International Electronic Journal of Pure and Applied Mathematics

Volume 3 No. 1 2011, 11-28

LIOUVILLE TYPE RESULTS FOR A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

Nedra Belhaj Rhouma[1], Mounir Hsini[2 §]

[1]Department of Mathematics
Faculty of Sciences
University of Tunis
El-Manar, Campus Universitaire, Tunis, 2092, TUNISIA
e-mail: Nedra.Belhajrhouma@fst.rnu.tn

[2]Department of Mathematics
Institute of Engineering
2, Rue Jawaherlal Nehru, Montfleury, 1008, TUNISIA
e-mail: mounir.hsini@ipeit.rnu.tn

Abstract: The goal of this paper is to study the properties of solutions of $\Delta u + f_1(u) - f_2(u) = 0$ in all of $\mathbb{R}^n$. We obtain Liouville type boundedness results for the solutions. We show that either u is bounded on $\mathbb{R}^n$ if it changes sign or u is a constant if it does not change sign.

AMS Subject Classification: 35B40, 35B05, 35J60
Key Words: nonlinear elliptic PDE, Liouville Theorem, comparison technique, Green function, harmonic kernel, Harnack inequality

1. Introduction

In this paper, we consider the following problem

$$\Delta u + f(u) = 0 \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 2, \tag{1.1}$$

where $f = f_1 - f_2$. The functions f_1 and f_2 are measurable and allowed to satisfy the following conditions

$$\forall u \geq 0, \quad f_1(u) \leq \alpha u^{p-1}, \quad f_2(u) \geq \beta u^m, \ (f_1 - f_2)(0) = 0, \tag{1.2}$$

for some $0 < \alpha \leq \beta$ and $m > p - 1 \geq 1$.

Received: September 27, 2010

§Correspondence author

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

When $f_1(u) = u$ and $f_2(u) = u^3$, the problem is related to the Ginzburg-Landau equation

$$\Delta u + u - u^3 = 0. \tag{1.3}$$

Let $F \in C^{2+\epsilon}(\mathbb{R})$ be a nonnegative function such that

$$F(\pm 1) = 0 \text{ and } F^{"}(\pm 1) \geq \mu > 0 \tag{1.4}$$

for some constant μ. A more general form of (1.3) is the equation

$$\Delta u - F'(u) = 0. \tag{1.5}$$

It is known in Gilbarg and Trudinger [8] that any bounded solution of the equation (1.5) is $C^2(\mathbb{R}^n)$. Modica [11], has shown that the solution satisfies the gradient bound:

$$|\nabla u(x)|^2 \leq 2F(u(x)) \qquad \text{for all} \quad x \in \mathbb{R}^n. \tag{1.6}$$

Farina [7], considered the problem

$$\Delta u + f(u) = 0, \text{ on } \mathbb{R}^N, \tag{1.7}$$

where f is a locally Lipschitz continuous function of bistable type.

More recently, the Liouville Theorem was further generalized to solutions of quasilinear elliptic equations (Serrin [13]): Let u be an entire solution of the equation

$$\triangle u + f(u, \nabla u) = 0, \quad \text{in } \mathbb{R}^n.$$

Supose that $\frac{\partial f}{\partial u} \leq 0$ and bouth u and ∇u are bounded. Then u must be constant.

Under further assumptions, it can be shown by Peletier and Serrin [12] and Serrin [14] that ∇u is necessarily bounded on $\mathbb{R}^n$. Using this fact, one gets a standard Liouville Theorem for bounded solutions, see also Caffarelli, Garofalo and Segala [4] and their references.

Serrin and Zou in [15] considered the nonhomogenous degenerate elliptic equation of the form

$$\triangle_m u + f(u) = 0, \;\; u \geq 0, \; x \in \Omega, \tag{1.8}$$

where Ω is a domain in $\mathbb{R}^n$, $n \geq 2$, $\triangle_m$ is the m-Laplace operator and the function f is subcritical.

Special cases of problem (1.8) have been considered by Du and Guo [5], when $f(u) \geq \sigma u^{m-1}$ near 0. Du and Ma [6] recently removed the boundeness condition $|u(x)| \leq 1$ in De Giorgi's conjecture and they proved the following Liouville Theorem: If $\lambda \in (-\infty, \infty)$ and $u \in C^2(\mathbb{R}^n)$ be a non negative stationary solution of $u_t - \triangle u = \lambda u - u^p, \;\; u \geq 0$, then u must be a constant.

Berestycki, Hamel and Rossi [2] gave Liouville type results for the equations

$$-a_{i,j}(x)\partial_{i,j}u(x) - q_i(x)\partial_i u(x) = f(x, u), \quad \text{in } \mathbb{R}^n,$$

when $\frac{f(x,s)}{s}$ is decreasing in s.

In our case, we will consider a more general form of problem (1.1). Note that in our situation condition (1.4) is not necessary satisfied.

The model example $f_1(u) = \alpha|u|^{p-2}u\exp(-|u|)$ and $f_2(u) = \beta|u|^{m-1}u\exp|u|$, leads to equation

$$\Delta u + \alpha|u|^{p-2}u\exp(-|u|) - \beta|u|^{m-1}u\exp|u| = 0, \tag{1.9}$$

where $m > p-1 \geq 1$.

When $f_1(u) = \alpha|u|^{p-2}u$ and $f_2(u) = \beta|u|^{m-1}u$, equation (1.1) is equivalent to

$$\Delta u + \alpha|u|^{p-2}u - \beta|u|^{m-1}u = 0, \quad \text{where} \quad m > p-1 \geq 1. \tag{1.10}$$

In this note, we will study the qualitative properties of solutions of (1.10) and we will deduce those of (1.1). In particular we will prove the following results.

Theorem 1.1. (Global Boundedness) *If u is a solution of the problem* (1.1) *on $\mathbb{R}^n$, then $u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$. Moreover, if f is odd we have $|u| \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$.*

Remark 1.1. When $\alpha = \beta = 1$ and $p = 2$, we generalize the result of Zhao [16]. Brezis, Merle and Rivière [3], Hervé [9] have shown, by different methods, that any solution u of (1.3) satisfies the condition $|u| \leq 1$.

