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VARIATIONAL PRINCIPLES FOR THE CONFORMAL CAPACITY PROBLEM

Valentina Proycheva
Department of Mathematics
Technical University - Plovdiv
61, Sanct Peterburg Str., 4000, Plovdiv, BULGARIA
e-mail: vproicheva@hotmail.com

Abstract: This note is concerned with the conformal capacity problem. Some variational principles for the conformal capacity are established, extending the well know Thomson and Dirichlet principles, as well as the Poincaré isoperimetric inequality for the electrostatic capacity. Some applications are mentioned.

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1. Introduction

Let $\Omega_k,\ k=0,1$ be two bounded domains in $\mathrm{R}^N,\ N\geq 3$, with smooth boundaries $\partial\Omega_k$, such that $\bar{\Omega}_0\subset\Omega_1$.

Let $v\,(x)$be the dimensional electrostatic potential associated to the condenser $\Omega:=\Omega_1/\bar{\Omega}_0$ defined as

$$\begin{cases}\Delta v=0, & x\in\Omega,\\ v=0, & x\in\partial\Omega_0,\quad v=1,\quad x\in\partial\Omega_1.\end{cases}\tag{1}$$

The electrostatic capacity associated to Ω is defined as

$$C\,(\Omega):=\int\limits_{\Omega}|\nabla v|^2dx.\tag{2}$$

The Thomson and Dirichlet principles may be formulated as follows

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$$\frac{\left\{\int_{\Gamma} \vec{f}.\vec{n}\, ds\right\}^2}{\int_{\Omega} \left|\vec{f}\right|^2 dx} \leq C(\Omega) \leq \int_{\Omega} |\nabla\varphi|^2 dx. \tag{3}$$

In the first inequality, (the Thomson principle), the admissible vector field $\vec{f}(x)$ is required to be divergence free in Ω (i.e. $div\, \vec{f} = 0$ in Ω), Γ is an arbitrary smooth closed surface separating the two boundaries Ω_0 and Ω_1, and $\vec{n}$ is the outward normal unit vector defined on Γ.

In the second inequality, (the Dirichlet principle), $\varphi(x)$ is an arbitrary C^1 - function required to satisfy the same boundary conditions as $v(x)$, i.e. $\varphi = 0\,, \quad x \in \partial\Omega_{0,}\ \varphi = 1\,, \quad x \in \partial\Omega_1$.

We have equality in the Thomson principle if and only if $\vec{f} := \nabla v$, and equality in the Dirichlet principle if and only if $\varphi := v$.

We refer to Plya and Szegö [3] and to Weinberger [5] for the proofs of these principles as well as for many applications.

In this note we consider some extensions of these principles when the electrostatics problem (1) is replaced by the conformal capacity problem defined as

$$\begin{cases} \nabla\left(|\nabla u|^{N-2}\ \nabla u\right) = 0\,, & x \in \Omega, \\ u = 0\,, \quad x \in \partial\Omega_{0,}\ u = 1\,, & x \in \partial\Omega_1. \end{cases} \tag{4}$$

The conformal capacity of Ω is then defined as

$$C_N(\Omega) := \int_{\Omega} |\nabla u|^N dx = \int_{\Gamma} \left|\frac{\partial u}{\partial n}\right|^{N-1} ds. \tag{5}$$

Problem (4) plays a central role in the theory of quasi-conformal mapping. We refer to a paper of Anderson [1] about this topics.

In Section 2 we formulate and prove the extensions of the Thomson and Dirichlet principles to the conformal capacity problem.

In Sections 3 and 4 we indicate some possible applications of these principles, and in Section 5 we establish a generalization of the Poincaré isoperimetric inequality for the electrostatic capacity.

2. Dirichlet and Thomson Principles

The conformal capacity $C_N(\Omega)$ defined in (5) may be bounded from below and from above by the following inequalities

$$\frac{\left\{\int\limits_{\Gamma}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\overrightarrow{n}\,ds\right\}^{N}}{\left(\int\limits_{\Omega}\left|\overrightarrow{f}\right|^{N}dx\right)^{N-1}} \le C_N(\Omega) \le \int\limits_{\Omega}|\nabla\varphi|^{N}dx. \tag{6}$$

In the first inequality (6), (the Thomson principle), the admissible vector field $\vec{f}(x)$ is required to satisfy the equation

$$div\left(\overrightarrow{f}\left|\overrightarrow{f}\right|^{N-2}\right)=0\,,\quad x\in\Omega\,, \tag{7}$$

Γ is an arbitrary smooth closed surface separating the two boundaries $\partial\Omega_0$ and $\partial\Omega_1$, and $\overrightarrow{n}$ is the outward unit normal vector defined on Γ.

