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A FIXED POINT THEOREM FOR TWO MAPPINGS ON TWO COMPLETE FUZZY METRIC SPACES USING IMPLICIT RELATIONS

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Abstract: A fixed point theorem for mappings satisfying implicit relations was proved. This result gives fuzzy versions of known fixed point theorems for mappings in two metric spaces.

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1. Introduction and Preliminaries

Fisher [5], Popa [11] and Nešić [10] proved some fixed point theorems on two metric spaces. Using the implicit relation, other authors unified and generalized some of the well-known theorems. So Telci [14] and later Aliouche and Fisher [1] realized the generalization for two mappings on two metric spaces. Our aim is to unify, generalize and extend all the above theorems in fuzzy metric spaces.

The concept of fuzzy sets was introduced initially by Zadeh [15]. George and Veeramani [6] modified the concept of fuzzy metric space which was introduced by Kramosil and Michalek [8] and defined a Hausdorff topology in this space. Grabiec [7] extended the well known fixed point theorems of Banach [2] and Edelstein [4] in fuzzy metric spaces.

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In this paper, using a new class of implicit relations, we prove a theorem as a corollary of which are taken a fuzzy version of theorems: Fisher [5], Popa [11], Nešić [10], Telci [14], the theorem of Aliouche and Fisher [1], etc.

In [1] and [10] the following theorems are proved:

Theorem 1.1. (Theorem of Aliouche and Fisher, see [1]) *Let $(X,d),(Y,\rho)$ be complete metric spaces, T a mapping of X into Y and S a mapping of Y into X satisfying the inequalities:*

$$\begin{aligned} f(\rho(Tx,TSy),d(x,Sy),\rho(y,Tx),\rho(y,TSy)) &\le 0,\\ g(d(Sy,STx),\rho(y,Tx),(d(x,Sy),d(x,STx)) &\le 0, \end{aligned}$$

for all $x\in X$ and $y\in Y$, where $f,g\in F$. Then ST has a unique fixed point $u\in X$ and TS has a unique fixed point $v\in Y$. Further, $Tu=v$ and $Sv=u$.

Theorem 1.2. (Theorem of Nešić, see [10]) *Let (X,d) and (Y,ρ) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities:*

$$\begin{aligned} \rho^p(Tx,TSy) &\le c_1\max M_1(x,y)+F_1(\min M_1(x,y)),\\ d^p(Sy,STx) &\le c_2\max M_2(x,y)+F_2(\min M_2(x,y)), \end{aligned}$$

for all $x\in X$ and $y\in Y$, where $0\le c_1,c_2<1$, then ST has a unique fixed point $z\in X$ and TS has a unique fixed point $w\in Y$. Further, $Tz=w$ and $Sw=z$.

Before we prove main results, we recall the following definitions and lemmas.

Definition 1.3. (see [15]) A fuzzy set A in X is a function with domain X and values in $[0,1]$.

Definition 1.4. (see [13]) A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm, if $([0,1],*)$ is an Abelian topological monoid with the unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all a, b, c, $d\in[0,1]$.

Two typical examples of continuous t-norm are $a*b=ab$ and $a*b=\min(a,b)$.

Definition 1.5. (see [6]) The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions:

For all $x,y,z\in X$ and $t,s>0$:

(FM-1) $M(x,y,t)>0$,

(FM-2) $M(x,y,t)=1$ if and only if $x=y$,

(FM-3) $M(x,y,t)=M(y,x,t)$,

(FM-4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$

(FM-5) $M(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous.

Example 1.6. (see [6]) Let (X, d) be a metric space. Define $a * b = ab$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, \quad k, m, n \in R^+.$$

Then $(X, M, *)$ is a fuzzy metric space. In the above example by taking $k = m = n = 1$ we get

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Definition 1.7. (see [7]) Let $(X, M, *)$ be a fuzzy metric space. Then:

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \text{for all } t > 0 \text{ and } p > 0.$$

3. A fuzzy metric space in which every Cauchy sequence is convergent is called complete.

Lemma 1.8. (see [7]) *For all $x, y \in X$, $M(x, y, \cdot)$ is no decreasing.*

Remark 1.9. Throughout this paper, $(X, M, *)$ will denote the fuzzy metric space in the sense of Definition 2.3 with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Lemma 1.10. (see [12]) *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Lemma 1.11. (see [3, 9]) *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$ for all $t > 0$ and $n = 1, 2, ...$, then $\{y_n\}$ is a Cauchy sequence in X.*

Lemma 1.12. (see [9]) *Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that*

$$M(x, y, kt) \geq M(x, y, t), \quad \text{then } x = y.$$

2. Implicit Relations

Before stating the main Theorem 3.1 we define a new class of implicit relations which will give a general character to this theorem.

