 10}}=Δ_3, the orbit containing exactly three of 1, 2, 3, 4, 5.

5. $G_{\{1,2,3,4,5\}}$\{1, 2, 3, 4, 6}={{1, 2, 3, 4, 6}, {1, 2, 3, 4, 7}, {1, 2, 3, 4, 8}, {1, 2, 3, 4, 9}, {1, 2, 3, 4, 10}, {1, 2, 3, 5, 6}, {1, 2, 3, 5, 7}, {1, 2, 3, 5, 8}, {1, 2, 3, 5, 9}, {1, 2, 3, 5, 10}, {1, 2, 4, 5, 6}, {1, 2, 4, 5, 7}, {1, 2, 4, 5, 8}, {1, 2, 4, 5, 9}, {1, 2, 4, 5, 10}, {1, 3, 4, 5, 6}, {1, 3, 4, 5, 7}, {1, 3, 4, 5, 8}, {1, 3, 4, 5, 9}, {1, 3, 4, 5, 10}, {2, 3, 4, 5, 6}, {2, 3, 4, 5, 7}, {2, 3, 4, 5, 8}, {2, 3, 4, 5, 9}, {2, 3, 4, 5, 10}}=Δ_4, the orbit containing exactly four of 1, 2, 3, 4, 5.

6. $G_{\{1,2,3,4,5\}}\{6,7,8,9,10\} = \{6,7,8,9,10\} = \Delta_5$, the orbit containing none of 1, 2, 3, 4, 5.

From Lemmas 3.12 and 3.13, we deduce the following result:

Theorem 3.14. *If $n \geq 2r$, the rank of $G = S_n$ acting on $X^{(r)}$ is $r+1$.*

Proof. To start with, $G_{\{1,2,3,...,r\}}$ has $r+1$ orbits. The $r+1$ suborbits are:

1. $Orb_{G_{\{1,2,3,...,r\}}}\{1,2,3,...,r\} = \Delta_0$, the trivial orbit.
2. $Orb_{G_{\{1,2,3,...,r\}}}\{1, r+1, r+2, \ldots, 2r-1\} = \Delta_1$, the orbit containing exactly one of 1, 2, 3,..., r.
3. $Orb_{G_{\{1,2,3,...,r\}}}\{1,2,r+1,...,2r-2\} = \Delta_2$, the orbit containing exactly two of 1, 2, 3,..., r.
4. $Orb_{G_{\{1,2,3,...,r\}}}\{1,2,3,r+1,\ldots,2r-3\} = \Delta_3$, the orbit containing exactly three of 1, 2, 3,..., r.

$\vdots$

5. $Orb_{G_{\{1,2,3,...,r\}}}\{1,2,...,r-1,r+1\} = \Delta_{r-1}$, the orbit containing exactly $r-1$ of either 1, 2, 3,..., r.
6. $Orb_{G_{\{1,2,3,...,r\}}}\{r+1,r+2,...,2r\} = \Delta_r$, the orbit containing none of 1, 2, 3,..., r.

Finally, the proof that this is possible only if $n \geq 2r$ is given. Suppose $n-r=0$, then $G_{\{1,2,3,...,r\}}$ has only one orbit, the trivial one; and if $n-r=1$, $G_{\{1,2,3,...,r\}}$ has two orbits, namely, the trivial orbit and the one containing exactly $r-1$ of 1, 2, 3,. . . , and r. If $n-r=2$, $G_{\{1,2,3,...,r\}}$ has three orbits, namely, the trivial orbit, the one containing exactly $r-1$ of 1, 2, 3,. . . , and r, and the one containing exactly $r-2$ of 1, 2, 3,. . . , and r. Continuing with this argument, we find that if $n-r=r-1$, then $G_{\{1,2,3,...,r\}}$ has r orbits i.e the trivial one, the one containing exactly $r-1$ of 1, 2, 3,...,and r, the orbit containing exactly $r-2$ of 1, 2, 3,. . . ,r and so on up to the orbit containing exactly one of 1, 2, 3,. . . , r. In a similar manner, if $n-r \geq r$, $G_{\{1,2,3,...,r\}}$ will have an additional orbit, the one containing none of 1, 2, 3,. . . , and r. This makes it have $r+1$ orbits in total. We can rewrite $n-r \geq r$ as $n \geq 2r$ and thus completing the proof.

Example 3.15. Let $G = S_6$ acting on $X^{(3)}$. Then rank of $G = 4$.

The four suborbits of G are:

1. $G_{\{1,2,3\}}$\{1,2,3\}=\{\{1,2,3\}\}=Δ_0, the trivial orbit.