Theorem 1.2. *Assume that f is odd and let u be a non constant solution of the problem* (1.1). *Then $|u| < (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$.*

Theorem 1.2 gives a generalisation of the results proved in [3] and [9] for the Ginzburg-Landau equation $\Delta u + u(1-|u|^2) = 0$.

Theorem 1.3. (Liouville Type Property) *Assume that $u \in C^2(\mathbb{R}^n)$ is a positive solution of* (1.10) *such that $\inf_{\mathbb{R}^n} u > 0$. Then u is a constant ($u \equiv (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$).*

We note that such result was proved when f is a real-valued C^1 function f satisfying

$$(H_1)\begin{cases} f(0) = f(1) = 0, \\ f(u) > 0, \ \forall\, u \in (0,1), \\ f(u) < 0, \ \forall\, u > 1, \end{cases}$$

and the Keller-Osserman conditions: For some large constant $M > 1$,

$$(H_2)\begin{cases} \lim_{u\to 0^+} f(u)/u^{1+2/n} \in (0,\infty], \\ f(u) \leq g(u) < 0, \text{ in } [M,\infty), \\ g \text{ is decreasing in } [M,+\infty) \text{ and } \int_M^\infty (\int_M^u |g(s)|ds)^{-1/2}du < \infty. \end{cases}$$

Under the assumptions (H_1) and (H_2), Du and Ma [6] proved that every positive solution in $C^2(\mathbb{R}^n)$ of

$$\Delta u + f(u) = 0$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

is a constant ($u = 1$).

Note that, in our case, the proof is more difficult. Indeed the comparison principle is not directly applied and the first hypothesis of Keller-Osserman conditions (H_2) on the function f is not satisfied for $p > 2$.

Note that we do not require a priori global boundness for the solution u, and this is different from other Liouville-type theorems. Since the problem (1.10) has a unique positive constant solution $u = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$, we see that Theorem 1.3 implies that the entire space elliptic problem behaves analogously to the bounded domain problem.

Remark 1.2. For the dimension $n = 2$, Hervé and Hervé [9] showed that there exists $C = C(R) > 0$ such that

$$1 - |u(z)| \leq C\Big[1 - |u(z_0)|\Big], \ \forall\, z \in B(z_0, r), \text{ and } 0 < r < R,$$

for any solution u of problem

$$-\triangle u = u(1 - |u|^2) \text{ in } B(z_0, R), |u| < 1 \text{ on } B(z_0, R). \tag{1.11}$$

Such result will be generalized in this paper when f is odd and for any dimension $n \geq 2$. This is the object of the following two theorems.

Theorem 1.4. *Case $n = 2$. Assume that f is odd and that f is nonnegative on $[0, (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}]$. Let $x \in \mathbb{R}^2$. Then for every $R > 0$ there exists $C = C(R) > 0$ such that*

$$\Big[(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}} - |u(y)|\Big] \leq C\Big[(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}} - |u(x)|\Big], \tag{1.12}$$

for every $y \in B(x, R)$ and any solution u of (1.1).

Theorem 1.5. *Case $n \geq 3$. Assume that f is odd and that f is nonnegative on $[0, (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}]$. Let $x \in \mathbb{R}^n$. Then, for every $q > \frac{n}{2}$ and for every $R > 0$, there exists $C = C(R) > 0$ such that*

$$(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}} - |u(y)| \leq C\Big[(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}} - |u(x)|\Big]^{\frac{1}{q}},$$

for every $y \in B(x, R)$ and any solution u of (1.1).

Remark 1.3. We notice that the conditions of Theorem 1.4 and Theorem 1.5 are trivially satisfied when $f(x) = \alpha|x|^{p-2}x - \beta|x|^{m-1}x$.

This paper is organized as follows: in Section 2, we will prove some results which allow us to prove Theorems 1.1 and 1.2 in Section 3. The proof of the Liouville-type theorem, is the object of Section 4. The last section is devoted to the proof of Theorems 1.4 and 1.5. At beginning, we introduce these notations.

Notation. For $x_0 \in \mathbb{R}^n$, $B(x_0, R)$ denotes the ball with center x_0 and radius R and we denote by $S(x_0, R) := \{x \in \mathbb{R}^n, |x - x_0| = R\}$.

$u \vee v$ and $u \wedge v$ design respectively the supremum and the infinimum of u and v.

2. Preliminary

In this section, we will be concerned with the solutions of the problem

$$\triangle u + \alpha|u|^{p-2}u - \beta|u|^{m-1}u = 0, \tag{2.1}$$

where $m > p - 1 \geq 1$. To obtain information about the solution, we will need the following comparison result. To this end, let Ω denote a bounded domain of $\mathbb{R}^n$.

Definition 2.1. Consider the problem

$$\triangle u + f(u) = 0, \text{ in } \Omega. \tag{2.2}$$

We say that a function $u \in W^{1,2}_{loc}(\Omega)$ is a (weak) supersolution (resp. subsolution) of (2.2) if

$$\int_\Omega \nabla u \nabla \varphi - \int_\Omega f(u)\varphi \geq 0 \quad \text{(resp. } \leq 0) \tag{2.3}$$

for every nonnegative function $\varphi \in W^{1,2}_0(\Omega)$.

Next, we recall the following result proved in [1].

Proposition 2.1. *Let u and v be two subsolutions of (2.2) in Ω. Then, $u \vee v$ is also a subsolution. A similar statement holds for the minimum of two supersolutions.*

Proposition 2.2. *Let u be a subsolution of* (2.2) *and v be a supersolution of* (2.2) *such that*

$$\liminf_{x \to z} (v(x) - u(x)) \geq 0$$

for all $z \in \partial\Omega$ and both sides of the inequality are not simultaneously $+\infty$ or $-\infty$. Moreover assume that $f(v) \geq f(u)$ on Ω, then, $v \geq u$ on Ω.

