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ON OPERATORS FOR MEROMORPHIC FUNCTIONS

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Abstract: In this article we define various types of differential operators for meromorphic functions of fractional power. New classes of meromorphic $\Phi-$like and **E** typed of functions are established. Further, some subordination theorems for such classes in punctured unit disk are introduced.

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1. Introduction and Preliminaries

Let $\mathbf{E}_\alpha^+$ be the class of functions $F(z)$ of the form

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^{n+\alpha-1}, \ \ \alpha \geq 1,$$

which are analytic in the punctured unit disk $U := \{z \in \mathbf{C},\, 0 < |z| < 1\}$. And let $\mathbf{E}_\alpha^-$ be the class of functions of the form

$$F(z) = \frac{1}{z} - \sum_{n=0}^{\infty} a_n z^{n+\alpha-1}, \ \ \alpha \geq 1,$$

which are analytic in the punctured unit disk U. Recalling the principle of subordination between analytic functions, let the functions f and g be analytic in the unit

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disk then the function f is *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in $\triangle$ such that $f(z) = g(w(z))$. We denote this subordination by $f \prec g$.

Let $\mathcal{E}^+$ be the class of analytic functions, in U, of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$. And let $\mathcal{E}^-$ be the class of analytic functions, in U, of the form $f(z) = \frac{1}{z} - \sum_{n=0}^{\infty} a_n z^n$, $a_n \geq 0$, $n = 0, 1, ...$ Then we have the following classes:

Definition 1. Let Φ be analytic function in a domain containing $F(U)$ and $\Phi(\omega) \neq 0$ for $\omega \in F(U)$. Let $q(z)$ be a fixed analytic function in U. The function $F \in \mathcal{E}_\alpha^+(\mathcal{E}_\alpha^-)$ is called $\Phi-$like with respect to q if $-\frac{zF'(z)}{\Phi(F(z))} \prec q(z),\ z \in U.$

Definition 2. Let $F(z) \in \mathcal{A}_\alpha^-$, we define the meromorphic family $\mathcal{E}_\alpha^-(\Phi, \Psi)$, where $\Phi(z) = \frac{1}{z} - \sum_{n=0}^{\infty} \varphi_n z^{n+\alpha-1}$ and $\Psi(z) = \frac{1}{z} - \sum_{n=0}^{\infty} \psi_n z^{n+\alpha-1}$ are analytic in U under the conditions $\varphi_n \geq 0$, $\psi_n \geq 0, \varphi_n \geq \psi_n$ for $n \geq 0$ and $F(z) * \Psi(z) \neq 0$.

Now we define differential operators as follows. Let $F \in \mathbf{E}_\alpha^+$ then

$$D_\alpha^1 F(z) = \frac{(z^2 F(z))'}{z} = \frac{1}{z} + \sum_{n=0}^{\infty}(n+\alpha+1)a_n z^{n+\alpha-1}, \cdots, D_\alpha^k F(z)$$
$$= \frac{1}{z} + \sum_{n=0}^{\infty}(n+\alpha+1)^k a_n z^{n+\alpha-1}. \quad (1)$$

$$\mathcal{D}_\alpha^1 F(z) = (\alpha+1)F(z) + \alpha z F'(z), \cdots, \mathcal{D}_\alpha^k F(z)$$
$$= \frac{1}{z} + \sum_{n=0}^{\infty}[(\alpha+1) + \alpha(n+\alpha-1)]^k a_n z^{n+\alpha-1}, \quad (2)$$

and

$$D_{\alpha,\lambda}^1 F(z) = (2\alpha-\lambda+1)F(z) + (2\alpha-\lambda)zF'(z), \cdots, D_{\alpha,\lambda}^k F(z) =$$
$$\frac{1}{z} + \sum_{n=0}^{\infty}[(2\alpha-\lambda)(n+\alpha)+1]^k a_n z^{n+\alpha-1}. \quad (3)$$

In this paper, we establish some sufficient conditions for functions $F \in \mathbf{E}_\alpha^+$ and $F \in \mathcal{E}_\alpha^-$ to satisfy

$$-\frac{z(D_\alpha^k F(z))'(z)}{\Phi(D_\alpha^k F(z))} \prec q(z), \quad z \in U, \quad (4)$$

$$-\frac{D_\alpha^k F(z) * \Phi(z)}{D_\alpha^k F(z) * \Psi(z)} \prec q(z), \quad z \in U, \quad (5)$$

$$-\frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)} \prec q(z), \quad z \in U,\ n = 1, 2, ..., \quad (6)$$

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and

$$-(\frac{1}{zF(z)})^{\mu} \prec q(z), \quad z \in U, \quad \mu \neq 0, \tag{7}$$

where $q(z)$ is a given univalent function in U. We shall need the following known results.