Definition 2.1. Let Φ_4 the set of all functions with 4 variables $\varphi : [0,1]^4 \to R$ satisfying the following conditions:

(a) φ is lower semi-continuous in each coordinate variable t_1, t_2, t_3, t_4.

(b) If for some constant $0 < c < 1$ we have $\varphi(u(ct), 1, u(t), v(t)) \geq 0$ or $\varphi(u(ct), u(t), 1, v(t)) \geq 0$ for any $t > 0$ and any functions $u, v : (0, \infty) \to [0,1]$ where the function u is no decreasing and $u(ct) = u(t) \Rightarrow u(t) = 1$, then $u(ct) \geq v(t)$.

Every such function φ will be called Φ_4-function with constant c.

Some examples of Φ_4-function are as follows:

Example 2.2. Let $F : R \to R$ be a continuous function with $F(1) = 0$. The function $\varphi(t_1, t_2, t_3, t_4) = t_1^p - \min\{t_2^p, t_3^p, t_4^p\} - F(\max\{t_2^p, t_3^p, t_4^p\})$, where $p > 0$, is Φ_4-function.

Proof. (a) is clear since φ is continuous.

Suppose that $0 < c < 1$; $u, v : (0, \infty) \to [0,1]$, u is no decreasing, $u(ct) \geq u(t) \Rightarrow u(t) = 1$ and then

$$\begin{aligned}\varphi(u(ct), 1, u(t), v(t)) &= u^p(ct) - \min\{1, u^p(t), v^p(t)\} - F(\max\{1, u^p(t), v^p(t)\})\\ &= u^p(ct) - \min\{1, u^p(t), v^p(t)\} - F(1)\\ &= u^p(ct) - \min\{1, u^p(t), v^p(t)\} \geq 0. \qquad (*)\end{aligned}$$

We have $u^p(t) \geq v^p(t)$, since in contrary (if $u^p(t) < v^p(t)$), from (*), we get $u^p(ct) \geq \min\{1, u^p(t), v^p(t)\} = u^p(t)$ and so $u(ct) = u(t) = 1$ (because u is non decreasing and $u(ct) \geq u(t) \Rightarrow u(t) = 1$), a contradiction. Therefore,

$$u^p(ct) \geq \min\{1, u^p(t), v^p(t)\} = v^p(t) \quad \text{and so } u(ct) \geq v(t).$$

Similarly, if $\varphi(u(ct), u(t), 1, v(t)) \geq 0$, then $u(ct) \geq v(t)$. The proof of (b) is completed. □

Definition 2.3. The set of all continuous functions with 3 variables $f : [0,1]^3 \to R$ satisfying the properties:

(a') f is non decreasing with respect to each variable.

(b') $f(t,t,t) \geq t, t \in [0,1]$ will be noted $\mathbb{F}_3$ and every such function will be called a $\mathbb{F}_3$-function.

Some examples of $\mathbb{F}_3$-function are as follows:

1. $f(t_1, t_2, t_3) = \min\{t_1, t_2, t_3\}$.
2. $f(t_1, t_2, t_3) = [\min\{t_1t_2, t_2t_3, t_3t_1\}]^{1/2}$.
3. $f(t_1, t_2, t_3) = [\min\{t_1^p, t_2^p, t_3^p\}]^{1/p}, p > 0$.
4. $f(t_1, t_2, t_3) = (a_1t_1^p + a_2t_2^p + a_3t_3^p)^{1/p}$, where $p > 0$ and $a_i \geq 0$, $\sum_{i=1}^{3} a_i \geq 1$.

The proof is done for Example 4:

(a') It is obvious that the function f is no decreasing with respect to each variable.