Proof. Let $\varepsilon > 0$ and K be a compact subset of Ω such that $u - v \leq \varepsilon$ on $\Omega \setminus K$. Then, the function $\varphi = (u - v - \varepsilon)^+ \in W^{1,2}_0(\Omega)$. Testing by φ in (2.3), we obtain that

$$0 \leq \int_{\{u>v+\varepsilon\}} (\nabla v - \nabla u)\nabla(u - v) - \int_{\{u>v+\varepsilon\}} (f(v) - f(u))(u - v - \varepsilon).$$

On the other hand, we have

$$\int_{\{u>v+\varepsilon\}} (\nabla v - \nabla u)\nabla(u - v) \leq 0 \quad \text{and} \quad \int_{\{u>v+\varepsilon\}} (f(v) - f(u))(u - v - \varepsilon) \geq 0.$$

Hence, $\nabla(u-v-\varepsilon)^+ = 0$ and $(u-v-\varepsilon)^+ = 0$ a.e. in Ω. It follows that $u \leq v+\varepsilon$ *a.e.* in Ω and therefore $u \leq v$ *a.e.* in Ω. $\square$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

Corollary 2.1. *Let $\delta > 0$. Assume that f is decreasing on $[\delta, \infty[$ and let u be a subsolution of* (2.2) *and v be a supersolution of* (2.2) *such that $\liminf_{x\to z}(v-u)(x) \geq 0$ $\forall z \in \partial\Omega$ and $v \geq \delta$ on Ω. Then, $v \geq u$ on Ω.*

Proof. Assume that the set $U = \{x \in \Omega; v(x) < u(x)\}$ is nonempty. We have $\liminf_{x\to z}(v-u)(x) \geq 0$, $\forall z \in \partial U$. Since $f(u) \leq f(v)$ on U, we get from Proposition 2.2 that $v \geq u$ on U which is impossible. Hence $u \leq v$ on Ω. □

In the sequel, we consider a real $R \geq 1$ and $x_0 \in \mathbb{R}^n$.

Lemma 2.1. *Let*

$$v_\lambda(x) = \frac{\lambda}{(R^2 - |x-x_0|^2)^{\frac{2}{m-1}}}, \quad x \in B(x_0, R),$$

where λ is a positive constant. We have

$$\lim_{x\to z} v_\lambda(x) = +\infty \text{ for all } z \in \partial B(x_0, R), \tag{2.4}$$

and

$$\Delta v_\lambda + \alpha v_\lambda^{p-1} - \beta v_\lambda^m \leq 0 \text{ in } B(x_0, R), \tag{2.5}$$

for some $\lambda > 0$.

Proof. For each $x \in B(x_0, R)$, we denote $r = |x - x_0|$ and we consider the function v_λ defined by $v_\lambda = \frac{\lambda}{(R^2-r^2)^{\frac{2}{m-1}}}$, for some $\lambda > 1$. So it is an easy task to show that v_λ fulfills the first part of the lemma. By mean of a straightforward calculation we verify that (2.5) is equivalent to

$$\beta\lambda^{m-1} - \alpha\lambda^{p-2}(R^2 - r^2)^{\frac{2(m-p+1)}{m-1}} \geq \frac{4n}{m-1}(R^2 - r^2) + 8\frac{m+1}{(m-1)^2}r^2. \tag{2.6}$$

Clearly (2.6) implies (2.5) if

$$\beta\lambda^{m-1} - \alpha\lambda^{p-2}R^{\frac{4(m-p+1)}{m-1}} \geq \frac{4n}{m-1}R^2 + 8\frac{m+1}{(m-1)^2}R^2. \tag{2.7}$$

Set

$$h(\lambda) = \beta\lambda^{m-1} - \alpha\lambda^{p-2}R^{\frac{4(m-p+1)}{m-1}}, \tag{2.8}$$

and for some $\gamma > 0$, let

$$\lambda_0 = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}[R^4 + \gamma R^2]^{\frac{1}{m-1}}, \tag{2.9}$$

we obtain,

$$h(\lambda_0) = \beta(\frac{\alpha}{\beta})^{\frac{m-1}{m-p+1}}(R^4 + \gamma R^2) - \alpha(\frac{\alpha}{\beta})^{\frac{p-2}{m-p+1}}R^4(1 + \frac{\gamma}{R^2})^{\frac{p-2}{m-1}}.$$

Using the fact that $(1+\frac{\gamma}{R^2})^{\frac{p-2}{m-1}} \leq 1+\frac{p-2}{m-1}\frac{\gamma}{R^2}$, we get

$$h(\lambda_0) \geq \gamma R^2 \alpha (\frac{\alpha}{\beta})^{\frac{p-2}{m-p+1}} (\frac{m-p+1}{m-1}).$$

Then, it is easily checked that (2.7) holds for λ_0 if

$$\gamma \geq \frac{1}{\alpha}(\frac{\beta}{\alpha})^{\frac{p-2}{m-p+1}}(\frac{m-1}{m-p+1})(\frac{4n}{m-1}+8\frac{m+1}{(m-1)^2}). \tag{2.10}$$

□

Remark 2.1. Let us observe that for every $\lambda \geq \lambda_0$ given in (2.9), the function $v_\lambda = \frac{\lambda}{(R^2-r^2)^{\frac{2}{m-1}}}$ is a supersolution of (2.5). Indeed, since the function h defined in (2.8) is nondecreasing on $[(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}} R^{\frac{4}{n-1}}, \infty)$, then for every $\lambda \geq \lambda_0$, (2.7) holds provided γ satisfying (2.10). Moreover, we have for $\lambda \geq \lambda_0$

$$v_\lambda \geq v_{\lambda_0} = \frac{(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}[R^4+\gamma R^2]^{\frac{1}{m-1}}}{(R^2-r^2)^{\frac{2}{m-1}}} \geq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}. \tag{2.11}$$

Proposition 2.3. *Let $x_0 \in \mathbb{R}^n$ and let u be subsolution of* (2.1) *on* $B(x_0, R)$ *for some $R > 0$. Then*

$$u(x_0) \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}(1+\frac{\gamma}{R^2})^{\frac{1}{m-1}}.$$

Proof. Let $\lambda_0 = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}[R^4+\gamma R^2]^{\frac{1}{m-1}}$ given in (2.9), then the function v_{λ_0} is a sursolution of (2.1) on $B(x_0, R)$ satisfying $\lim_{x\to z} v_{\lambda_0}(x) = +\infty \geq \limsup_{x\to z} u(x)$, $\forall z \in S(x_0, R)$. On the other hand, due to equation (2.11) we have, $v_{\lambda_0} \geq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$. Since the function f defined by $f(y) = \alpha y^{p-1} - \beta y^m$ is decreasing on $[(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}, +\infty[$, it follows from Corollary 2.1 that

$$u(x_0) \leq v_{\lambda_0}(x_0) = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}(1+\frac{\gamma}{R^2})^{\frac{1}{m-1}}.$$

□

An easy consequence of Proposition 2.3 is the following corollary.

