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ON THE EXISTENCE AND PROPERTIES OF THE POSITIVE DEFINITE SOLUTION OF THE MATRIX EQUATION $X = I + A^*\sqrt{X^{-1}}A$

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Abstract: The main goal of this paper is to discuss the matrix equation $X = I + A^*\sqrt{X^{-1}}A$. This type of the nonlinear matrix equations has many applications, such as in the analysis of ladder networks, optimal control theory, dynamic programming. An iterative method to find a positive definite solution is introduced. Special attention for necessary and sufficient conditions of the existence of a positive definite solution of the problem is implemented. Numerical examples are given to ensure the efficiency of the proposed method.

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1. Introduction

This paper is concerned with an iterative method to find a positive definite solution of the nonlinear matrix equation:

$$X - A^*F(X)A = I, \qquad X,\, A \in P(n), \tag{1}$$

with $F(\cdot)$ being a continuous operator from $P(n)$ to itself, where $P(n)$ denotes the set of all positive definite $n \times n$ matrices. This type of nonlinear matrix equations has many applications, such as in the analysis of ladder networks, optimal control theory,

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Andrenson et al [1], dynamic programming, stochastic filtering and statistics, Zhan et al [9]. Several authors (see [1], [3]-[5], [8]) have considered such a nonlinear matrix equation problem. Anderson et al [1] discussed the existence of the positive definite solution to the matrix equation (1) when $F(X) = X^{-1}$ and with right hand side an arbitrary matrix, while Engwerda et al [3] established and proved theorems for the necessary and sufficient conditions of existence of a positive definite solution of the matrix equation as in Andrenson et al [1]. They discussed both the real and complex case and established recursive algorithms to compute the largest and smallest solution of the equation. Also Engwerda [4] proved the existence of the positive definite solution of the real matrix equation (1) when the right hand side is the identity matrix, and also found the algorithm to calculate the solution. Salah [7], obtained necessary and sufficient conditions for existence of a positive definite solution of the matrix equation (1) with several forms of $F(\cdot)$, without any conditions on the equation, for example $F(X) = \sqrt{X}$.

We will introduce an iterative method which is suitable for obtaining a positive definite solution of (1). The following notations are used throughout the rest of the paper, the notation $A \geq 0$ ($A > 0$) is used to indicate that A is positive semi definite (positive definite), A^* denotes the complex conjugate transpose of A, and I is the identity matrix, and $A \geq B(A > B)$ is used as a different notation for $A - B \geq 0(A - B > 0)$. All the norms used in this paper are spectral norm. The sequence of matrices generated by every iteration is denoted by $\{X_n\}$ where $n \in N$.

2. Preliminaries

In this section we introduce some intermediate results which are interesting to suggest workable method for (1).

Definition 1. (see [2]) The square root of a matrix A is the matrix $A^{1/2}$ having the property that $A = A^{1/2}\,A^{1/2}$. If A and $A^{1/2}$ are both required to be positive definite or positive semi-definite then $A^{1/2}$ is unique, and the square root is a well-defined function.

Lemma 1. *Let A, B, $C \in C^{n\times n}$ and A, B, $C > 0$. In this case, if the unique solution of the equation $AX + XB = C$ is Hermitian, then it is positive definite.*

Lemma 2. (see [7]) *Let A, B, $C > 0$ and the unique solution of the equation $AXB = C$ is Hermitian, then it is positive definite.*

Lemma 3. *If $A \geq B > 0$, then we have $\sqrt{A} \geq \sqrt{B}(\sqrt{A}, \sqrt{B} > 0)$.*

Theorem 1. (see [6]) *Let the matrices A, B and $C \in P(n)$, such that the integral $\int_0^\infty e^{At}Ce^{Bt}dt$ exists and $\lim_{t\to\infty} e^{At}Ce^{Bt} = 0$, in this case the matrix $X = -\int_0^\infty e^{At}Ce^{Bt}dt$ is a solution of the matrix equation $AX + XB = C$.*

The main goal of this paper is to discuss the matrix equation (1) with $F(X) =$

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$\sqrt{X^{-1}}$.

3. Necessary and Sufficient Conditions

In this section we construct number of necessary and sufficient conditions of existence of the positive definite solution of the problem (1) with $F(X) = \sqrt{X^{-1}}$, i.e., this equation reduces to the following form:

$$X - A^*\sqrt{X^{-1}}A = I, \qquad X,\, A \in P(n). \tag{2}$$

Also, we obtain the rate of convergence to the iteration sequence of approximate solutions and a stopping criterion. The results in this section depend on two facts:

(I) The spectral norm is monotonic, i.e., if $0 < A < B$ then $||A|| < ||B||$.

(II) If the matrix sequence $\{X_n\}_{n=1}^{\infty}$ is a monotonically non-decreasing sequence and has upper bound, i.e., $X_k \le X_{k+1} \le C_1$ or is a monotonically non-increasing and has lower bound, i.e., $C_2 \le X_{k+1} \le X_k$ then this sequence is convergent [4] and the limit is positive definite if $X_k > 0, \quad k \in N$.

