International Electronic Journal of Pure and Applied Mathematics
Volume 3 No. 1 2011, 83-90

ON f-DERIVATIONS OF INCLINE ALGEBRAS

Şule Ayar Özbal[1], Alev Firat[2 §]
[1]Department of Mathematics
Faculty of Science
Yaşar University
Izmir, 35100, TURKEY
e-mail: sule.ayar@yasar.edu.tr
[2]Department of Mathematics
Faculty of Science
Ege University
Izmir, 35100, TURKEY
e-mail: alev.firat@ege.edu.tr

Abstract: In this paper some properties of derivation of an incline algebra are extended to an ideal of an incline algebra. Additionally, as a generalization of derivation of an incline algebra, the notion of f-derivation in an incline algebra is introduced and some of its properties are investigated in an incline and an integral incline algebra.

AMS Subject Classification: 16Y60
Key Words: (sub)incline, integral incline, ideal, derivation, f-derivation

1. Introduction

The notion of derivation on a ring has an important role for the characterization of rings. As generalizations of derivations, α-derivations on prime and semi-prime rings are studied by a lot of researchers, see [7], [5]. In [8] the notion of derivation on a lattice was defined and some of its related properties were examined firstly by G. Szasz. In [4] the problems initiated by Szasz are pursued and completed by Luca Ferrari. Later, derivations, f-derivations on a lattice and some properties related with these derivations were discussed by X.L. Xin, T.Y. Li and J.H. Lu [9], and M.A. Öztürk, Y. Çeven in [3].

Received: January 24, 2011
§Correspondence author

In [2] the notion of inclines is introduced and their applications are studied by Z.Q. Cao, K.H. Kim, and F.W. Roush. The notion of derivation for an incline algebra is introduced by N.O. Al-Shehri in [1] and he discussed some of its properties. In this paper, as a generalization of derivation of an incline algebra, the notion of f-derivation in an incline algebra is introduced and some properties of derivation and f-derivation of an incline algebra are investigated.

2. Preliminaries

Definition 2.1. (see [2]) An incline algebra is a non-empty set R with binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in R$:
(RI) $x + y = y + x$,
(RII) $x + (y + z) = (x + y) + z$,
(RIII) $x * (y * z) = (x * y) * z$,
(RIV) $x * (y + z) = (x * y) + (x * z)$,
(RV) $(y + z) * x = (y * x) + (z * x)$,
(RVI) $x + x = x$,
(RVII) $x + (x * y) = x$,
(RVIII) $y + (x * y) = y$.

Furthermore, an incline algebra R is said to be commutative if $x * y = y * x$ for all $x, y \in R$.

For convenience, we pronounce "+" (resp."$*$") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in R$ (see [3, Proposition 1.1.1]). A subincline of an incline R is a nonempty subset M of R which is closed under addition and multiplication. An ideal in an incline R is a subincline $M \subseteq R$ such that if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra R is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$ for any $x \in R$. An element 1 ($\neq$ zero element) in an incline algebra R is called a multiplicative identity if for any $x \in R$, $x * 1 = 1 * x = x$. A non-zero element a in an incline algebra R with zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a * b = 0$ (resp. $b * a = 0$). A zero divisor is an element of R which is both a left zero divisor and a right zero divisor. An incline algebra R with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors.

Note that if $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on R and that for any $x, y \in R$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$. It follows that:
(1) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in R$.

(2) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in R$.
(3) If $x \leq y$, $a \leq b$, then $x + a \leq y + b$, $x * a \leq y * b$.

Definition 2.2. (see [1]) Let R be an incline and $d : R \to R$ be a function. d is called as a derivation of R if it satisfies the following condition $d(x * y) = (d(x) * y) + (x * d(y))$ for all $x, y \in R$.

Definition 2.3. (see [1]) Let d be a derivation of an incline R. If $x \leq y$ implies that $d(x) \leq d(y)$ for all $x, y \in R$, then d is called an isotone derivation.

