International Electronic Journal of Pure and Applied Mathematics
Volume 3 No. 2 2011, 111-117

GENERALIZATIONS OF SOME POLYNOMIAL INEQUALITIES

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Abstract: In this paper, we consider lacunary polynomials of degree n having some of the zeros at origin and rest of the zeros lying on or outside the boundary of a prescribed disk our results generalize some earlier well known results.

AMS Subject Classification: 30A10, 30C10, 30C15
Key Words: polynomials, zeros, boundary and exterior of circle, lacunary

1. Introduction and Statements of Results

Let $p(z)$ be a polynomial of degree n. Then we have the following well-known Bernstein's inequality [2].

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

Equality holds in (1) if and only if $p(z)$ has all of its zeros at the origin.

It we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, the inequality (1) can be sharpened. In fact it was conjectured by Erdös and later proved by Lax [8] that if $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

Inequality (2) is best possible and equality holds for $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

Aziz and Dawood [1] improved the inequality (2) under the same hypothesis by

Received: September 13, 2010
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proving

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\Big[\max_{|z|=1}|p(z)| - \min_{|z|=1}|p(z)|\Big]. \tag{3}$$

Equality in (3) holds for $p(z) = \beta + \alpha z^n$, $|\beta| \geq |\alpha|$.

For the class of polynomials $p(z)$ for degree n not vanishing in $|z| < k$, $k \geq 1$, Malik [9] proved

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{4}$$

Inequality (4) was further improved by Govil [7] under the same hypothesis by proving

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\Big\{\max_{|z|=1}|p(z)| - \min_{|z|=1}|p(z)|\Big\}. \tag{5}$$

The inequalities (4) and (5) are sharp and extremal polynomial is $p(z) = (z+k)^n$.

Chan and Malik [3] considered the lacunary type of polynomials and obtained the following generalization of (4).

Theorem A. *If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq= \frac{n}{1+k^\mu}\max_{|z|=1}|p(z)|. \tag{6}$$

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of μ.

The following theorem was proved by Pukhta [5], which is an improvement of Theorem A and a generalization of the inequality (5).

Theorem B. *If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^\mu}\Big\{\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\Big\}. \tag{7}$$

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^{n/\mu}$, where n is a multiple of u.

For the polynomials having all its zeros on $|z| = k$, $k \leq 1$, Govil [6] proved

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{k^n + k^{n-1}}\max_{|z|=1}|p(z)|. \tag{8}$$

Recently; Dewan and Hans [4] generalized the inequality (8) for the polynomials of the type $p(z) = c_n z^n + \sum_{v=\mu}^{n} c_{n-v} z^{n-v}$, $1 \leq \mu < n$ and proved the following

Theorem C. *If* $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \leq \mu < n$ *is a polynomial of degree* n*, having all its zeros on* $|z| = k$, $k \leq 1$*, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)|. \qquad (9)$$

In this paper, first we extended Theorem A and B to the class of polynomials of degree n of the type $p(z) = z^s(a_0 + \sum_{v=\mu}^{n-s} a_v z^v), 1 \leq \mu \leq n-s$, where $0 \leq s \leq n-1$.

Theorem 1. *Let* $p(z) = z^s \left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$ *is a polynomial of degree* n*, having* s*-fold zeros at origin and the remaining* $n-s$ *zeros in* $|z| \geq k$, $k \geq 1$ *then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n + sk^\mu}{1 + k^\mu} \max_{|z|=1} |p(z)|.$$

The following corollary follows by taking $k = 1$ in Theorem 1.

Corollary 1. *If* $p(z) = z^s \left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, *is a polynomial of degree* n *having* s *zeros at origin and the remaining* $n-s$ *zeros in* $|z| \geq 1$*, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n+s}{2} \max_{|z|=1} |p(z)|.$$

Remark 1. If we take $s = 0$ in Theorem 1 it reduces to Theorem A.

Remark 2. For $\mu = 1$, Theorem 1, reduces to a result of Mir [10, Theorem 1.6].

Instead of proving Theorem 1, we prove the following more general result.

Theorem 2. *If* $p(z) = z^s \left(a_0 + \sum_{v=\mu}^{n-s} a_v z^v \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, *is a polynomial of degree* n *having* s*-fold zeros at origin and the remaining then* $n-s$ *zeros in* $|z| \geq k$, $k \geq 1$

$$\max_{|z|=1} |p'(z)| \leq \frac{n + sk^\mu}{1 + k^\mu} \max_{|z|=1} |p(z)| - \frac{(n-s)}{k^s(1 + k^\mu)} \min_{|z|=k} |p(z)|$$

If we take $\mu = 1$ in Theorem 2, we get the following result which is an improvement of a result of Mir [10, Theorem 1.6].