Lemma 1. (see [1]) *Let $q(z)$ be convex univalent in the unit disk $\triangle$ and ψ and $\gamma \in \mathbf{C}$ with $\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\}>0$. If $p(z)$ is analytic in $\triangle$ and $\psi p(z)+\gamma zp'(z) \prec \psi q(z)+\gamma zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 2. (see [2]) *Let $q(z)$ be univalent in the unit disk $\triangle$ and θ and ϕ be analytic in a domain D containing $q(\triangle)$ with $\phi(w) \neq 0$ when $w \in q(\triangle)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z))+Q(z)$. Suppose that $Q(z)$ is starlike univalent in $\triangle$, and $\Re\frac{zh'(z)}{Q(z)}>0$ for $z \in \triangle$. If $\theta(p(z))+zp'(z)\phi(p(z)) \prec \theta(q(z))+zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

2. Subordination Results

In this section, we establish some sufficient conditions for subordination of analytic functions in the classes $\mathbf{E}_{\alpha}^{+}$ and $\mathcal{E}_{\alpha}^{-}$.

Theorem 1. *Let the function $q(z)$ be convex univalent in U such that $q'(z) \neq 0$ and $\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\}>0$, $\gamma \neq 0$. Suppose that $-\frac{z(D_{\alpha}^{k}F(z))'(z)}{\Phi(D_{\alpha}^{k}F(z))}$ is analytic in U. If $F \in \mathbf{E}_{\alpha}^{+}$ satisfies the subordination*

$$-\frac{z(D_{\alpha}^{k}F(z))'(z)}{\Phi(D_{\alpha}^{k}F(z))}\{\psi+\gamma[1+\frac{z(D_{\alpha}^{k}F(z))''(z)}{(D_{\alpha}^{k}F(z))'(z)}-\frac{z\Phi'(D_{\alpha}^{k}F(z))}{\Phi(D_{\alpha}^{k}F(z))}]\} \prec \psi q(z)+\gamma zq'(z).$$

Then $-\frac{z(D_{\alpha}^{k}F(z))'(z)}{\Phi(D_{\alpha}^{k}F(z))} \prec q(z)$, $z \in U$, and $q(z)$ is the best dominant.

Proof. Let the function $p(z)$ be defined by $p(z) := -\frac{z(D_{\alpha}^{k}F(z))'}{\Phi(D_{\alpha}^{k}F(z))}$, $z \in U$. It can easily observed that

$$\psi p(z)+\gamma zp'(z) = -\frac{z(D_{\alpha}^{k}F(z))'}{\Phi(D_{\alpha}^{k}F(z))}\{\psi+\gamma[1+\frac{z(D_{\alpha}^{k}F(z))''}{(D_{\alpha}^{k}F(z))'}-\frac{z\Phi'(D_{\alpha}^{k}F(z))}{\Phi(D_{\alpha}^{k}F(z))}]\} \prec \psi q(z)+\gamma zq'(z).$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1. □

For $\Phi(\omega)=\omega$, in Theorem 1, we have the following result.

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Corollary 2. *If the subordination*

$$-\frac{z(D_\alpha^k F(z))'}{D_\alpha^k F(z)}\{\psi+\gamma[1+\frac{z(D_\alpha^k F(z))''}{(D_\alpha^k F(z))'}-\frac{z(D_\alpha^k F(z))'}{D_\alpha^k F(z)}]\}\prec \psi q(z)+\gamma z q'(z)$$

satisfies then $-\frac{z(D_\alpha^k F(z))'}{D_\alpha^k F(z)}\prec q(z)$.

For $\Phi(\omega)=\omega, k=0$ in Theorem 1, we have the following result.

Corollary 3. *If the subordination*

$$-\frac{z(F(z))'}{F(z)}\{\psi+\gamma[1+\frac{z(F(z))''}{(F(z))'}-\frac{z(F(z))'}{F(z)}]\}\prec \psi q(z)+\gamma z q'(z)$$

satisfies, then $-\frac{z(F(z))'}{F(z)}\prec q(z)$.