3. Some Results on Derivations of Incline Algebras

Theorem 3.1. *If d is a nonzero derivation and $M \neq 0$ is an ideal of an integral incline R, then $d^2(M) \neq 0$.*

Proof. Assume that $d^2(M) = 0$. Then for $m, n \in R$ we have

$$\begin{aligned} 0 &= d^2(m * n) = d((d(m) * n) + (m * d(n))) \\ &= (d^2(m) * n) + (d(m) * d(n)) + (m * d^2(n)) + (m * d^2(n)) = d(m) * d(n). \end{aligned}$$

From here we get $d(m)*d(n) = 0$. Since R is an integral incline we have $d(m) = 0$ or $d(n) = 0$. Hence $d(M) = 0$.

By Theorem 3.1.13 stated in [1] we get d is a zero derivation on R. This contradicts with or hypothesis that $d \neq 0$. Hence d^2 must be different than zero on M. □

Theorem 3.2. *Let d be a nonzero derivation of an integral incline R. If M is a nonzero ideal of R, and $a \in R$ such that $a * d(M) = 0$, then $a = 0$.*

Proof. By Theorem 3.1.13 stated in [1] we know that there is an element m in M such that $d(m) \neq 0$. Assume that M is a nonzero ideal of R, and $a \in R$ such that $a * d(M) = 0$. Then for $n, m \in M$ we can write

$$0 = a*d(m*n) = a*(d(m)*n+m*d(n)) = a*d(m)*n+a*m*d(n) = a*m*d(n).$$

Since R is an integral incline, d is a nonzero derivation of R and M is a nonzero ideal of R we have $a = 0$. □

4. The f-Derivations of Incline Algebras

The following definition introduces the notion of f-derivation for an incline algebra.

Definition 4.1. Let R be an incline algebra. A function d: $R \to R$ is called an f-derivation of R, if there exists a function $f : R \to R$ such that

$$d(x * y) = (d(x) * f(y)) + (f(x) * d(y)) \quad \text{for all} \quad x, y \in R.$$

Example 4.1. Let $R = \{0, a, b, c, d, 1\}$, and the sum " + " and product " $*$ " be defined on R:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	1	1
b	b	1	b	1	b	1
c	c	1	1	c	1	1
d	d	1	b	c	d	1
1	1	1	1	1	1	1

*	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	d	d	d	d
c	0	0	d	d	d	d
d	0	0	d	d	d	d
1	0	0	d	d	d	d

Then $(R, +, *)$ is an incline but not a distributive lattice, see [6].
We define a function $d : R \to R$ for all x in R by

$$d(x) = \begin{cases} 0, & x = 0, a \\ d, & \text{otherwise.} \end{cases}$$

Then d is a derivation of R. If we define a function f of R by $f(x) = 0$ for all $x \in R$. Then d is not an f-derivation of X since $d(b*1) = d(d) = d$, but $(d(b)*f(1))+(f(b)*d(1)) = (d*0)+(0*d) = 0+0 = 0$ and thus $d(b*1) \neq (d(b)*f(1))+(f(b)*d(1))$. But if we define f as an identity function then d satisfies the equation in Definition 4.1, i.e. d is an f-derivation of R.

Remark 4.1. Every derivation of R could be made an f-derivation of R by an identity function of R.

Example 4.2. Let R be the incline algebra in Example 4.1.
If we define a function

$$d(x) = \begin{cases} d, & x = 0, b, d, \\ b, & \text{x=a, c, 1,} \end{cases}$$

then d is not a derivation of R, since $d(0*0) = d(0) = d$, but $d(0 * 0) = (d((0)* 0))+(0*d(0)) = (d*0)+(0*d) = 0+0 = 0$, and thus $d(0*0) \neq d((0)*0)+(0*d(0))$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

Now we define a function f of R for all $x \in R$ by

$$f(x) = \begin{cases} d, & x = 0, a, c, d, 1, \\ b, & \text{otherwise,} \end{cases}$$

then it is easily checked that d is an f-derivation of R.

Example 4.3. Let R be the incline algebra in Example 4.1.
If we define a function

$$d(x) = \begin{cases} 0, & x = 0, d, \\ b, & x = a, \\ d, & \text{otherwise.} \end{cases}$$

Then d is not a derivation of R, since $d(a*d) = d(0) = 0$, but $d(a * d) = (d((a)* d)) + (a * d(d)) = (b * d) + (a * d) = d + 0 = d$, and thus $d(a * d) \neq d((a) * d) + (a * d(d))$.
Now we define a function f of R for all $x \in R$ by

$$f(x) = \begin{cases} 0, & x = 0, \\ a, & \text{otherwise.} \end{cases}$$

Then it is easily checked that d is an f-derivation of R.