Corollary 2. *If* $p(z)=z^s\left(a_0+\sum_{v=1}^{n-s}a_vz^v\right)$ *is a polynomial of degree* n *having* s *zeros at origin and remaining* $n-s$ *zeros in* $|z|\geq k$, $k\geq 1$, *then*

$$\max_{|z|=1}|p'(z)|\leq\frac{n+sk}{1+k}\max_{|z|=1}|p(z)|-\frac{(n-s)}{1+k}\times\frac{1}{k^s}\min_{|z|=k}|p(z)|.$$

When we take $k=1$ in Theorem 2, we get the following refinement of Corollary 1.

Corollary 3. *If* $p(z)=z^s\left(a_0+\sum_{v=\mu}^{n-s}a_vz^v\right)$, $1\leq\mu\leq n-s$, $0\leq s\leq n-1$, *is a polynomial of degree* n *having* s *zeros at origin and the remaining* $n-s$ *zeros in* $|z|\geq 1$, *then*

$$\max_{|z|=1}|p'(z)|\leq\frac{(n+s)}{2}\max_{|z|=1}|p(z)|-\frac{(n-s)}{2}\min_{|z|=1}|p(z)|$$

Remark 3. If we put $s=0$ in Theorem 2, it reduces to Theorem B.

Now we obtain the extension of Theorem C to the polynomials of the type $p(z)=z^s\left(a_{n-s}z^{n-s}+\sum_{v=\mu}^{n-s}a_{n-s-v}z^{n-s-v}\right)$, $1\leq\mu<n-s$, $0\leq s\leq n-1$.

Theorem 3. *Let* $p(z)=z^s\left(a_{n-s}z^{n-s}+\sum_{v=\mu}^{n-s}a_{n-s-v}z^{n-s-v}\right)$, $1\leq\mu<n-s$, $0\leq s\leq n-1$, *is a polynomial of degree* n, *having* s*-fold zeros at origin and remaining* $n-s$ *zeros an* $|z|=k$, $k\leq 1$, *then*

$$\max_{|z|=1}|p'(z)|\leq\frac{n+s(k^{n-s-2\mu+1}+k^{n-s-\mu+1}-1)}{k^{n-s-2\mu+1}+k^{n-s-\mu+1}}\max_{|z|=1}|p(z)|$$

when we take $\mu=1$ *in Theorem* 3, *we get the following*

Corollary 4. *If* $p(z)=z^s\left(a_{n-s}z^{n-s}+\sum_{v=1}^{n-s}a_{n-s-v}z^{n-s-v}\right)$, $0\leq s\leq n-1$ *is a polynomial of degree* n *having* s*-fold zeros at origin and the remaining* $n-s$ *zeros on* $|z|=k$, $k\leq 1$, *then*

$$\max_{|z|=1}|p'(z)|\leq\frac{n+s(k^{n-s-1}+k^{n-s}-1)}{k^{n-s-1}+k^{n-s}}\max_{|z|=1}|p(z)|$$

Remark 4. If we take $s=0$, in Theorem 3, it reduces to Theorem C.

Remark 5. If in corollary 4, we put $s=0$ then we get a result of Govil [6].

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 2 (2011)

2. Proofs

Proof of Theorem 2. Let

$$p(z) = z^s \phi(z) \tag{10}$$

where

$$\phi(z) = a_0 + \sum_{v=\mu}^{n-s} a_v z^v$$

is a polynomial of degree $n-s$ having no zero in $|z| < k$, $k \geq 1$.

Now, from (10) we have

$$\begin{aligned} zp'(z) &= sz^s\phi(z) + z^{s+1}\phi'(z) \\ &= sp(z) + z^{s+1}\phi'(z) \end{aligned}$$

For $|z| = 1$, i.e. $0 \leq \theta < 2\pi$, we have

$$|p'(z)| \leq s|p(z)| + |\phi'(z)|$$

The above inequality holds for all points on $|z| = 1$ i.e. $0 \leq \theta < 2\pi$, therefore

$$\max_{|z|=1} |p'(z)| \leq s \max_{|z|=1} |p(z)| + \max_{|z|=1} |\phi'(z)|. \tag{11}$$

Now let $m = \min_{|z|=k} |\phi(z)|$, then $m \leq |\phi(z)|$ for $|z| = k$.