Corollary 2.2. *If u is a subsolution of the problem* (2.1) *on* $\mathbb{R}^n$, *then* $u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$.

Now, we are able to prove our first two theorems.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. First, we remark from conditions (1.2) that 0 is a solution of (1.1). Let u be a solution of the problem (1.1). By means of Proposition 2.1, we get that $\widetilde{u} = u \vee 0$ is also a subsolution of (1.1). Using (1.2), we get $\widetilde{u}$ is a subsolution of (2.1). Thus, we obtain from Corollary 2.2 that $\widetilde{u} \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$ which leads to $u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$.

The assumption f is odd implies that if u is a solution of (1.1) then $-u$ is also a solution of (1.1), hence $-u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$. This proves Theorem 1.1. □

Proof of Theorem 1.2. By Theorem 1.1, we know that $|u| \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$. There exist $R > 0$ and $\gamma > 0$ such that u is non constant on $B(x_0, R)$ and $(\frac{\alpha}{\beta})^{\frac{1}{m-p+1}} \geq u > \gamma$ in $B(x_0, R)$. For the convenience of the proof, take $x_0 = 0$, $B = B(0, R)$, and set $v = u^2$. Then

$$(\frac{\alpha}{\beta})^{\frac{2}{m-p+1}} \geq v \geq \gamma^2 \text{ in } B. \tag{3.1}$$

In the sequel, for $0 \leq t \leq R$, we denote by $S(0,t)$ the boundary of the ball centered at 0 of radius t, let $d\sigma_t$ denotes the surface measure on $S(0,t)$, $d\sigma$ denotes the surface measure of $S(0,1)$ and let $\sigma_n = \sigma(S(0,1))$. So

$$\sigma_t(S(0,t)) = \sigma_n t^{n-1}, \;\; t > 0.$$

We denote by $\mu(t)$ the mean value of v, i.e.

$$\mu(t) = \frac{1}{\sigma_n t^{n-1}} \int_{S(0,t)} v(y) d\sigma_t(y) = \frac{1}{\sigma_n} \int_{S(0,1)} v(t\sigma) d\sigma.$$

We notice that,

$$\triangle v = 2|\nabla u|^2 + 2u\Delta u \geq 2u(f_2(u) - f_1(u)). \tag{3.2}$$

From (1.2), we have

$$u(f_2(u) - f_1(u)) \geq \beta|u|^{m+1} - \alpha|u|^p = \beta|u|^p(v^{\frac{m-p+1}{2}} - \frac{\alpha}{\beta}). \tag{3.3}$$

Estimations (3.1), (3.2) and (3.3) lead to

$$\triangle v \geq 2\beta\gamma^p \Big[v^{\frac{m-p+1}{2}} - \frac{\alpha}{\beta} \Big] \text{ in } B. \tag{3.4}$$

Hence, for $0 \leq r \leq R$,

$$\frac{1}{\sigma_n} \int_{B(0,r)} \Delta v(y) dy \geq 2\beta\gamma^p \Big[\frac{1}{\sigma_n} \int_{B(0,r)} v^{\frac{m-p+1}{2}}(y) dy - \frac{\alpha}{\beta\sigma_n} \int_{B(0,r)} dy \Big]$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

$$\geq 2\beta\gamma^p\Big[\int_0^r t^{n-1}\Big(\frac{1}{\sigma_n}\int_{S(0,1)} v^{\frac{m-p+1}{2}}(t\sigma)\,d\sigma\Big)\,dt - \frac{\alpha}{\beta\sigma_n}\int_0^r t^{n-1}\int_{S(0,1)} d\sigma dt\Big].$$

Using Hölder inequality, we obtain

$$\int_{S(0,1)} v d\sigma(y) \leq \Big[\int_{S(0,1)} v^{\frac{m-p+1}{2}}\,d\sigma(y)\Big]^{\frac{2}{m-p+1}}\Big(\sigma_n\Big)^{\frac{m-p-1}{m-p+1}},$$

which yields that,

$$\frac{1}{\sigma_n}\int_{S(0,1)} v^{\frac{m-p+1}{2}}(y)d\sigma(y) \geq \Big[\mu(t)\Big]^{\frac{m-p+1}{2}}. \tag{3.5}$$

On the other hand, by means of Green formula, we have

$$\frac{1}{\sigma_n}\int_{B(0,r)} \Delta v(y)dy = \frac{1}{\sigma_n}\int_{S(0,r)} \frac{\partial v(y)}{\partial n}d\sigma_r(y) = \frac{r^{n-1}}{\sigma_n}\int_{S(0,1)} \nabla v(r\sigma).\sigma d\sigma$$
$$= r^{n-1}\mu^{'}(r).$$

Combing equations (3.1), (3.5) and the last equality, we get

$$r^{n-1}\mu^{'}(r) \geq 2\beta\gamma^p\Big[\int_0^r t^{n-1}\Big(\mu(t)^{\frac{m-p+1}{2}} - \frac{\alpha}{\beta}\Big)dt\Big]. \tag{3.6}$$

Now, we consider the function F defined by

$$F(r) = \int_0^r \Big(\frac{\alpha}{\beta} - \mu(t)^{\frac{m-p+1}{2}}\Big)\,dt,\ \ 0 \leq r \leq R. \tag{3.7}$$

By (3.1), we have $\mu(t) \leq (\frac{\alpha}{\beta})^{\frac{2}{m-p+1}}$ for $0 \leq t \leq R$.