In the second inequality (6), (the Dirichlet principle), $\varphi(x)$ is an arbitrary C^1 - function required to satisfy the same boundary conditions as $u(x)$, i.e.

$$\varphi=0\,,\quad x\in\partial\Omega_{0,}\ \varphi=1\,,\quad x\in\partial\Omega_1. \tag{8}$$

We have equality in the Thomson principle if and only if $\vec{f}:=\nabla u$, and equality in the Dirichlet principle if and only if $\varphi:=u$, where u is the solution of (4).

For the proof of the Thomson principle, we make use of condition (7) to write

$$\int\limits_{\partial\Omega_1}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\overrightarrow{n}\,ds=\int\limits_{\Gamma}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\overrightarrow{n}\,ds. \tag{9}$$

Moreover, integrating the equation

$$div\left(\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}\,u\right)=\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\nabla u \tag{10}$$

over Ω, where u is the solution of the conformal capacity problem (4) , we obtain

$$\int\limits_{\partial\Omega_1}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\overrightarrow{n}\,ds=\int\limits_{\Omega}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\nabla u\,dx. \tag{11}$$

Combining (10) and (11), and making use of the Schwarz inequality, we obtain

$$\left(\int\limits_{\Gamma}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\overrightarrow{n}\,ds\right)^{2}=\left(\int\limits_{\Omega}\left|\overrightarrow{f}\right|^{N-2}\overrightarrow{f}.\nabla u\,dx\right)^{2} \tag{12}$$

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$$\leq \int_{\Omega} \left|\overrightarrow{f}\right|^{N-2} |\nabla u|^2 \, dx \int_{\Omega} \left|\overrightarrow{f}\right|^{N} dx.$$

Finally it follows from Hölder's inequality that

$$\int_{\Omega} \left|\overrightarrow{f}\right|^{N-2} |\nabla u|^2 \, dx \leq \left(\int_{\Omega} \left|\overrightarrow{f}\right|^{N} dx\right)^{\frac{N-2}{2}} \left(\int_{\Omega} |\nabla u|^N dx\right)^{\frac{2}{N}}. \tag{13}$$

Combining (12) and (13) , we obtain the first inequality in (6).

For the proof of the Dirichlet principle, we note that

$$\left\{\int_{\Omega} |\nabla u|^{N-2} \nabla u \,.\, \nabla \varphi \, dx\right\}^2 \leq \int_{\Omega} |\nabla u|^N dx \int_{\Omega} |\nabla u|^{N-2} |\nabla \varphi|^2 \, dx \tag{14}$$

by the Schwarz inequality, and that

$$\int_{\Omega} |\nabla u|^{N-2} |\nabla \varphi|^2 \, dx \leq \left(\int_{\Omega} |\nabla u|^N dx\right)^{\frac{N-2}{2}} \left(\int_{\Omega} |\nabla \varphi|^N dx\right)^{\frac{2}{N}} \tag{15}$$

by Hölder's inequality.

Combining (14) and (15), and making use of the condition (5), we obtain

$$\left\{\int_{\Omega} |\nabla u|^{N-2} \nabla u \,.\, \nabla \varphi \, dx\right\}^2 \leq [C_N(\Omega)]^{\frac{2(N-1)}{N}} \left(\int_{\Omega} |\nabla \varphi|^N dx\right)^{\frac{2}{N}}. \tag{16}$$

Moreover, it follows from (4) that

$$\nabla \left(\varphi \, |\nabla u|^{N-2} \nabla u\right) = |\nabla u|^{N-2} \nabla u \,.\, \nabla \varphi, \quad x \in \Omega\,. \tag{17}$$

Integrating (17) over Ω and making use of the divergence theorem , we obtain since $\varphi = 0$ on $\partial\Omega_0$ and $\varphi = 1$ on $\partial\Omega_1$

$$\int_{\Omega} |\nabla u|^{N-2} \nabla u \,.\, \nabla \varphi \, dx = \int_{\partial\Omega_1} |\nabla u|^{N-1} ds = C_N(\Omega)\,. \tag{18}$$

Combining (16) and (18) leads to the second inequality (6).

3. Applications of the Dirichlet Principle

1. Let us consider $\omega := \omega_1/\overline{\omega}_0$ and $\Omega := \Omega_1/\bar{\Omega}_0$, with $\Omega_0 \subseteq \omega_0 \subset \omega_1 \subseteq \Omega_1$, so that we have $\omega \subseteq \Omega$.