Theorem 4. *Let the function* $q(z)$ *be univalent in* U*such that* $q(z)\neq 0,\ z\in U$, $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U *and* $\Re\{\frac{a}{b}q(z)+[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}]\}>0,\ b\neq 0,\ q'(z)\neq 0,\ \ z\in U$. *If* $F\in \mathbf{E}_\alpha^-$ *satisfies*

$$a[-\frac{z(D_\alpha^k F(z))'}{\Phi(D_\alpha^k F(z))}]+b[1+\frac{z(D_\alpha^k F(z))''}{(D_\alpha^k F(z))'}-\frac{z\Phi'(D_\alpha^k F(z))}{\Phi(D_\alpha^k F(z))}]\prec aq(z)+b\frac{zq'(z)}{q(z)}$$

then $-\frac{z(D_\alpha^k F(z))'}{D_\alpha^k F(z)}\prec q(z)$ *and* $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ be defined by $p(z):=-\frac{z(D_\alpha^k F(z))'}{\Phi(D_\alpha^k F(z))},\ \ z\in U$. By setting $\theta(\omega):=a\omega$ *and* $\phi(\omega):=\frac{b}{\omega},\ b\neq 0$, it can easily observed that $\theta(\omega)$ is analytic in $\mathbf{C}$, $\phi(\omega)$ is analytic in $\mathbf{C}\backslash\{0\}$ and that $\phi(\omega)\neq 0,\ \omega\in\mathbf{C}\backslash\{0\}$. Also we obtain $Q(z)=zq'(z)\phi(q(z))=b\frac{zq'(z)}{q(z)}$ *and* $h(z)=\theta(q(z))+Q(z)=aq(z)+b\frac{zq'(z)}{q(z)}$. It is clear that $Q(z)$ is starlike univalent in U and $\Re\{\frac{zh'(z)}{Q(z)}\}=\Re\{\frac{a}{b}q(z)+[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}]\}>0$. Straightforward computation, we have

$$ap(z)+b\frac{zp'(z)}{p(z)}=a[-\frac{z(D_\alpha^k F(z))'}{\Phi(D_\alpha^k F(z))}]+b[1+\frac{z(D_\alpha^k F(z))''}{(D_\alpha^k F(z))'}-\frac{z\Phi'(D_\alpha^k F(z))}{\Phi(D_\alpha^k F(z))}]$$
$$\prec aq(z)+b\frac{zq'(z)}{q(z)}.$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 2. □

For $\Phi(\omega)=\omega, k=0$ in Theorem 2, we have the following result.

Corollary 5. *If the subordination* $a[-\frac{zF'(z)}{F(z)}]+b[1+\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}]\prec aq(z)+b\frac{zq'(z)}{q(z)}$ *satisfies then* $-\frac{zF'(z)}{F(z)}\prec q(z)$.

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In the similar way, we have the following results:

Theorem 6. *Let the function* $q(z)$ *be convex univalent in* U *such that* $q'(z) \neq 0$ *and* $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0,\ \ \gamma \neq 0$. *Suppose that* $-\frac{\mathbf{D}_\alpha^k F(z)*\Phi(z)}{\mathcal{D}_\alpha^k F(z)*\Psi(z)}$ *is analytic in* U. *If* $F \in \mathbf{A}_\alpha^+$ *satisfies the subordination*

$$-\frac{\mathbf{D}_\alpha^k F(z)*\Phi(z)}{\mathbf{D}_\alpha^k F(z)*\Psi(z)}\{\psi + \gamma[\frac{z(\mathbf{D}_\alpha^k F(z)*\Phi(z))'}{D_\alpha^k F(z)*\Phi(z)} - \frac{z(\mathbf{D}_\alpha^k F(z)*\Psi(z))'}{\mathbf{D}_\alpha^k F(z)*\Psi(z)}]\} \prec \psi q(z) + \gamma z q'(z).$$

Then $-\frac{\mathbf{D}_\alpha^k F(z)*\Phi(z)}{\mathbf{D}_\alpha^k F(z)*\Psi(z)} \prec q(z),\ \ z \in U,$ *and* $q(z)$ *is the best dominant.*

Theorem 7. *Let the function* $q(z)$ *be univalent in* U*such that* $q(z) \neq 0,\ z \in U,$ $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U *and* $\Re\{\frac{a}{b}q(z) + [1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}]\} > 0,\ b \neq 0,\ q'(z) \neq 0,\ \ z \in U$. *If* $F \in \mathbf{A}_\alpha^-$ *satisfies*

$$a[-\frac{\mathbf{D}_\alpha^k F(z)*\Phi(z)}{\mathbf{D}_\alpha^k F(z)*\Psi(z)}] + b[\frac{z(\mathbf{D}_\alpha^k F(z)*\Phi(z))'}{\mathbf{D}_\alpha^k F(z)*\Phi(z)} - \frac{z(\mathbf{D}_\alpha^k F(z)*\Psi(z))'}{\mathbf{D}_\alpha^k F(z)*\Psi(z)}] \prec aq(z) + b\frac{zq'(z)}{q(z)},$$