Theorem 2. (Main Theorem) *If there exists a real number $\alpha > 1$ such that:*

1. $\sqrt{\alpha}(\alpha - 1)I < A^*A$;
2. $\frac{\sqrt{\alpha}}{(\alpha-1)^2}(AA^*)^2 - A^*A > \sqrt{\alpha}I$;
3. $||A||^2 < 2\alpha\sqrt{\alpha}$;

then the equation (2) has a positive definite solution, and:

$$||X_{2k+1} - X_{2k}|| = q^{2k}||X_1 - X_0||, \qquad \textit{where} \quad q = ||A||^2/2\alpha\sqrt{\alpha}.$$

Proof. Take the initial matrix $X_0 = \alpha I$ and the iteration process:

$$X_{k+1} = I + A^*\sqrt{X_k^{-1}}A, \qquad \forall \quad k \in N. \tag{3}$$

The first elements of the sequence $\{X_k\}$ from (3) are:

$$\text{for } k = 0\text{: } X_1 = I + \frac{A^*A}{\sqrt{\alpha}} > I + (\alpha - 1)I = \alpha I = X_0.$$

This means that $X_1 > X_0$ and therefore $X_2 < X_1$. Also, we get:

$$\text{for } k = 1\text{: } X_2 = I + A^*\sqrt{X_1^{-1}}A = I + A^*\sqrt{(I + \frac{A^*A}{\sqrt{\alpha}})^{-1}}A,$$

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by using condition 2 of Theorem 2 we find that:

$$\frac{1}{\sqrt{\alpha}}A^*A + I < \frac{1}{(\alpha-1)^2}(AA^*)^2 \text{ then } \sqrt{\left(I + \frac{A^*A}{\sqrt{\alpha}}\right)^{-1}} > (\alpha-1)A^{-*}A^{-1}$$

and we have:

$$X_2 > \alpha I = X0, \text{ i.e., } X_2 > X_0 \text{ therefore, } X_0 < X_2 < X_1.$$

By the same way we can see that $X_1 > X_3$ and $X_3 > X_2$, ... hence:

$$X_0 < X_2 < X_3 < X_1.$$

In general the sequence $\{X_k\}$ satisfies the following inequality:

$$X_0 < X_{2k} < X_{2k+2} < X_{2r+3} < X_{2r+1} < X_1, \qquad \forall \quad k, r \in N. \tag{4}$$

Immediately by using (II) then we can show that the two sequences $\{X_{2k}\}$ and $\{X_{2k+1}\}$ are convergent to positive definite matrices. To prove that these two sequences have the same limit which is (positive definite) solution to (2), it is sufficient to prove that:

$$X_{2k+1} - X_{2k} \to 0, \text{ when } \quad k \to \infty.$$

Indeed, if we use (I) and Lemma 2 we get:

$$\begin{aligned} ||X_{2k+1} - X_{2k}|| &= \left\| A^* \left(\sqrt{X_{2k}^{-1}} - \sqrt{X_{2k-1}^{-1}} \right) A \right\| \\ &= \left\| A^* \sqrt{X_{2k}^{-1}} \left(\sqrt{X_{2k-1}} - \sqrt{X_{2k}} \right) \sqrt{X_{2k-1}^{-1}} A \right\| \\ &\le \frac{||A||^2}{\alpha} \left\| \sqrt{X_{2k-1}} - \sqrt{X_{2k}} \right\|. \end{aligned} \tag{5}$$

By using Theorem 1, the matrix $Y = \sqrt{X_{2k-1}} - \sqrt{X_{2k}}$ is the solution to the matrix equation $\sqrt{X_{2k-1}}\,Y + Y\sqrt{X_{2k}} = X_{2k-1} - X_{2k}$, this solution has the following form:

$$Y = \int_0^\infty e^{-\sqrt{X_{2k-1}}\,t} \left(X_{2k-1} - X_{2k}\right) e^{-\sqrt{X_{2k}}\,t} dt. \tag{6}$$

Because the integral exists and $e^{-\sqrt{X_{2k-1}}\,t}\left(X_{2k-1} - X_{2k}\right)e^{-\sqrt{X_{2k}}\,t} \to 0$ when $t \to \infty$,