Hence F is a nonnegative function satisfying $F(0) = 0$. Similarly the function $\Big(F'(r) = \Big(\frac{\alpha}{\beta} - \mu(r)^{\frac{m-p+1}{2}}\Big)\Big)$ is nonnegative and $F'(0) = 0$, from the assumption $v(0) = (\frac{\alpha}{\beta})^{\frac{2}{m-p-1}}$. Furthermore, for $r \in]0, R[$, we have $F^{''}(r) = -\frac{m-p+1}{2}\mu'(r)\mu(r)^{\frac{m-p-1}{2}}$. Using inequality (3.6), we get

$$-\frac{m-p-1}{2}r^{n-1}\mu^{'}(r)\mu(r)^{\frac{m-p-1}{2}} \leq (m-p-1)\beta\gamma^p\mu(r)\Big(\int_0^r t^{n-1}\Big(\frac{\alpha}{\beta} - \mu(t)^{\frac{m-p+1}{2}}\Big)dt\Big). \tag{3.8}$$

The fact that $\mu(r) \leq (\frac{\alpha}{\beta})^{\frac{2}{m-p+1}}$ leads to

$$F^{''}(r) \leq \delta\int_0^r \frac{t^{n-1}}{r^{n-1}}\Big(\frac{\alpha}{\beta} - (\mu(t))^{\frac{m-p+1}{2}}\Big)dt \leq \delta\int_0^r \Big(\frac{\alpha}{\beta} - (\mu(t))^{\frac{m-p+1}{2}}\Big)dt,$$

where $\delta = (m-p+1)\beta\gamma^p(\frac{\alpha}{\beta})^{\frac{m-p-1}{m-p+1}}$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

Hence,

$$F''(r) \leq \delta F(r). \tag{3.9}$$

Since $\Big(F' \geq 0,\ F(0) = F'(0) = 0\Big)$, equation (3.9) implies

$$F'(r) \leq \delta F(r), \quad \forall\ 0 \leq r \leq R.$$

Hence

$$\Big(\frac{F(r)}{e^{\delta r}}\Big)' = \frac{1}{e^{2\delta r}}\Big(F'(r)e^{\delta r} - \delta F(r)e^{\delta r}\Big) \leq 0, \quad \forall\ 0 \leq r \leq R.$$

Consequently, we deduce $\frac{F(r)}{e^{\delta r}} \leq 0$, which means that $F(r) = 0$, for all $0 \leq r \leq R$. Using (3.7), we conclude that

$$\mu(t) = \Big(\frac{\alpha}{\beta}\Big)^{\frac{2}{m-p+1}}, \ \ \forall\ t \in [0, R].$$

Since $\mu(t)$ is the mean value of v on $S(0,t)$, for all $0 \leq t \leq R$ and $v \leq \Big(\frac{\alpha}{\beta}\Big)^{\frac{2}{m-p+1}}$, we conclude that $v = \Big(\frac{\alpha}{\beta}\Big)^{\frac{2}{m-p+1}}$ on $B(0,R)$. Then $v = u^2$ is constant, which is a contradiction with our hypotheses. Thus, we must have $|u| < (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$.

Examples. 1) For $\alpha = \beta = 1$ and $p = 2$, we obtain the result of Zhao [16].

2) For the Ginzburg-Landau equation $\Delta u + u(1 - |u|^2) = 0$, if u is a global solution, then $|u| < 1$, see Brezis, Merle and Rivière [3], also [9].

4. Proof of Theorem 1.3

These lemmas are given in much more general form than is required in the proof of Theorem 1.3.

Lemma 4.1. (Comparison Principle) *Suppose that Ω is a bounded domain in $\mathbb{R}^n$, $\alpha(x)$ and $\beta(x)$ are continuous functions on Ω with $\|\alpha\| < \infty$ and $\beta(x)$ is non-negative and not identically zero. Let $u_1, u_2 \in C^2(\Omega)$ and satisfy*

$$\Delta u_1 + \alpha(x)u_1 - \beta(x)g(u_1) \leq 0 \leq \Delta u_2 + \alpha(x)u_2 - \beta(x)g(u_2), \quad x \in \Omega$$

and $\limsup_{x\to\partial\Omega}(u_2 - u_1) \leq 0$, where $g(u)$ is continuous and such that $g(u)/u$ is strictly increasing for u in the range $\inf_{\Omega}\{u_1, u_2\} < u < \sup_{\Omega}\{u_1, u_2\}$. Then $u_2 \leq u_1$ in Ω.

Lemma 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Suppose that α and β are smooth positive functions on $\overline{\Omega}$, and let μ_1 denote the* *first eigenvalue of $-\Delta u = \mu\alpha(x)u(x)$ on Ω under Dirichlet boundary conditions on $\partial\Omega$. Then the problem*

$$\begin{cases} \Delta v + \mu(\alpha v - \beta v^{m-p+2}) = 0 & on \quad \Omega, \\ v = 0 & on \quad \partial\Omega, \end{cases} \tag{4.1}$$

has a unique positive solution u_μ for every $\mu > \mu_1$, and $u_\mu(x) \longrightarrow (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$ uniformly on any compact subset of Ω as $\mu \longrightarrow \infty$.

For a proof, we can see Lemma 2.1 and Lemma 2.2 of [6].

Now, we are ready to prove Theorem 1.3.

Our proof of Theorem 1.3 relies essentially on a comparison principle for concave sublinear problems (Lemma 4.1) and involves boundary blow-up solutions. Note that our case is more general than the one considered by Du and Ma [6], since the comparison principle and Lemma 4.2 cannot directly be applied.

Proof of Theorem 1.3. Let $\lambda > 0$, and u be an arbitrary positive solution of (1.10) such that $\delta = \min_{\mathbb{R}^n} u > 0$.