then $-\frac{\mathbf{D}_\alpha^k F(z)*\Phi(z)}{\mathbf{D}_\alpha^k F(z)*\Psi(z)} \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Theorem 8. *Let the function* $q(z)$ *be convex univalent in* U *such that* $q'(z) \neq 0$ *and* $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0,\ \ \gamma \neq 0$. *Suppose that* $-\frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)}$ *is analytic in* U. *If* $F \in \mathbf{E}_\alpha^+$ *satisfies the subordination*

$$-\frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)}\Big\{\psi + \gamma\Big[1 + \frac{z[D_{\alpha,\lambda}^k F(z)]''}{[D_{\alpha,\lambda}^k F(z)]'} - \frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)}\Big]\Big\} \prec \psi q(z) + \gamma z q'(z).$$

Then $-\frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)} \prec q(z),\ \ z \in U,$ *and* $q(z)$ *is the best dominant.*

Theorem 9. *Let the function* $q(z)$ *be univalent in* U*such that* $q(z) \neq 0,\ z \in U,$ $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U *and* $\Re\{\frac{a}{b}q(z) + [1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}]\} > 0,\ b \neq 0,\ q'(z) \neq 0,\ \ z \in U$. *If* $F \in \mathbf{E}_\alpha^-$ *satisfies the subordination*

$$a[-\frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)}] + b[1 + \frac{z[D_{\alpha,\lambda}^k F(z)]''}{[D_{\alpha,\lambda}^k F(z)]'} - \frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)}] \prec aq(z) + b\frac{zq'(z)}{q(z)}$$

then $-\frac{z[D_{\alpha,\lambda}^k F(z)]'}{D_{\alpha,\lambda}^k F(z)} \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Theorem 10. *Let the function $q(z)$ be convex univalent in U such that $q'(z) \neq 0$ and $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$, $\gamma \neq 0$. Suppose that $-(\frac{F(z)}{z})^{\mu}$ is analytic in U. If $F \in \mathbf{E}_{\alpha}^{+}$ satisfies the subordination*

$$-(\frac{F(z)}{z})^{\mu}\{\psi + \gamma\mu[\frac{zF'(z)}{F(z)} - 1]\} \prec \psi q(z) + \gamma z q'(z).$$

Then $-(\frac{F(z)}{z})^{\mu} \prec q(z)$, $z \in U$, and $q(z)$ is the best dominant.

Theorem 11. *Let the function $q(z)$ be univalent in U such that $q(z) \neq 0$, $z \in U$, $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and $\Re\{\frac{a}{b}q(z) + [1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}]\} > 0$, $b \neq 0$, $q'(z) \neq 0$, $z \in U$. If $F \in \mathbf{E}_{\alpha}^{-}$ satisfies*

$$a[-(\frac{F(z)}{z})^{\mu}] + b\mu[\frac{zF'(z)}{F(z)} - 1] \prec aq(z) + b\frac{zq'(z)}{q(z)}$$

then $-(\frac{F(z)}{z})^{\mu} \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 12. *Let the function $q(z)$ be convex univalent in U such that $q'(z) \neq 0$ and $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$, $\gamma \neq 0$. Suppose that $-(\frac{1}{zF(z)})^{\mu}$ is analytic in U. If $F \in \mathbf{E}_{\alpha}^{+}$ satisfies the subordination*

$$-(\frac{1}{zF(z)})^{\mu}\{\psi + \mu\gamma[F^2(z) - \frac{zF'(z)}{F(z)}]\} \prec \psi q(z) + \gamma z q'(z),$$

where μ is a positive real number. Then $-(\frac{1}{zF(z)})^{\mu} \prec q(z)$, $z \in U$, and $q(z)$ is the best dominant.

Theorem 13. *Let $q(z) \neq 0$ be univalent in U with $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and $\Re\{aq(z) + [1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}]\} > 0$, $a \geq 0$, $q'(z) \neq 0$, $z \in U$. If $F \in \mathbf{E}_{\alpha}^{-}$ satisfies the subordination*

$$a[-(\frac{1}{zF(z)})^{\mu}] + \mu[F^2(z) - \frac{zF'(z)}{F(z)}] \prec aq(z) + \frac{zq'(z)}{q(z)}$$

then $-(\frac{1}{zF(z)})^{\mu} \prec q(z)$ and $q(z)$ is the best dominant.

Note that the authors introduced and studied different classes of analytic functions of fractional power in the unit disk (see [3-9]). Further, for different studies on fractional power and other operators, see for examples in ([10]-[12]).

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