2. $G_{\{1,2,3\}}\{1,4,5\}$={{1,4,5}, {1,4,6}, {1,5,6}, {2,4,5}, {2,4,6}, {2,5,6}, {3,4,5}, {3,4,6}, {3,5,6}}=Δ_1, the orbit containing exactly one of 1, 2, and 3.

3. $G_{\{1,2,3\}}\{1,2,4\}$={{1,2,4}, {1,2,5}, {1,2,6}, {1,3,4}, {1,3,5}, {1,3,6}, {2,3,4}, {2,3,5}, {2,3,6}}=Δ_2, the orbit containing exactly two of 1, 2, and 3.

4. $G_{\{1,2,3\}}\{4,5,6\} = \{\{4,5,6\}\} = \Delta_3$, the orbit containing none of 1, 2, and 3.

Example 3.16. Let $G = S_{10}$ acting on $X^{(6)}$. Rank of $G = 5$.

The five suborbits of G are:

1. $G_{\{1,2,3,4,5,6\}}\{1,2,3,4,5,6\} = \Delta_0$, the trivial suborbit.

2. $G_{\{1,2,3,4,5,6\}}\{1,2,7,8,9,10\}$={{1, 2, 7, 8, 9, 10}, {1, 3, 7, 8, 9, 10}, {1, 4, 7, 8, 9, 10}, {1, 5, 7, 8, 9, 10}, {1, 6, 7, 8, 9, 10}, {2, 3, 7, 8, 9, 10}, {2, 4, 7, 8, 9, 10}, {2, 5, 7, 8, 9, 10}, {2, 6, 7, 8, 9, 10}, {3, 4, 7, 8, 9, 10}, {3, 5, 7, 8, 9, 10}, {3, 6, 7, 8, 9, 10}, {4, 5, 7, 8, 9, 10}, {4, 6, 7, 8, 9, 10}, {5, 6, 7, 8, 9, 10}}=Δ_2, the suborbit containing exactly two of 1, 2, 3, 4, 5, and 6.

3. $G_{\{1,2,3,4,5,6\}}\{1,2,3,7,8,9\}$={{1, 2, 3, 7, 8, 9}, {1, 2, 3, 7, 8, 10}, {1, 2, 3, 7, 9, 10}, {1, 2, 3, 8, 9, 10}, {1, 2, 4, 7, 8, 9}, {1, 2, 4, 7, 8, 10}, {1, 2, 4, 7, 9, 10}, {1, 2, 4, 8, 9, 10}, {1, 2, 5, 7, 8, 9}, {1, 2, 5, 7, 8, 10}, {1, 2, 5, 7, 9, 10}, {1, 2, 5, 8, 9, 10}, {1, 2, 6, 7, 8, 9}, {1, 2, 6, 7, 8, 10}, {1, 2, 6, 7, 9, 10}, {1, 2, 6, 8, 9, 10}, {1, 3, 4, 7, 8, 9}, {1, 3, 4, 7, 8, 10}, {1, 3, 4, 7, 9, 10}, {1, 3, 4, 8, 9, 10}, {1, 3, 5, 7, 8, 9}, {1, 3, 5, 7, 8, 10}, {1, 3, 5, 7, 9, 10}, {1, 3, 5, 8, 9, 10}, {1, 3, 6, 7, 8, 9}, {1, 3, 6, 7, 8, 10}, {1, 3, 6, 7, 9, 10}, {1, 3, 6, 8, 9, 10}, {1, 4, 5, 7, 8, 9}, {1, 4, 5, 7, 8, 10}, {1, 4, 5, 7, 9, 10}, {1, 4, 5, 8, 9, 10}, {1, 4, 6, 7, 8, 9}, {1, 4, 6, 7, 8, 10}, {1, 4, 6, 7, 9, 10}, {1, 4, 6, 8, 9, 10}, {1, 5, 6, 7, 8, 9}, {1, 5, 6, 7, 8, 10}, {1, 5, 6, 7, 9, 10}, {1, 5, 6, 8, 9, 10}, {2, 3, 4, 7, 8, 9}, {2, 3, 4, 7, 8, 10}, {2, 3, 4, 7, 9, 10}, {2, 3, 4, 8, 9, 10}, {2, 3, 5, 7, 8, 9}, {2, 3, 5, 7, 8, 10}, {2, 3, 5, 7, 9, 10}, {2, 3, 5, 8, 9, 10}, {2, 3, 6, 7, 8, 9}, {2, 3, 6, 7, 8, 10}, {2, 3, 6, 7, 9, 10}, {2, 3, 6, 8, 9, 10}, {2, 4, 5, 7, 8, 9}, {2, 4, 5, 7, 8, 10}, {2, 4, 5, 7, 9, 10}, {2, 4, 5, 8, 9, 10}, {2, 4, 6, 7, 8, 9}, {2, 4, 6, 7, 8, 10}, {2, 4, 6, 7, 9, 10}, {2, 4, 6, 8, 9, 10}, {2, 5, 6, 7, 8, 9}, {2, 5, 6, 7, 8, 10}, {2, 5, 6, 7, 9, 10}, {2, 5, 6, 8, 9, 10}, {3, 4, 5, 7, 8, 9}, {3, 4, 5, 7, 8, 10}, {3, 4, 5, 7, 9, 10}, {3, 4, 5, 8, 9, 10}, {3, 4, 6, 7, 8, 9}, {3, 4, 6, 7, 8, 10}, {3, 4, 6, 7, 9, 10}, {3, 4, 6, 8, 9, 10}, {3, 5, 6, 7, 8, 9}, {3, 5, 6, 7, 8, 10}, {3, 5, 6, 7, 9, 10}, {3, 5, 6, 8, 9, 10}, {4, 5, 6, 7, 8, 9}, {4, 5, 6, 7, 8, 10}, {4, 5, 6, 7, 9, 10}, {4, 5, 6, 8, 9, 10}}=Δ_3, the suborbit containing exactly three of 1, 2, 3, 4, 5, and 6.