(b') We have:

$$\begin{aligned} f(t,t,t) &= (a_1t^p + a_2t^p + a_3t^p)^{\frac{1}{p}} = [(a_1 + a_2 + a_3)t^p]^{\frac{1}{p}} \\ &= (a_1 + a_2 + a_3)^{\frac{1}{p}} t \geq t, t \in [0,1]. \end{aligned}$$

The proof of (b') is completed.

Definition 2.4. The set of all continuous functions with 3 variables $g : [0,1]^3 \to R$ satisfying the property:

$$(t_1 - 1)(t_2 - 1)(t_3 - 1) = 0 \Rightarrow g(t_1, t_2, t_3) = 1,$$

(the function g takes the value 1 at the points for which at least one of the coordinates is 1) will be noted G_3 and every such function will be called a G_3 -function.

Some examples of G_3 -function are as follows:

5. $g(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$.
6. $g(t_1, t_2, t_3) = \max\{t_1^p, t_2^p, t_3^p\}$, $p > 0$, etc.

Definition 2.5. Let $\mathcal{F}$ be the set of continuous functions $F : R \to R$ such that $F(1) = 0$.

For example $F = 0; F(t) = (t - 1)^3$, etc.

The following relationship between $\mathbb{F}_3$-functions and Φ_4-functions holds:

Lemma 2.6. *If $f \in \mathbb{F}_3$, then the function $\varphi(t_1, t_2, t_3, t_4) = t_1 - f(t_2, t_3, t_4)$ is Φ_4-function.*

Proof. (a) is clear.

Suppose that $0 < c < 1$; $u, v : (0, \infty) \to [0, 1]$, u non decreasing, $u(ct) \geq u(t) \Rightarrow u(t) = 1$ and then

$$\varphi(u(ct), 1, u(t), v(t)) = u(ct) - f(1, u(t), v(t)) \geq 0. \qquad (**)$$

We have $u(t) \geq v(t)$, since in contrary (if $u(t) < v(t)$), by using the properties of f we get: $f(1, u(t), v(t)) \geq f(u(t), u(t), u(t)) \geq u(t)$ and by (**) it follows

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$u(ct) \geq u(t)$, a contradiction. Therefore, after replacing the coordinates of the point $(1, u(t), v(t))$ by $v(t)$ and using the properties of f we get $u(ct) \geq v(t)$. Similarly, if $\varphi(u(ct), u(t), 1, v(t)) \geq 0$, then $u(ct) \geq v(t)$. The proof of (b) is completed. □

Lemma 2.7. *If $f \in \mathbb{F}_3$, $g \in G_3$ and $F \in F$, then the function $\varphi(t_1, t_2, t_3, t_4) = t_1 - f(t_2, t_3, t_4) - F(g(t_2, t_3, t_4)$ is Φ_4-function.*

Proof. (a') is clear.

Suppose that $0 < c < 1$; $u, v : (0, \infty) \to [0, 1]$; u non decreasing, $u(ct) \geq u(t) \Rightarrow u(t) = 1$ and then

$$\begin{aligned}\varphi(u(ct), 1, u(t), v(t)) =& u(ct) - f(1, u(t), v(t)) - F(g(1, u(t), v(t)) \\ =& u(ct) - f(1, u(t), v(t)) - F(1) \\ =& u(ct) - f(1, u(t), v(t)) \geq 0.\end{aligned}$$

Further, we continue in the same way as in Lemma 2.6. □

The above lemmas give us the opportunity to construct other functions of type Φ_4:

Example 2.8. $\varphi(t_1, t_2, t_3, t_4) = t_1 - [\min\{t_2 t_3, t_3 t_4, t_4 t_2\}]^{1/2} - F(\max\{t_2, t_3, t_4\})$.

Example 2.9. $\varphi(t_1, t_2, t_3, t_4) = t_1 - (a_2 t_2^p + a_3 t_3^p + a_4 t_4^p)^{1/p} - \max\{t_2^p, t_3^p, t_4^p\}$, where $p > 0$ and $a_i \geq 0$, $\sum_{i=2}^{4} a_i \geq 1$ etc.

3. Main Results

We prove now a new theorem which is general theorem due to using of implicit relations considered above.