The conformal capacities associated to ω and to Ω then satisfy the following inequality

$$C_N(\Omega) \le C_N(\omega), \tag{19}$$

with equality if and only if $\Omega = \omega$.

Note that the same inequality holds for the electrostatic capacity.

For the proof of (19), we define in Ω an auxiliary function by

$$\varphi(x) := \begin{cases} u(x), & x \in \omega, \\ 1, & x \in \Omega_1/\omega_1, \\ 0, & x \in \omega_0/\Omega_0, \end{cases} \tag{20}$$

where $u(x)$ is the solution of the conformal capacity problem in ω,

$\varphi(x)$ is admissible for the Dirichlet problem in Ω.

This leads to the desired result

$$C_N(\Omega) \le \int_\Omega |\nabla\varphi|^N dx = \int_\omega |\nabla u|^N dx = C_N(\omega). \tag{21}$$

2. The following inequalities hold between electrostatic and conformal capacities:

$$C(\Omega) \le [C_N(\Omega)]^{\frac{2}{N}} \ |\Omega|^{\frac{N-2}{N}}, \tag{22}$$

$$mC(\Omega) \le C_N(\Omega) \le MC(\Omega). \tag{23}$$

In (22), $|\Omega|$ is the N- volume of Ω, and in (23), m and M are constants defined as

$$m := \left(\min_\Omega |\nabla v|\right)^{N-2}, \qquad M := \left(\max_\Omega |\nabla v|\right)^{N-2}. \tag{24}$$

For the proof of (22), we use the fact that the solution $u(x)$ of (4) is admissible in the Dirichlet principle for $C(\Omega)$.

It then follows from the second inequality (3) that $C(\Omega) \le \int_\Omega |\nabla u|^2 dx$.

This inequality may be continued by means of Hölder's inequality as follows

$$\begin{aligned} C(\Omega) &\le \int_\Omega |\nabla u|^2 dx \le \left(\int_\Omega |\nabla u|^N dx\right)^{\frac{2}{N}} \left(\int_\Omega dx\right)^{\frac{N-2}{N}} \\ &= [C_N(\Omega)]^{\frac{2}{N}} \ |\Omega|^{\frac{N-2}{N}}. \end{aligned} \tag{25}$$

For the proof of the second inequality in (23),we observe that the electrostatic potential $v(x)$ associated to Ω is admissible in the Dirichlet principle associated to the conformal capacity problem, so that we have

$$C_N(\Omega) \leq \int_\Omega |\nabla v|^N dx = \int_\Omega |\nabla v|^{N-2} \ |\nabla v|^2 \, dx \leq M \, C(\Omega) . \tag{26}$$

For the proof of the first inequality in (23) , we make use of the divergence theorem to write

$$C(\Omega) := \int_\Omega |\nabla v|^2 dx = \int_{\partial\Omega_1} \frac{\partial v}{\partial n} \, ds = \int_\Omega \nabla u \, . \, \nabla v \, dx. \tag{27}$$

Moreover, it follows from Hölder's inequality that

$$\int_\Omega \nabla u \, . \, \nabla v \, dx \leq \left(\int_\Omega |\nabla u|^N \, dx \right)^{\frac{1}{N}} \left(\int_\Omega |\nabla v|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} . \tag{28}$$

The last integral in (28) may be bounded as follows

$$\int_\Omega |\nabla v|^{\frac{N}{N-1}} dx = \int_\Omega |\nabla v|^2 \ |\nabla v|^{\frac{2-N}{N-1}} \ dx \leq \left(\min_\Omega |\nabla v| \right)^{\frac{2-N}{N-1}} C(\Omega) . \tag{29}$$

Combining (27), (28), and (29), we obtain the first inequality in (23).

4. Applications of the Thomson Principle

Taking the origin O in Ω_0, and noting that the vector field defined in Ω as

$$\vec{f} := \frac{\overrightarrow{r}}{r^2} \tag{30}$$

satisfies the admissibility condition (7) with

$$\overrightarrow{r} := (x_1, x_2, ..., x_N) , \quad r^2 := \sum_{k=1}^N x_k^2,$$

the first inequality in (6) reduces to

$$C_N(\Omega) \geq \frac{\left(\int_\Gamma r^{-N} \, \overrightarrow{r} . \overrightarrow{n} \, ds \right)^N}{\left(\int_\Omega r^{-N} dx \right)^{N-1}} . \tag{31}$$

Since r^{2-N} is harmonic in $\mathrm{R}^N / \{0\}$ for $N \geq 3$ (or $\ln r$ is harmonic in $\mathrm{R}^2 / \{0\}$), we have

$$\int\limits_{\Gamma} r^{-N}\, \overrightarrow{r}.\overrightarrow{n}\, ds = \int\limits_{r=1} r^{-N}\, \overrightarrow{r}.\overrightarrow{n}\, ds = \int\limits_{r=1} ds = \omega_N, \tag{32}$$

where ω_N is the surface area of the unit ball in R^N.