4. $G_{\{1,2,3,4,5,6\}}\{1,2,3,4,7,8\}$={{1, 2, 3, 4, 7, 8}, {1, 2, 3, 4, 7, 9}, {1, 2, 3, 4, 7, 10}, {1, 2, 3, 4, 8, 9}, {1, 2, 3, 4, 8, 10}, {1, 2, 3, 4, 9, 10}, {1, 2, 3, 5, 7, 8},

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{1, 2, 3, 5, 7, 9}, {1, 2, 3, 5, 7, 10}, {1, 2, 3, 5, 8, 9}, {1, 2, 3, 5, 8, 10}, {1, 2, 3, 5, 9, 10}, {1, 2, 3, 6, 7, 8}, {1, 2, 3, 6, 7, 9}, {1, 2, 3, 6, 7, 10}, {1, 2, 3, 6, 8, 9}, {1, 2, 3, 6, 8, 10}, {1, 2, 3, 6, 9, 10}, {1, 2, 4, 5, 7, 8}, {1, 2, 4, 5, 7, 9}, {1, 2, 4, 5, 7, 10}, {1, 2, 4, 5, 8, 9}, {1, 2, 4, 5, 8, 10}, {1, 2, 4, 5, 9, 10}, {1, 2, 4, 6, 7, 8}, {1, 2, 4, 6, 7, 9}, {1, 2, 4, 6, 7, 10}, {1, 2, 4, 6, 8, 9}, {1, 2, 4, 6, 8, 10}, {1, 2, 4, 6, 9, 10}, {1, 2, 5, 6, 7, 8}, {1, 2, 5, 6, 7, 9}, {1, 2, 5, 6, 7, 10}, {1, 2, 5, 6, 8, 9}, {1, 2, 5, 6, 8, 10}, {1, 2, 5, 6, 9, 10}, {1, 3, 4, 5, 7, 8}, {1, 3, 4, 5, 7, 9}, {1, 3, 4, 5, 7, 10}, {1, 3, 4, 5, 8, 9}, {1, 3, 4, 5, 8, 10}, {1, 3, 4, 5, 9, 10}, {1, 3, 4, 6, 7, 8}, {1, 3, 4, 6, 7, 9}, {1, 3, 4, 6, 7, 10}, {1, 3, 4, 6, 8, 9}, {1, 3, 4, 6, 8, 10}, {1, 3, 4, 6, 9, 10}, {1, 3, 5, 6, 7, 8}, {1, 3, 5, 6, 7, 9}, {1, 3, 5, 6, 7, 10}, {1, 3, 5, 6, 8, 9}, {1, 3, 5, 6, 8, 10}, {1, 3, 5, 6, 9, 10}, {1, 4, 5, 6, 7, 8}, {1, 4, 5, 6, 7, 9}, {1, 4, 5, 6, 7, 10}, {1, 4, 5, 6, 8, 9}, {1, 4, 5, 6, 8, 10}, {1, 4, 5, 6, 9, 10}, {2, 3, 4, 5, 7, 8}, {2, 3, 4, 5, 7, 9}, {2, 3, 4, 5, 7, 10}, {2, 3, 4, 5, 8, 9}, {2, 3, 4, 5, 8, 10}, {2, 3, 4, 5, 9, 10}, {2, 3, 4, 6, 7, 8}, {2, 3, 4, 6, 7, 9}, {2, 3, 4, 6, 7, 10}, {2, 3, 4, 6, 8, 9}, {2, 3, 4, 6, 8, 10}, {2, 3, 4, 6, 9, 10}, {2, 3, 5, 6, 7, 8}, {2, 3, 5, 6, 7, 9}, {2, 3, 5, 6, 7, 10}, {2, 3, 5, 6, 8, 9}, {2, 3, 5, 6, 8, 10}, {2, 3, 5, 6, 9, 10}, {2, 4, 5, 6, 7, 8}, {2, 4, 5, 6, 7, 9}, {2, 4, 5, 6, 7, 10}, {2, 4, 5, 6, 8, 9}, {2, 4, 5, 6, 8, 10}, {2, 4, 5, 6, 9, 10}, {3, 4, 5, 6, 7, 8}, {3, 4, 5, 6, 7, 9}, {3, 4, 5, 6, 7, 10}, {3, 4, 5, 6, 8, 9}, {3, 4, 5, 6, 8, 10}, {3, 4, 5, 6, 9, 10}}= Δ_4, the suborbit containing exactly four of 1, 2, 3, 4, 5, and 6.