Theorem 3.1. *Let $(X, M_1, *_1)$ and $(Y, M_2, *_2)$ be two complete fuzzy metric spaces and $T : X \to Y$, $S : Y \to X$ two maps which satisfy the conditions:*

$$\varphi_1(M_1(Sy, STx, ct), M_1(x, Sy, t), M_1(x, STx, t), M_2(y, Tx, t)) \geq 0, \tag{1}$$

$$\varphi_2(M_2(Tx, TSy, ct), M_2(y, Tx, t), M_2(y, TSy, t), M_1(x, Sy, t)) \geq 0, \tag{2}$$

for all $x \in X$, $y \in Y$, $t > 0$ where $c \in (0, 1)$ and $\varphi_1, \varphi_2 \in \Phi_4$. Then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Moreover, $T\alpha = \beta$ and $S\beta = \alpha$.

Proof. Let x_0 be a arbitrary point in X. Construct the sequences $\{x_n\}$, $\{y_n\}$ in X and Y, respectively, as follows:

$$x_n = (ST)^n x_0, \quad y_n = Tx_{n-1}, \quad n \in N.$$

We will show that, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Denote:

$$d_n(t) = M_1(x_n, x_{n+1}, t), \quad \rho_n(t) = M_2(y_n, y_{n+1}, t).$$

By the inequality (2) for $x = x_{n-1}$ and $y = y_n$ we get:

$$\begin{aligned} &\varphi_2(M_2(Tx_{n-1}, TSy_n, ct), M_2(y_n, Tx_{n-1}, t), M_2(y_n, TSy_n, t), M_1(x_{n-1}, Sy_n, t)) \\ =&\varphi_2(M_2(y_n, y_{n+1}, ct), M_2(y_n, y_n, t), M_2(y_n, y_{n+1}, t), M_1(x_{n-1}, x_n, t)) \\ =&\varphi_2(\rho_n(ct), 1, \rho_n(t), d_{n-1}(t)) \geq 0. \end{aligned} \tag{3}$$

Next, from (3), after the application of property (b) of φ_2 we have

$$\rho_n(ct) \geq d_{n-1}(t), \quad \forall n \in N. \tag{4}$$

(We emphasize that the function $u : u(t) = \rho_n(t)$ satisfies the conditions of Definition 2.1).

In a similar way, by the inequality (1) for $x = x_n$ and $y = y_n$ we find:

$$\begin{aligned} &\varphi_1(M_1(x_n, x_{n+1}, ct), M_1(x_n, x_n, t), M_1(x_n, x_{n+1}, t), M_2(y_n, y_{n+1}, t)) \\ &\qquad = \varphi_1(d_n(ct), 1, d_n(t), \rho_n(t)) \geq 0. \end{aligned}$$

And so

$$d_n(ct) \geq \rho_n(t). \tag{5}$$

By the inequalities (5), (4) and $\rho_n(t) \geq \rho_n(ct)$, because ρ_n is no decreasing, we get:

$$d_n(ct) \geq d_{n-1}(t), \quad \forall n \in N. \tag{6}$$

By induction, applying (6) and (4), we obtain

$$d_n(t) \geq d_1(\frac{t}{c^{n-1}}) \text{ and } \rho_n(t) \geq d_1(\frac{t}{c^{n-1}}),$$

or

$$M_1(x_n, x_{n+1}, t) \geq M_1(x_1, x_2, \frac{t}{c^{n-1}}) \text{ and } M_2(y_n, y_{n+1}, t) \geq M_1(x_1, x_2, \frac{t}{c^{n-1}}).$$

But $\lim\limits_{n\to\infty} \frac{t}{c^{n-1}} = \infty$ because $c \in (0, 1)$ and applying (FM-6) we get

$$\lim_{n\to\infty} M_1(x_n, x_{n+1}, t) = \lim_{n\to\infty} M_2(y_n, y_{n+1}, t) = 1.$$

Now, for all n and p, using the Definition 1.5 (FM-4) obtaining

$$M_1(x_n, x_{n+p}, t) \geq \underbrace{M_1(x_n, x_{n+1}, \frac{t}{p}) * M_1(x_{n+1}, x_{n+2}, \frac{t}{p}) * ... * M_1(x_{n+p-1}, x_{n+p}, t)}_{p}.$$

When n tends to infinity, we have

$$\lim_{n\to\infty} M_1(x_{n+p}, x_n, t) \geq \underbrace{1 * 1 * ... * 1}_{p} = 1.$$

This shows that $\{x_n\}$ is a Cauchy sequence with a limit α in X. We can show in the same way that $\{y_n\}$ is also Cauchy sequence with a limit β in Y.