We will show that $u(x_0) = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$ for x_0 an arbitrary point in $\mathbb{R}^n$. Let Ω be a smooth bounded domain containing x_0 and let us define

$$\tilde{u}_\mu(x) = u((x_0 + \mu(x - x_0)).$$

It is easily checked that $\tilde{u}_\mu$ satisfies

$$\begin{aligned} -\Delta\tilde{u}_\mu = -\mu^2\Delta\tilde{u}((x_0 + \mu(x - x_0)) &= \mu^2(\alpha\tilde{u}_\mu^{p-1} - \beta\tilde{u}_\mu^m) \\ &= \mu^2\tilde{u}_\mu^{p-2}(\alpha\tilde{u}_\mu - \beta\tilde{u}_\mu^{m-p+2}). \end{aligned}$$

By the estimation of u given in Theorem 1.1, we obtain

$$-\Delta\tilde{u}_\mu \geq \mu^2\delta^{p-2}(\alpha\tilde{u}_\mu - \beta(\tilde{u}_\mu)^{m-p+2}). \tag{4.2}$$

By Lemma 4.2, for μ large, the problem

$$\begin{cases} \Delta v + \mu^2\delta^{p-2}(\alpha v - \beta v^{m-p+2}) = 0 & \text{on} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.3}$$

has a unique positive solution v_μ such that $v_\mu(x) \longrightarrow (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$ uniformly on any compact subset of Ω as $\mu \longrightarrow \infty$.

Let K be a compact of Ω which contains x_0. By the comparison principle, we have $\tilde{u}_\mu \geq v_\lambda$ on Ω. So, we can write

$$u(x_0) = \tilde{u}_\mu(x_0) \geq v_\mu(x_0) \longrightarrow (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}.$$

That is $u(x_0) \geq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$. On the other hand, we have by Theorem 1.1 that $u \leq (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$, therefore we must have $u = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$ as required. □

5. Proofs of Theorems 1.4 and 1.5

In this section, we assume that f is odd, then from assumption (1.2) we have

$$xf(x) \leq \alpha|x|^p - \beta|x|^{m+1}, \quad \forall x \in \mathbb{R}.$$

Let us set $v = u^2$, $\gamma = (\frac{\alpha}{\beta})^{\frac{1}{m-p+1}}$ and $\delta = \frac{1}{2}\gamma^{m-1}\beta(m+1)$. Using Theorem 1.1, we observe that any solution u of (1.1) on $\mathbb{R}^n$, satisfies

$$uf(u) \leq \alpha|u|^p - \beta|u|^{m+1} \leq \alpha\gamma^p - \beta v^{\frac{m+1}{2}}. \tag{5.1}$$

On the other hand, we remark that the function h defined by

$$h(y) = \alpha\gamma^p - \beta y^{\frac{m+1}{2}} - \delta(\gamma^2 - y)$$

is nondecreasing on $[0, \gamma^2]$ and $h(\gamma^2) = 0$. Since $v \in [0, \gamma^2]$, (5.1) allows us to get

$$uf(u) \leq \delta(\gamma^2 - v), \text{ for every solution } u \text{ of (1.1)}. \tag{5.2}$$

To prove Theorem 1.4, we introduce the following proposition.

Proposition 5.1. *Let $x \in \mathbb{R}^n$ and u be a solution of* (1.1). *Then for every $r \in]0, 1[$, there exists $C > 0$ such that*

$$\frac{1}{\sigma_n r^{n-1}} \int_{S(x,r)} (\gamma - |u(y)|)\, d\sigma_r(y) \leq C(\gamma - |u(x)|). \tag{5.3}$$

Proof. Note that from (5.2)

$$\triangle v = \Delta u^2 = 2|\nabla u|^2 + 2u\triangle u \geq -2uf(u) \geq -2\delta(\gamma^2 - v).$$

Hence

$$\triangle(\gamma^2 - v) \leq 2\delta(\gamma^2 - v). \tag{5.4}$$

It follows that the function $(\gamma^2 - v)$ is superharmonic for the linear operator $L = \triangle - 2\delta$. Let H (resp. $\widetilde{H}$) denotes the harmonic kernel associated to $\triangle$ (resp. L). It is well known (see [10]), that there exists $C \geq 1$ such that for every $x \in \mathbb{R}^n$, $r \in]0, 1[$ and every nonnegative continuous function f on $S(x, r)$, we have

$$\frac{1}{C} H_{B(x,r)} f \leq \widetilde{H}_{B(x,r)} f \leq C H_{B(x,r)} f. \tag{5.5}$$

Next, we recall that

$$H_{B(x,r)} f(z) = r^{n-2} \int_{S(x,r)} f(y) \frac{r^2 - |z-x|^2}{|z-y|^n} \frac{d\sigma_r(y)}{\sigma_n r^{n-1}}, \quad z \in B(x, r).$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

So for $z = x$, and $f = \gamma^2 - v$ we obtain

$$H_{B(x,r)}(\gamma^2 - v)(x) = \int_{S(x,r)} (\gamma^2 - v(y)) \frac{d\sigma_r(y)}{\sigma_n r^{n-1}}. \tag{5.6}$$

On the other hand, since $(\gamma^2 - v)$ is L-superharmonic, we get that

$$\widetilde{H}_{B(x,r)}(\gamma^2 - v)(x) \le (\gamma^2 - v(x)). \tag{5.7}$$

So, combining (5.5), (5.6) and (5.7), we get

$$\int_{S(x,r)} (\gamma^2 - v(y)) \frac{d\sigma_r(y)}{\sigma_n r^{n-1}} \le C(\gamma^2 - v)(x). \tag{5.8}$$

Using the fact that $|u| \le \gamma$, we have

$$\gamma - |u(y)| = \gamma\Big[1 - \frac{|u(y)|}{\gamma}\Big] \le \gamma\Big[1 - \frac{u^2(y)}{\gamma^2}\Big] = \frac{1}{\gamma}(\gamma^2 - v(y)). \tag{5.9}$$

From (5.8) and (5.9), we obtain

$$\int_{S(x,r)} (\gamma - |u(y)|) \frac{d\sigma_r(y)}{\sigma_n r^{n-1}} \le \frac{1}{\gamma} \int_{S(x,r)} (\gamma^2 - v(y)) \frac{d\sigma_r(y)}{\sigma_n r^{n-1}} \le 2C\Big[\gamma - |u(x)|\Big].$$

□

Let us now consider $R \ge 0$. We denote by G the Green function on $B(x, R)$ associated to the Laplacian operator $\triangle$, i.e $\triangle G(x, y) = -\delta_x$ where δ_x is the Dirac measure in x. Then the following result holds.