Proposition 4.2. *Let R be an incline algebra and d be an f-derivation of R. Then the followings hold for all x, y in R:*
(i) $d(x * y) \leq d(x) + d(y)$.
(ii) If $x \leq y$ and f an order preserving mapping then $d(x * y) \leq f(y)$.
(iii) If R is a distributive lattice then $d(x) \leq f(x)$.

Proof. (i) Let $x, y \in R$. We know that from (1) we have $d(x) * f(y) \leq d(x)$ and $f(x)*d(y) \leq d(y)$. Then by using (3) we get $d(x*y) = (d(x)*f(y))+(f(x)*d(y)) \leq d(x) + d(y)$. Hence we find $d(x * y) \leq d(x) + d(y)$.

(ii) Let $x \leq y$ and f be an order preserving function. Then by using (3) and (1) we get $f(x) * d(y) \leq f(y) * d(y) \leq f(y)$. Similarly, we can get $d(x) * f(y) \leq f(y)$. Then we obtain, $d(x * y) = (d(x) * f(y)) + (f(x) * d(y)) \leq f(y)$. Hence we have $d(x * y) \leq f(y)$.

(iii) Let R be a distributive lattice, then we have $d(x*x) = (d(x)*f(x))+(f(x)*d(x))$ and so we get $d(x) + f(x) = (d(x) * f(x)) + ((f(x) * d(x)) + f(x))$. By using $(RVII)$ we can write $d(x) + f(x) = (d(x) * f(x)) + f(x)$ and also by using $(RVIII)$ we can write $d(x) + f(x) = f(x)$, hence we have $d(x) \leq f(x)$. □

Proposition 4.3. *Let R be an incline algebra with a zero element and d be an f-derivation of R. Then $d(0) = 0$ with $f(0) = 0$.*

Proof. Since R is an incline algebra with a zero element we have $x*0 = 0*x = 0$ for all $x \in R$ then we can write $d(0) = d(x * 0) = (d(x) * f(0)) + (f(x) * d(0)) = f(x) * d(0)$.

If we take $x = 0$ in this equation then we get $d(0) = 0$.

Proposition 4.4. *Let R be an incline algebra with a multiplicative identity element and d be an f-derivation of R where f is a function satisfying that $f(1) = 1$. Then the followings hold for all $x \in R$:*
(i) $f(x) * d(1) \leq d(x)$.
(ii) If $d(1) = 1$, then $f(x) \leq d(x)$.
(iii) If R is a distributive lattice, then $d(1) = 1$ if and only if $d = f$.

Proof. (i) Since R is an incline algebra with a multiplicative identity element we have $x * 1 = 1 * x = x$ for all $x \in R$, then we can write $d(x) = d(x * 1) = (d(x) * f(1)) + (f(x) * d(1))$. Then by our assumption we have $d(x) = d(x) + (f(x) * d(1))$. Therefore we get, $f(x) * d(1) \leq d(x)$.

(ii) It can be derived from (i).

(iii) Let R be a distributive lattice. If $d = f$ then by our assumption it is clear that $d(1) = 1$. Now we will prove the necessity. Let $d(1) = 1$. By using (2), we have $f(x) \leq d(x)$ for all $x \in R$. But since R is a distributive lattice, by Proposition 4.2(iii) we have $d(x) \leq f(x)$. Hence we get $d = f$. □

Proposition 4.5. *Let R be an integral incline and d be an f-derivation of R where f is a function satisfying that $f(1) = 1$ and a be an element of R. Then for all $x \in R$ we have:*
(i) $a * d(x) = 0$ implies that $a = 0$ or $d = 0$.
(ii) $d(x) * a = 0$ implies that $a = 0$ or $d = 0$.