Applying inequality (1) we now have

$$\varphi_1 \left(M_1(S\beta, x_{n+1}, ct), M_1(x_n, S\beta, t), M_1(x_n, x_{n+1}, t), M_2(\beta, y_{n+1}, t),\right) \geq 0.$$

Letting n tends to infinity and using the property (a) of φ_1, we have

$$\varphi_1(M_1(S\beta, \alpha, ct), M_1(\alpha, S\beta, t), 1, 1) \geq 0.$$

And so $M_1(S\beta, \alpha, ct) \geq M_1(\alpha, S\beta, t)$. This means (Lemma 1.12) that

$$S\beta = \alpha. \tag{7}$$

Applying inequality (2) and considering (7) we have

$$\varphi_2 \left(M_2(T\alpha, y_{n+1}, ct), M_2(y_n, T\alpha, t), M_2(y_n, y_{n+1}, t) M_1(S\beta, x_n, t)\right) \geq 0.$$

Letting n tend to infinity we take

$$\varphi_2(M_2(T\alpha, \beta, kt), M_2(\beta, T\alpha, t), 1, 1) \geq 0.$$

Thus,

$$T\alpha = \beta. \tag{8}$$

Next, from (7) and (8), we have

$$TS\beta = T\alpha = \beta \quad \text{and} \quad ST\alpha = S\beta = \alpha.$$

So, α is a fixed point for ST and β is a fixed point for TS.

To prove the uniqueness, we suppose that α' is another fixed point of ST.

Applying (1) for $y = T\alpha$ and $x = \alpha'$, we have

$$\begin{aligned}\varphi_1 &\left(M_1(ST\alpha, ST\alpha', ct), M_1(\alpha', ST\alpha, t), M_1(\alpha', ST\alpha', t), M_2(T\alpha, T\alpha', t)\right) \\ &= \varphi_1(M_1(\alpha, \alpha', ct), M_1(\alpha, \alpha', t), 1, M_2(T\alpha, T\alpha', t)) \geq 0.\end{aligned}$$

Applying now the property (b) for φ_1, we obtain

$$M_1(\alpha, \alpha', ct) \geq M_2(T\alpha, T\alpha', t). \tag{9}$$

Next, from (2) it follows that

$$\varphi_2(M_2(TST\alpha, TST\alpha', ct), \\ M_2(T\alpha', TST\alpha, t), M_2(T\alpha', T\alpha', t), M_1(RST\alpha, RST\alpha', t)) \\ = \varphi_2(M_2(T\alpha, T\alpha', ct), M_2(T\alpha', T\alpha, t), 1, M_1(\alpha, \alpha', t)) \geq 0.$$

Thus, we have

$$M_2(T\alpha, T\alpha', ct) \geq M_1(\alpha, \alpha', t). \tag{10}$$

Now, from (9) and (10) and from the fact that $M_2(T\alpha, T\alpha', t) \geq M_2(T\alpha, T\alpha', ct)$, we have

$$M_1(\alpha, \alpha', ct) \geq M_1(\alpha, \alpha', t). \tag{11}$$

Therefore (see Lemma 1.12)

$$\alpha = \alpha'.$$

Thus, α is the unique fixed point for ST. In the same way we show that β is the unique fixed point for TS. □

4. Corollaries

Theorem 3.1 extends in fuzzy metric spaces Theorem 1.1 (Theorem of Aliouche and Fisher [1]). At the same time it unifies and generalizes the fuzzy version of the theorems of Nešic′ [10], Fisher [5], Popa [11], Telci [14], etc.

Corollary 4.1. *Let $(X, M_1, *_1)$ and $(Y, M_2, *_2)$ be two complete fuzzy metric spaces and $T : X \to Y$, $S : Y \to X$ two maps which satisfy the conditions:*

$$\begin{aligned} M_1(Sy, STx, ct) &\geq f_1(M_2(y, Tx, t), M_1(x, Sy, t), M_1(x, STx, t)) \\ &+ F_1(g_1(M_2(y, Tx, t), M_1(x, Sy, t), M_1(x, STx, t)), \\ M_2(Tx, TSy, ct) &\geq f_2(M_1(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t)) \\ &+ F_2(g_2(M_1(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t)), \end{aligned}$$

for all $x \in X$, $y \in Y$, $t > 0$ where $c \in (0, 1)$ and $f_1, f_2 \in \mathbb{F}_3$; $g_1, g_2 \in G_3$; $F_1, F_2 \in F$.