Proposition 5.2. *Let $x \in \mathbb{R}^n$ and $R' > R > 0$. Then there exists $C = C(R) > 0$ such that*

$$\gamma - |u(y)| \le C(\gamma - |u(x)|) + C \int_{B(x,R')} G(y, z)(\gamma - |u(z)|)dz \tag{5.10}$$

for every $y \in B(x, R)$ and any solution u of (1.1).

Proof. Let u be a solution of (1.1) and set

$$h(y) = u(y) - \int_{B(x,R')} G(y, z) f(u(z))\, dz.$$

Since h is harmonic on $B(x, R')$ and $h = u$ on $S(x, R')$, we get $|h| \le \gamma$ on $B(x, R')$. On the other hand, using the fact that $(\gamma - h)$ is a nonnegative harmonic function, we get from Harnack inequality that

$$\gamma - h(y) \le C_1(\gamma - h(x)), \text{ for every } y \in B(x, R). \tag{5.11}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

So

$$\gamma - u(y) = (\gamma - h)(y) - I(y), \ \forall\, y \in B(x,R), \tag{5.12}$$

where

$$I(y) = \int_{B(x,R')} G(y,z) f(u(z))\, dz.$$

Hence, from (5.11) and (5.12) we obtain

$$\begin{array}{rcl} (\gamma - u)(y) & \leq & C(\gamma - h(x)) - I(y) \\ & \leq & C(\gamma - u(x) + I(x)) - I(y), \end{array} \tag{5.13}$$

for every $y \in B(x,R)$.

Arguing as in the proof of (5.2) we have

$$\alpha x^{p-1} - \beta x^m \leq \gamma x^{p-1} - \beta x^m \leq \delta'(\gamma - x), \quad \text{for every } x \in [0,\gamma], \tag{5.14}$$

where $\delta' = \beta(m)\gamma^{m-1}$.

First, we observe that the function $f \geq 0$ on $[0,\gamma]$. So if $u(z) \geq 0$, then

$$0 \leq f(u(z)) \leq \alpha\gamma^{p-1} - \beta|u(z)|^m \leq \delta'(\gamma - |u(z)|).$$

Since f is odd and satisfies (1.2), if $u(z) \leq 0$ we have $0 \leq |f(u(z))| = f(-u(z)) \leq \delta'(\gamma - |u(z)|)$, which yields that

$$I(y) \leq C \int_{B(x,R')} G(y,z)(\gamma - |u(z)|) dz, \quad \forall y \in B(x,R'). \tag{5.15}$$

Finally, combining (5.13) and (5.15) we get

$$\gamma - u(y) \leq C(\gamma - u(x)) + C \int_{B(x,R')} G(y,z)(\gamma - |u(z)|) dz.$$

Since $(-u)$ is a solution of (1.1), we get also the same inequality with $-u$ which completes the proof. □

Now, we are able to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. First we suppose $R \in (0,1)$, then we choose R' such that $0 < R < R' < 1$. It is obvious that to get the proof of the theorem, we should give an estimation of $\int_{B(x,R')} |G(y,z)|(\gamma - |u(z)|)\, dz$. For that purpose, we recall that in $\mathbb{R}^2$ the Green function of $B(x,R')$ is given by the following expression

$$G(y,z) = \frac{1}{2\pi} \log \frac{\left||y-x|(z-x) - \frac{R'^2}{|y-x|}(y-x)\right|}{R'|z-y|}, \ z \neq x, \ z \neq y.$$

$$= \frac{1}{2\pi} \log \frac{|R'^2 - (y-x)\overline{(z-x)}|}{R'|z-y|},$$

where for $t = (t_1, t_2) \in \mathbb{R}^2$, $t = t_1 + it_2$ and $\overline{t} = t_1 - it_2$, $t\overline{t} = t_1^2 + t_2^2$. So, we can also write

$$G(z,y) = \frac{1}{4\pi} \log(1 + \frac{(R'^2 - |y-x|^2)(R'^2 - |z-x|^2)}{R'^2|z-y|^2}). \tag{5.16}$$

Now, using (5.16), we see that for every $z \in S(x,t)$ and $y \in B(x,R')$, we have

$$G(y,z) \leq \frac{1}{4\pi} \log(1 + \frac{R'^2}{||y-x|-t|^2}).$$

Hence

$$\int_{B(x,R')} |G(y,z)|(\gamma - |u(z)|)\, dz = \int_0^{R'} t(\int_{S(x,t)} G(y,z)(\gamma - |u(z)|) \frac{d\sigma_t(z)}{2\pi t}) dt,$$

from (5.3) we get

$$|I(y)| \leq C(\gamma - |u(x)|) \int_0^{R'} \log(1 + \frac{R'^2}{||y-x|-t|^2}) dt \tag{5.17}$$

for every $y \in B(x,R')$.

Next, we will estimate $\int_0^{R'} \log(1 + \frac{R'^2}{||y-x|-t|^2}) dt$.

Let $0 \leq m \leq R'$, then

$$\int_0^{R'} \log(1 + \frac{R'^2}{|m-t|^2}) dt \leq \int_0^{R'} \log(R'^2 + |m-t|^2) dt - \frac{1}{2} \int_0^{R'} \log(|m-t|) dt$$

$$\leq R' \log(2R'^2) - \frac{1}{2}\big[m \log(m) - R' + (R'-m)\log(R'-m)\big].$$

It is obvious that there exists a constant $C > 0$, such that

$$0 \leq \int_0^{R'} \log(1 + \frac{R'^2}{|m^2 - t^2|}) dt \leq C$$

for every $m \in [0, R']$. It follows

$$\int_{B(x,R')} |G(y,z)|(\gamma - |u(z)|)\, dz \leq C(\gamma - |u(x)|) \tag{5.18}$$

for every $y \in B(x,R')$. Finally, combining equations (5.10) and (5.18) we obtain

$$\gamma - |u(y)| \leq C(\gamma - |u(x)|), \quad \forall y \in B(x,R).$$

It is also easy to establish the result for any $R > 0$ by using the fact $B(x,R) \subset \bigcup_{i=1}^k B(x_i, R_i), \quad R_i \in (0,1)$.