If $\phi(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Theorem 1. Hence forth we suppose that all $n-s$ zeros of $\phi(z)$ lie in $|z| > k$, $k \geq 1$. Therefore for every complex number α such that $|\alpha| < 1$, it follows by Rouche's Theorem that all the zeros of the polynomial $\phi(z) - \alpha m$ of degree $n-s$ lie in $|z| > k$, $k \geq 1$.

Applying the inequality (6) to the polynomial $\phi(z) - \alpha m$ of degree $n-s$, we get

$$\max_{|z|=1} |\phi'(z)| \leq \frac{n-s}{1+k^\mu} \max_{|z|=1} |\phi(z) - \alpha m| \tag{12}$$

Now choosing the argument of α such that

$$|\phi(z) - \alpha m| = |\phi(z)| - |\alpha| m \quad \text{for } |z| = 1, \tag{13}$$

and letting $|\alpha| \to 1$, we get from (12) in view of (13)

$$\max_{|z|=1} |\phi'(z)| \leq \frac{n-s}{1+k^\mu} \max_{|z|=1} [|\phi(z)| - m] \tag{14}$$

combining the inequalities (11) and (14), we obtain

$$\max_{|z|=1}|p'(z)| \leq \frac{n-s}{1+k^{\mu}}\max_{|z|=1}|\phi(z)| - \frac{n-s}{1+k^{\mu}}m + s\max_{|z|=1}|p(z)|$$

Using the fact from (10) that on $|z|=1, |p(z)|=|\phi(z)|$, we get

$$\max_{|z|=1}|p'(z)| \leq \left(\frac{n-s}{1+k^{\mu}}+s\right)\max_{|z|=1}|p(z)| - \frac{(n-s)^m}{1+k\mu} \tag{15}$$

From (10) one can easily obtain that on $|z|=k$,

$$m = \min|\phi(z)| = \frac{1}{k^s}\min|p(z)|, \tag{16}$$

combining (15) and (16), we get

$$\max_{|z|=1}|p'(z)| \leq \left(\frac{n+sk^{\mu}}{1+k^{\mu}}\right)\max_{|z|=1}|p(z)| - \left(\frac{n-s}{1+k^{\mu}}\right)\frac{1}{k^s}\min_{|z|=k}|p(z)| \tag{17}$$

The proof of the Theorem 2 is completed. □

Proof of Theorem 3. Let

$$p(z) = z^s\phi(z) \tag{18}$$

where $\phi(z) = a_{n-s}z^{n-s} + \sum_{v=\mu}^{n-s} a_{n-s-v}z^{n-s-v}$ is a polynomial of degree $n-s$, having all its zeros on $|z|=k, k\leq 1$.

Now using the inequality (9) for $\phi(z)$, we get

$$\max_{|z|=1}|\phi'(z)| \leq \frac{n-s}{k^{n-s-2\mu+1}+k^{n-s-\mu+1}}\max_{|z|=1}|\phi(z)| \tag{19}$$

Now for $p(z)=z^s\phi(z)$, as in proof of previous theorem, we can prove that

$$\max_{|z|=1}|p'(z)| \leq s\max_{|z|=1}|p(z)| + \max_{|z|=1}|\phi'(z)| \tag{20}$$

combining the inequalities (18) and (19), we get

$$\max_{|z|=1}|p'(z)| \leq \frac{n-s}{k^{n-s-2\mu+1}+k^{n-s-\mu+1}}\max_{|z|=1}|\phi(z)| + s\max_{|z|=1}|p(z)|$$

Using the fact from (17) that on $|z|=1$, $|p(z)|=|\phi(z)|$, we get

$$\max_{|z|=1}|p'(z)| \leq \frac{n-s}{k^{n-s-2\mu+1}+k^{n-s-\mu+1}}\max_{|z|=1}|p(z)| + s\max_{|z|=1}|p(z)|$$

or

$$\max_{|z|=1}|p'(z)| \leq \frac{n+s(k^{n-s-2\mu+1}+k^{n-s-\mu+1}-1)}{(k^{n-s-2\mu+1}+k^{n-s-\mu+1})}\max_{|z|=1}|p(z)|$$

This completes the proof of the Theorem 3. □

Acknowledgments

The research of the author is supported by UGC, New Delhi; F.No.17-52/98(SA-I).

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