5. $G_{\{1,2,3,4,5,6\}}\{1,2,3,4,5,7\}$={{1, 2, 3, 4, 5, 7}, {1, 2, 3, 4, 5, 8}, {1, 2, 3, 4, 5, 9}, {1, 2, 3, 4, 5, 10}, {1, 2, 3, 4, 6, 7}, {1, 2, 3, 4, 6, 8}, {1, 2, 3, 4, 6, 9}, {1, 2, 3, 4, 6, 10}, {1, 2, 3, 5, 6, 7}, {1, 2, 3, 5, 6, 8}, {1, 2, 3, 5, 6, 9}, {1, 2, 3, 5, 6, 10}, {1, 2, 4, 5, 6, 7}, {1, 2, 4, 5, 6, 8}, {1, 2, 4, 5, 6, 9}, {1, 2, 4, 5, 6, 10}, {1, 3, 4, 5, 6, 7}, {1, 3, 4, 5, 6, 8}, {1, 3, 4, 5, 6, 9}, {1, 3, 4, 5, 6, 10}, {2, 3, 4, 5, 6, 7}, {2, 3, 4, 5, 6, 8}, {2, 3, 4, 5, 6, 9}, {2, 3, 4, 5, 6, 10}}=Δ_5, the suborbit containing exactly five of 1, 2, 3, 4, 5, and 6.

In this example, G does not have the suborbit Δ_1 and Δ_6. This is because $n < 2r$. This stresses the fact that for the rank of G to be $r+1$, then $n \geq 2r$.

3.5. Self Paired Suborbits of S_n

In the next Theorem we show that all the suborbits of S_n are self paired.

Theorem 3.17. *Let $n \geq 2r$, then the suborbits $\Delta_0, \Delta_1, \Delta_2, ..., \Delta_{r-1},\ \Delta_r$ of S_n acting on $X^{(r)}$ are self paired.*

Proof. The proof for Δ_0 is trivial. Consider an arbitrary member of Δ_1 say $\{1, r+1, ..., 2r-1\}$. By the definition of a self paired suborbit (see Section 2) if $g\{1, r+1, ..., 2r-1\} = \{1, 2, ..., r\}$, then we can take g to be $(1)(r+1\ 2)(r+2\ 3)...(2r-$

$1\ r)$ and $g\{1,2,...,r\} = \{1, r+1, r+3, ..., 2r-1\} \in \Delta_1$. This shows that Δ_1 is self paired.

Similarly consider an arbitrary member of Δ_2 say $\{1, 2, r+1, ..., 2r-2\}$. If $g\{1, 2, r+1, ..., 2r-2\} = \{1, 2, ..., r\}$, then we can take g to be $(1)(2)(r+1\ 3)...(2r-2\ r)$ and $g\{1, 2, ..., r\} = \{1, 2, r+1, \ldots, 2r-2\} \in \Delta_2$, showing that Δ_2 is also self paired. Using similar arguments, we can show that $\Delta_3, \Delta_4, ..., \Delta_{r-1}$ are self paired.