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by substituting from (6) in (5), we get:

$$\begin{aligned}
||X_{2k+1} - X_{2k}|| &\leq \frac{||A||^2}{\alpha} ||\sqrt{X_{2k+1}} - \sqrt{X_{2k}}|| \\
&\leq \frac{||A||^2}{\alpha} \left\| \int_0^\infty e^{-\sqrt{X_{2k-1}}\,t} \left(X_{2k-1} - X_{2k}\right) e^{-\sqrt{X_{2k}}\,t} dt \right\| \\
&\leq \frac{||A||^2}{\alpha} \int_0^\infty \left\| e^{-\sqrt{X_{2k-1}}\,t} \right\| \|X_{2k-1} - X_{2k}\| \left\| e^{-\sqrt{X_{2k}}\,t} \right\| dt \\
&\leq \frac{||A||^2}{\alpha} \|X_{2k-1} - X_{2k}\| \int_0^\infty e^{-2\sqrt{\alpha}\,t} dt \\
&= \frac{||A||^2}{\alpha} \|X_{2k-1} - X_{2k}\| \frac{1}{2\sqrt{\alpha}},
\end{aligned}$$

this implies that:

$$||X_{2k+1} - X_{2k}|| \leq \frac{||A||^2}{2\alpha\sqrt{\alpha}} \|X_{2k-1} - X_{2K}\| . \tag{7}$$

Similarly, we have

$$||X_{2k+1} - X_{2k}|| \leq q^{2k} \parallel X_1 - X_0 \parallel, \quad \text{where} \quad q = ||A||^2/2\,\alpha\sqrt{\alpha}.$$

Now, if $||A||^2 < 2\alpha\sqrt{\alpha}$; then we get $q = ||A||^2/2\,\alpha\sqrt{\alpha} \in (0,1)$.

This proves that the two sequences $\{X_{2k}\}$ and $\{X_{2k+1}\}$, $\forall\ k \in N$ have the same positive definite bound X which is the solution of the matrix equation (2). This completes the proof of the theorem. □

Corollary 1. *By using (7), it is easy to prove that:*

$$||X_{2k+1} - X_{2k}|| \leq q^{2k} \| \frac{A^*A}{\sqrt{\alpha}} + (1-\alpha)I \| \quad \textit{and}$$

$$\max\left(||X_{2k} - X||, ||X - X_{2k+1}||\right) \leq q^{2k} \Big|\Big| \frac{A^*A}{\sqrt{\alpha}} + (1-\alpha)I \Big|\Big| \quad \textit{where} \quad q = \frac{||A||^2}{2\alpha\sqrt{\alpha}}.$$

Corollary 2. *If ϵ is the convergence tolerance, then the number of iterations is:*

$$k = \frac{ln(\epsilon) - ln \left\| \frac{1}{\sqrt{\alpha}} A^*A + (1-\alpha)I \right\|}{2\, lnq} \quad \textit{where} \quad q = \frac{||A||^2}{2\alpha\sqrt{\alpha}}.$$

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n	k	ϵ_k
4	22	3.9253869e-9
8	15	5.54345458e-9
12	12	2.49345521e-9
16	10	2.23839515e-9
20	8	7.10738956e-9
24	8	7.10738956e-9

Table 1

4. Numerical Results

In this section, we report some numerical examples. These numerical examples describe the performance of the algorithm. The tables indicate the convergence pattern of the iterative sequence of approximate solutions.

In the following examples we take $\alpha = 1.01$. In Tables 1-3, n denotes the order of matrices, k denotes the number of iterations and ϵ_k denotes $||X_k - X_{k-1}||$.

Example 1.

$$A = (a_{ij}) = \begin{cases} \dfrac{2(i+j+n)}{n^3}, & i \neq j; \\ \dfrac{2(2n+i)}{n^2}, & i = j. \end{cases}$$

Example 2.

$$A = (a_{ij}) = \begin{cases} \dfrac{i+j}{3n^2}, & i \neq j; \\ \dfrac{i+j}{3n}, & i = j. \end{cases}$$

Example 3.

$$A = (a_{ij}) = \mathrm{diag}(\frac{j}{2n+1}).$$

5. Conclusion and Remarks

In this paper we have studied the existence and uniqueness of a positive definite solution X for the matrix equation $X - A^*\sqrt{X^{-1}}A = I$. This equation is the generalization of the scalar equation $x - a^2\sqrt{x^{-1}} = 1$. Since the necessary and sufficient conditions for the existence of a positive definite solution are difficult to find by direct methods, for this we formulated an iterative algorithm for which a

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n	k	ϵ_k
4	12	9.8371904e-9
8	13	2.44436945e-9
12	13	2.69474842e-9
16	13	2.82598108e-9
20	13	2.21033578e-9
24	13	2.21033578e-9

Table 2

n	k	ϵ_k
4	7	3.15889914e-9
8	7	5.56134738e-9
12	7	6.76661149e-9
16	7	7.47469575e-9
20	9	9.78456078e-9
24	9	9.78456078e-9

Table 3

solution of the matrix equation is obtained. Of course, the nonlinear equation has in general more than one solution, but here we have concentrated on finding solvability conditions which can be easily verified. We show that whenever $||A||^2 < 2\alpha\sqrt{\alpha}$ then the equation (2) is always solvable.

The results in Tables 1-3 show that with this method the efficiency and the accuracy achieved are acceptable.

The algorithm was implemented in *C++* to calculate the number of iterations k and the values of $\epsilon_k = ||X_k - X_{k-1}||$ with different order n of a matrix A.

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