Then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Moreover, $T\alpha = \beta$ and $S\beta = \alpha$.

Proof. The proof follows by Theorem 3.1 in the case $\varphi_i \in \Phi_4, f_i \in \mathbb{F}_3, g_i \in G_3, F_i \in F$ such that $\varphi_i(t_1, t_2, t_3, t_4) = t_1 - f_i(t_2, t_3, t_4) - F_i(g_i(t_2, t_3, t_4)$ for $i = 1, 2$. □

The next corollary fuzzyfies Theorem 1.2 (Theorem of Nešic′ [10]).

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Corollary 4.2. *Let $(X, M_1, *_1)$ and $(Y, M_2, *_2)$ be two complete fuzzy metric spaces and $T : X \to Y$, $S : Y \to X$ two maps which satisfy the conditions:*

$$\begin{aligned} M_1^p(Sy, STx, ct) &\geq \min\{M_2^p(y, Tx, t), M_1^p(x, Sy, t), M_1^p(x, STx, t)\} \\ &+ F_1(\max\{M_2^p(y, Tx, t), M_1^p(x, Sy, t), M_1^p(x, STx, t)\}), \\ M_2^p(Tx, TSy, ct) &\geq \min\{M_1^p(x, Sy, t), M_2^p(y, Tx, t), M_2^p(y, TSy, t)\} \\ &+ F_2(\max\{M_1^p(x, Sy, t), M_2^p(y, Tx, t), M_2^p(y, TSy, t)\}), \end{aligned}$$

for all $x \in X$, $y \in Y$, $t > 0$ where $c \in (0, 1)$, $p > 0$ and $F_1, F_2 \in F$. Then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Moreover, $T\alpha = \beta$ and $S\beta = \alpha$.

Proof. The proof follows by Theorem 3.1 in the case $\varphi_i \in \Phi_4$, such that

$$\varphi_i(t_1, t_2, t_3, t_4) = t_1^p - \min\{t_2^p, t_3^p, t_4^p\} - F_i(\max\{t_2^p, t_3^p, t_4^p\}),$$

for $i = 1, 2$. □

Remark 4.3. The constants c_1 and c_2 of Nešic′ Theorem [10] are replaced from $c = \max\{c_1, c_2\}$.

Corollary 4.4. *If in Theorem 3.1 we take $\varphi_1 = \varphi_2 = \varphi \in \Phi_4$ where*

$$\varphi(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, t_3, t_4\}$$

we obtain the theorem which fuzzyfies the Fisher Theorem (see Theorem 1 in [5]) for metric spaces.

Corollary 4.5. *If in Theorem 3.1 we take $\varphi_1 = \varphi_2 = \varphi \in \Phi_4$ where*

$$\varphi(t_1, t_2, t_3, t_4) = t_1 - [\min\{t_2t_3, t_3t_4, t_4t_2\}]^{\frac{1}{2}},$$

we obtain the theorem which fuzzyfies the Popa result (Theorem 2, [11]*) for metric spaces.*

Corollary 4.6. *If in Theorem 3.1 we take $\varphi_1 = \varphi_2 = \varphi \in \Phi_4$ where*

$$\varphi(t_1, t_2, t_3, t_4) = t_1 - [\min\{t_2^p, t_3^p, t_4^p\}]^{\frac{1}{2}}, \quad p > 0,$$

we obtain a generalization of Corollary 4.1.

Corollary 4.7. *If in Theorem 3.1 we take $\varphi_i \in \Phi_4$ where $\varphi_i(t_1, t_2, t_3, t_4) = t_1 - f_i(t_2, t_3, t_4)$, $f_i \in F_3$ for $i = 1, 2$ we obtain the theorem which fuzzyfies the Telci result* [14] *for metric spaces.*

Remark 4.8. We can obtain many other similar results for different φ.

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