Finally, consider an arbitrary member of Δ_r say $\{r+1, r+2, ..., 2r\}$. If $g\{r+1, r+2, ..., 2r\} = \{1, 2, ..., r\}$, then we can take g to be $(r+1\ 1)(r+2\ 2)...(2r\ r)$ and $g\{1, 2, ..., r\} = \{r+1, r+2 \ldots, 2r\} \in \Delta_r$, showing that Δ_r is a self paired.

Example 3.18. Let $G = S_7$ acting on $X^{(3)}$, then Δ_1, Δ_2 and Δ_3 are self paired.

This is because {1,4,5} $\in \Delta_1$, and we can take $g = (1)(4\ 2)(5\ 3)$. Consequently $gx = \{1, 4, 5\} \in \Delta_1$, where $x = \{1, 2, 3\}$. Similarly {1,2,4}$\in \Delta_2$, $g = (1)(2)(4\ 3)$ and $gx = \{1, 2, 4\} \in \Delta_2$. Lastly {4,5,6}$\in \Delta_3$, $g = (4\ 1)(5\ 2)(6\ 3)$ and $gx = \{4, 5, 6\} \in \Delta_3$.

3.6. Subdegrees of S_n Acting on $X^{(r)}$

The rank and the subdegrees of a group G are closely related in that while the rank is the number of the suborbits of G, the subdegrees are the sizes of these suborbits of G. The subdegrees of S_n acting on $X^{(r)}$ are given by Theorem 3.19.

Theorem 3.19. *Let $n \geq 2r$, then the subdegrees of S_n acting on $X^{(r)}$ are:*

$$1, r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \binom{r}{3}\binom{n-r}{r-3}, ...,$$
$$\binom{r}{r-1}\binom{n-r}{1}, \binom{n-r}{r}$$

Proof. The $r+1$ suborbits of G obtained in Theorem 3.14 are considered. The length of $\Delta_0 = 1$. Consider the suborbit Δ_1, which contains exactly one of $1, 2, 3, .., $ and r. One of 1, 2, 3,..., and r may be chosen in r ways while the remaining $r-1$ elements may be chosen from $n-r$ elements of X in $\binom{n-r}{r-1}$ ways. This makes the total number of selections to be $r\binom{n-r}{r-1}$. For the suborbit Δ_2, the two elements may be chosen in $\binom{r}{2}$ ways while the remaining $r-2$ elements may be chosen in $\binom{n-r}{r-2}$ ways, making a total of $\binom{r}{2}\binom{n-r}{r-2}$ ways. Continuing with the same argument, in the suborbit Δ_{r-1}, the $r-1$ elements may be chosen in $\binom{r}{r-1}$ and the remaining 1 element in $\binom{n-r}{1}$ ways making a total of $\binom{n-r}{1}\binom{r}{r-1}$

ways. Finally, for Δ_r which does not contain any of elements from $\{1, 2, 3,..., r\}$, the r elements from X can be chosen in $\binom{n-r}{r}$ ways.

Example 3.20. If $G = S_7$ acting on $X^{(3)}$, the subdegrees of G are:

$$1, 3\binom{4}{2}, 4\binom{3}{2}, \binom{4}{3},$$

that is 1, 18, 12, 4.

The four suborbits of G are:

1. $\Delta_0 = \{1,2,3\}$
2. $\Delta_1 = \{\{1,4,5\}, \{1,4,6\}, \{1,4,7\}, \{1,5,6\}, \{1,5,7\}, \{1,5,7\}, \{2,4,5\}, \{2,4,6\}, \{2,4,7\}, \{2,5,6\}, \{2,5,7\}, \{2,6,7\}, \{3,4,5\}, \{3,4,6\}, \{3,4,7\}, \{3,5,6\}, \{3,5,7\}, \{3,6,7\}\}$.
3. $\Delta_2 = \{\{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,2,7\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,3,7\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,3,7\}\}$.
4. $\Delta_3 = \{\{4,5,6\}, \{4,5,7\}, \{4,6,7\}, \{5,6,7\}\}$.

$|\Delta_0| = 1, |\Delta_1| = 18, |\Delta_2| = 12, \text{and} |\Delta_3| = 4$. So that the subdegrees are 1, 4, 12, and 18 as expected.

Example 3.21. Let $G = S_{10}$ acting on $X^{(5)}$. The subdegrees of G are: $1, 5\binom{5}{4}, \binom{5}{2}\binom{5}{3}, \binom{5}{3}\binom{5}{2}, \binom{5}{4}\binom{5}{1}, \binom{5}{5}$, that is $1, 25, 100, 100, 25, 1$.

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