Proof. (i) Let $a * d(x) = 0$ for all $x \in R$. If we replace x by $x * y$ for $y \in R$ we get

$$\begin{aligned} 0 = a * d(x) = a * d(x * y) &= a * [(d(x) * f(y)) + (f(x) * d(y))] \\ &= (a * (d(x) * f(y))) + (a * (f(x) * d(y))) \\ &= a * (f(x) * d(y)). \end{aligned}$$

In this equation by taking $x = 1$ we get $a * d(1) = 0$. Since R is an integral incline we have $a = 0$ or $d = 0$.

(ii) The proof can be made similarly with the previous one. □

Theorem 4.6. *Let M be a nonzero ideal of an integral incline R. If d is a nonzero f-derivation on R where f is a nonzero function on M, then d is nonzero f-derivation on M.*

Proof. Assume that d is a nonzero f-derivation on R but d is zero f-derivation on M where f is a nonzero function on M and $x \in M$. Then we have $d(x) = 0$. Let $y \in R$. By (1) $x * y \leq x$ and since M is an ideal of R, we have $d(x * y) = 0$, so we can write that

$$0 = d(x * y) = (d(x) * f(y)) + (f(x) * d(y)) = f(x) * d(y).$$

We know by our assumption that R has no zero divisors, so we have $f(x) = 0$ for all $x \in M$ or $d(y) = 0$ for all $y \in R$. Since f is a nonzero function on M, we get that $d(y) = 0$ for all $y \in R$. This contradicts with our assumption that is d is a nonzero f-derivation on R. Hence, we have d is nonzero on M. □

Theorem 4.7. *Let d be a nonzero f-derivation of an integral incline R where f is a nonzero function on M. If M is a nonzero ideal of R, and $a \in R$ such that $a * d(M) = 0$, then $a = 0$.*

Proof. By Theorem 4.6 we know that there is an element m in M such that $d(m) \neq 0$. Let M be a nonzero ideal of R, and $a \in R$ such that $a * d(M) = 0$. Then for $m, n \in M$ we can write

$$0 = a * d(m * n) = a * (d(m) * f(n) + f(m) * d(n)) \\ = a * d(m) * f(n) + a * f(m) * d(n) = a * f(m) * d(n).$$

Since R is an integral incline, d is a nonzero f-derivation of R and f is a nonzero function on M we have $a = 0$.

Proposition 4.8. *Let d be an f-derivation of an incline algebra R. If for all $x, y \in R$ we have $d(x + y) = d(x) + d(y)$, then the followings hold for all $x, y \in R$:*
(i) $d(x * y) \leq d(x)$.
(ii)$d(x * y) \leq d(y)$.
(iii) d is an isotone derivation.

Proof. (i) Let $x, y \in R$. By using $(RVII)$ we have $d(x) = d(x + x * y) = d(x) + d(x * y)$. Therefore we get that $d(x * y) \leq d(y)$.

(ii) The proof can be obtained similarly with the previous one.

(iii) Let $x \leq y$, then we have $x + y = y$, therefore we have $d(y) = d(x + y) = d(x) + d(y)$. Hence $d(x) \leq d(y)$. □

References

[1] N.O. Al-Shehri, On derivations of incline algebras, *Scientia Mathematicae Japonicae*, **71**, No. 3 (2010), 349-355.

[2] Z.Q. Cao, K.H. Kim, F.W. Roush, *Incline Algebra and Applications*, Ellis Horwood Series, Mathematics and its Applications, Ellis Horwood, Chichester, UK (1984).

[3] Y. Çeven, M.A. Öztürk, On f-derivations of lattices, *Bull. Korean Math. Soc.*, **45**, No. 4 (2008), 701-707.

[4] L. Ferrari, On derivations of lattices, *Pure Math. Appl.*, **12**, No. 4 (2001), 365-382.

[5] K. Kaya, Prime rings with α-derivations, *Hacettepe Bull. Natural Sci. and Eng.*, **16-17** (1987/1988), 63-71.

[6] A.R. Meebakshi, S. Anbalagan, On regular elements in an incline, *International JOurnal of Mathematics and Mathematical Sciences*, **2010**, Article ID 9030363, 12 pages.

[7] E. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093-1100.

[8] G. Szasz, Derivations of lattices, *Acta Sci. Math.*, Szeged, **37** (1975), 149-154.

[9] X.L. Xin, T.Y. Li, J.H. Lu, On derivations of lattices, *Information Sciences*, **178** (2008), 307-316.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)