Proof of Theorem 1.5. First, we suppose that $R \in (0,1)$ and we choose $0 < R < R' < 1$, we should estimate

$$\int_{B(x,R')} |G(y,z)|(\gamma - |u(z)|)\, dz.$$

Let $q' > 1$ such that $\frac{1}{q} + \frac{1}{q'} = 1$. We notice that when $n \geq 3$,

$$\int_{B(x,R')} (G(y,z))^{q'}\, dz \leq C \int_{B(x,R')} \frac{dz}{\|y-z\|^{q'(n-2)}} \leq C \int_0^{R'} \frac{1}{t^{1-n+q'(n-2)}} dt \leq C$$

where C is finite whenever $q > \frac{n}{2}$.

Now, from Hölder inequality, we get

$$\int_{B(x,R')} |G(y,z)|(\gamma - |u(z)|)\, dz \leq \Big[\int_{B(x,R')} (G(y,z))^{q'}\, dz\Big]^{\frac{1}{q'}} \Big[\int_{B(x,R')} (\gamma - |u(z)|)^q dz\Big]^{\frac{1}{q}}$$

$$\leq C\Big[\int_0^{R'} t^{n-1} \int_{S(x,t)} (\gamma - |u(z)|)^q \frac{d\sigma_t(z)}{\sigma_n t^{n-1}} dt\Big]^{\frac{1}{q}}.$$

Since,

$$\Big[\gamma - |u(z)|\Big]^q = \gamma^q \Big[1 - \frac{|u(z)|}{\gamma}\Big]^q \leq \gamma^q \Big[1 - \frac{|u(z)|}{\gamma}\Big] = \gamma^{q-1}\Big[\gamma - |u(z)|\Big],$$

then Proposition 5.1 implies

$$\int_{B(x,R')} |G(y,z)|(\gamma - |u(z)|)\, dz \leq C\Big[\gamma - |u(x)|\Big]^{\frac{1}{q}}. \tag{5.19}$$

On the other hand, due to the relation $0 < \frac{1}{q} < 1$, we get

$$\gamma - |u(x)| = \gamma(1 - \frac{|u(x)|}{\gamma}) \leq \gamma(1 - \frac{|u(x)|}{\gamma})^{\frac{1}{q}} \leq \gamma^{\frac{1}{p}}(\gamma - |u(x)|)^{\frac{1}{q}}. \tag{5.20}$$

Finally, by means of Proposition 5.2, (5.19) and (5.20) we get

$$\gamma - |u(y)| \leq C\Big[\gamma - |u(x)|\Big]^{\frac{1}{q}}, \quad \forall\, y \in B(x,R),$$

for some constant $C > 0$, as required. Similarly, the above estimation is easy to generalize for any $R > 0$. □

References

[1] A. Baalal, N. Belhaj Rhouma, Dirichlet problem for quasilinear elliptic equations, *EJDE*, **82** (2002), 1-18.

[2] H. Berestycki, F. Hamel, L. Rossi, Liouville type results for semilinear elliptic equations in unbounded domains, *Ann. Mat. Pura Appl.*, **4**, No. 3 (2007), 469-507.

[3] H. Brezis, F. Merle, T. Rivière, Quantization effects for $-\Delta u = u(1 - |u|^2)$ in $\mathbb{R}^2$, *Arch. Ration. Mech. Anal*, **126**, No. 1 (1994), 35-58.

[4] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.*, **47** (1994), 1457-1473.

[5] Y. Du, Z. Guo, Liouville type results and eventual flatness of positive solutions for p-Laplacian aquations, *Adv. Differ. Eq.*, **7** (2002), 1479-1512.

[6] Y. Du, L. Ma, Logistic type equations on $\mathbb{R}^n$ by a squeezing method involving boundary blow-up solutions, *J. Lond. Math. Soc.*, **64** (2001), 107-124.

[7] A. Farina, Symmetry for solutions of semilinear elliptic equationss in $\mathbb{R}^n$ and related conjectures, *Rend. Mat. App. Lincei*, **10**, No. 9 (1999), 255-265.

[8] D. Gilbarg, N.S. Trudinger, *Elliptic Partiel Differentiel Equations of second order*, Springer-Verlag, Berlin, NewYork (1983).

[9] R.M. Hervé, M. Hervé, Quelques proprietés des solutions de l'equation de Ginzburg-Landau sur un ouvert de $\mathbb{R}^2$, *Potential. Anal.*, **5** (1996), 591-609.

[10] H. Hueber, M. Sieveking, Uniform bounds for quotients of Green functions in $\mathcal{C}^{1,1}$-domain, *Ann. Inst. Fourier*, No. 1, **32** (1982), 105-117.

[11] L. Modica, A gradient bound and a Liouville theorem for nonlinear Poisson equations, *Commun. Pure Appl. Math.*, **38** (1985), 679-684.

[12] L. Peletier, J. Serrin, Gradient bounds and Liouville theorems for quasilinear elliptic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci*, **4**, No. 5 (1978), 65-104.

[13] J. Serrin, Entire solutions of nonlinear poisson equations, *Proc. Lond. Math. Soc.* **3**, **24** (1972), 348-366.

[14] J. Serrin, Liouville theorems for quasi-linear elliptic equations, In: *Atti del Conveguaderno Internazionale sui Metodi Valutative nella Fisica-Matematica* Rome (1975), 207-15.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 3, No. 1 (2011)

[15] J. Serrin, H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta Math.*, **189** (2002), 79-142.

[16] Z. Zhao, Qualitative properties of solutions for quasilinear elliptic equations, *EJDE*, **99** (2003), 1-18.