Combining (31) and (32), we obtain the lower bound

$$C_N(\Omega) \quad \geq \quad \frac{(\omega_N)^N}{\left(\int\limits_{\Omega} r^{-N} dx\right)^{N-1}} \quad . \tag{33}$$

This lower bound is exact if and only if Ω is radially symmetric.

5. An Extension of Poincaré's Isoperimetric Inequality

In this section we want to establish the isoperimetric inequality for $C_N(\Omega)$ formulated in the next theorem.

Theorem 1. *Let Ω_k^* be N - balls centered at the origin with the same volumes as Ω_k, $k = 0, 1$, and consider the radially symmetric domain $\Omega^* := \Omega_1^* / \overline{\Omega_0^*}$.*

Then we have the following isoperimetric inequality

$$C_N(\Omega) \quad \geq \quad \frac{(N)^{N-1}\, \omega_N}{\left(\ln \frac{|\Omega_1|}{|\Omega_0|}\right)^{N-1}} \; = \; C_N(\Omega^*), \tag{34}$$

where ω_N is the area of a unit ball in R^N.

Inequality (34) is exact if and only if $\Omega = \Omega^*$.

For the proof of Theorem 1, we consider the domain Ω_α defined as

$$\Omega_\alpha := \Omega_0 \cup \{\, x \in \Omega;\ \ 0 \leq u(x) < \alpha \,\}, \ \ 0 \leq \alpha \leq 1. \tag{35}$$

Making use of the co-area formula [2] , we have

$$|\Omega_\alpha| = |\Omega_0| \; + \int\limits_0^{\alpha} \int\limits_{\Gamma\eta} \frac{ds}{|\nabla u|}\, d\eta, \tag{36}$$

with $\Gamma_\eta := \{\, x \in \Omega\, ;\ \ u(x) = \eta \,\}$.

Differentiating and making use of Hölder's inequality, we obtain

$$\frac{d}{d\alpha}\left|\Omega_\alpha\right| = \int\limits_{\Gamma_\alpha} \frac{ds}{|\nabla u|} \geq \frac{\left(\int\limits_{\Gamma_\alpha} ds\right)^{\frac{N}{N-1}}}{\left(\int\limits_{\Gamma_\alpha} |\nabla u|^{N-1}\, ds\right)^{\frac{1}{N}}} = \frac{|\partial\Omega_\alpha|^{\frac{N}{N-1}}}{\left[C_N\left(\Omega\right)\right]^{\frac{1}{N}}}, \tag{37}$$

where $|\partial\Omega_\alpha|$ is the area of $\partial\Omega_\alpha$.

The above inequality may be continued by making use of the classical geometric isoperimetric inequality

$$\left(\int\limits_{\Gamma_\alpha} ds\right)^{\frac{N}{N-1}} = |\partial\Omega_\alpha|^{\frac{N}{N-1}} \geq N\left(\omega_N\right)^{\frac{1}{N-1}} \left|\Omega_\alpha\right|. \tag{38}$$

We are then led to the first order differential inequality

$$\frac{d\left|\Omega_\alpha\right|}{d\alpha} \geq k\left|\Omega_\alpha\right|, \tag{39}$$

with

$$k := \frac{N\left(\omega_N\right)^{\frac{1}{N-1}}}{\left[C_N\left(\Omega\right)\right]^{\frac{1}{N-1}}} . \tag{40}$$

Rewriting (39) as $\frac{d}{d\alpha}\left(|\Omega_\alpha|\, e^{-k\alpha}\right) \geq 0$ and integrating from

$\alpha = 0$ to $\alpha = 1$, we obtain the desired inequality (34).

Finally, we observe that the solution of problem (4) in Ω^* is

$$u\left(r\right) = \frac{\ln \frac{r}{R_0}}{\ln \frac{R_1}{R_0}}, \tag{41}$$

where R_k are the radii of Ω_k^*, $k = 0, 1$.

This leads to

$$C_N\left(\Omega^*\right) = \frac{\left(N\right)^{N-1}\omega_N}{\left(\ln \frac{|\Omega_1|}{|\Omega_0|}\right)^{N-1}}. \tag{42}$$

This achieves the proof of Theorem 1.

Further isoperimetric inequalities for $C_N\left(\Omega\right)$ based on the maximum principle may be found in [4].

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