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ON ORDERING OF AG-GROUPOIDS

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Abstract: Total ordering plays an important role in the theory of semigroups. In this study we extend this characteristic to AG*-groupoids as: If S is an M-torsion free and cancellative AG*-groupoid with left identity e with quotient group T, then S admits a total order compatible with its operation if and only if T has a total order.

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1. Introduction

Following Denes and Keedwell [1], a groupoid S is said to be Abel-Grassmann's groupoid (AG-groupoid) if for all $a, b, c \in S$, $(ab)c = (cb)a$. This structure is also known as left almost semigroup (abbreviated as LA-semigroup), a generalized form of a commutative semigroup (see [4]). It is known that in an AG-groupoid S the medial property (i.e. $(ab)(cd) = (ac)(bd)$, for all $a, b, c, d \in S$) holds. By [4], an AG-groupoid S is said to be a weak associative AG-groupoid, denoted AG*-groupoid if it satisfies one of the equivalent conditions: (i) $(ab)c = b(ca)$; (ii) $(ab)c = b(ac)$, for all $a, b, c, \in S$. An AG-groupoid S is said to be a locally associative if $(aa)a = a(aa)$ for all $a \in S$, see [6]. It is fairly easy to see that every AG*-groupoid is locally associative. In an AG-groupoid S with left identity e, if $ab = cd$, then $ba = dc$ for all $a, b, c, d \in S$ (cf. [7, Theorem 2.7]).

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Inspiration by the usefulness of totally ordered semigroups, in this study we extend it to the AG*-groupoids with left identity e and established that: An M-torsion free and cancellative AG*-groupoid S with quotient group T, admits a total order compatible with its operation if and only if T has a total order.

The techniques we used in this paper are mainly inspired by [2].

2. Main Results

We begin initially by the following theorem which is a generalization of [2, Theorem 1.2].

Theorem 1. *If S is an AG*-groupoid with left identity e and C is a left cancellative subAG*-groupoid of S, then there exists an embedding $\phi : S \to T$, where T is an Abelian monoid such that:*

(1) $\phi(c)$ has an inverse $(\phi(c))^{-1}$ in T for all $c \in C$ and

(2) $T = \{(\phi(c))^{-1}\phi(s) : s \in S, c \in C\}$.

If $S = C$, then monoid T is an Abelian group.

Proof. Define a relation $\sim$ on $A = C \times S$ by $(c_1, s_1) \sim (c_2, s_2)$ if and only if $c_1 s_2 = c_2 s_1$. We claim that $\sim$ is an equivalence relation. Indeed, the relation $\sim$ is reflexive, as $cs = cs$ implies $(c, s) = (c, s)$. Clearly $\sim$ is symmetric as $(c_1, s_1) \sim (c_2, s_2)$ implies $c_1 s_2 = c_2 s_1$, i.e. $c_2 s_1 = c_1 s_2$ and hence $(c_2, s_2) \sim (c_1, s_1)$. Now suppose $(c_1, s_1) \sim (c_2, s_2)$ and $(c_2, s_2) \sim (c_3, s_3)$. This implies $c_1 s_2 = c_2 s_1$ and $c_2 s_3 = c_3 s_2$. Now using [6, Lemma 4], we have $c_2(c_1 s_3) = c_1(c_2 s_3) = c_1(c_3 s_2) = c_3(c_1 s_2) = c_3(c_2 s_1) = c_2(c_3 s_1)$. This implies that $c_1 s_3 = c_3 s_1$ and hence $(c_1, s_1) \sim (c_3, s_3)$ and therefore $\sim$ is transitive. If $(c_1, s_1) \sim (c_2, s_2)$, then $c_1 s_2 = c_2 s_1$. By [7, Theorem 2.7],it implies that $s_2 c_1 = s_1 c$. Now $(c_3 s_4)(s_2 c_1) = (c_3 s_4)(s_1 c_2)$ implies $(c_3 s_2)(s_4 c_1) = (c_3 s_1)(s_4 c_2)$ and so $(c_3 s_2, s_4 c_2) \sim (c_3 s_1, s_4 c_1)$ or $(c_3, s_4)(c_2, s_2) \sim (c_3, s_4)(c_1, s_1)$ or $(c_3, s_4)(c_1, s_1) \sim (c_3, s_4)(c_2, s_2)$. This implies $\sim$ is left compatible. Again if $(c_1, s_1) \sim (c_2, s_2)$, then $c_1 s_2 = c_2 s_1$ and by [7, Theorem 2.7], $s_2 c_1 = s_1 c_2$. Now $(s_2 c_1)(c_3 s_4) = (s_1 c_2)(c_3 s_4)$, using medial law we have $(s_2 c_3)(c_1 s_4) = (s_1 c_3)(c_2 s_4)$ and so $(c_1 s_4)(s_2 c_3) = (c_2 s_4)(s_1 c_3)$. This implies, $(c_1 s_4, s_1 c_3) \sim (c_2 s_4, s_2 c_3)$ or $(c_1, s_1)(c_3, s_4) \sim (c_2, s_2)(c_3, s_4)$. Hence $\sim$ is right compatible. Thus $\sim$ is compatible. Now $T = C \times S/\sim = \{[c, s] : c \in C, s \in S\}$ is the set of all equivalence classes of $C \times S$ under "$\sim$". T is a commutative monoid under the binary operation "$*$" defined by

$$[(c_1, s_1)] * [(c_2, s_2)] = [(c_1 c_2, s_2 s_1)] \in T.$$

Clearly T is closed. Now we show that $(T, *)$ is an AG-groupoid. For this consider

$$([(c_1, s_1)] * [(c_2, s_2)]) * [(c_3, s_3)] \quad = \quad [(c_1 c_2, s_2 s_1)] * [(c_3, s_3)]$$

$$\begin{aligned} &= [(c_1c_2, s_2s_1)] * [(c_3, s_3)] \\ &= [((c_1c_2)\, c_3, s_3\,(s_2s_1))] \\ &= [((c_3c_2)\, c_1, s_2\,(s_3s_1))]\,. \end{aligned}$$

Now take

$$\begin{aligned} ([c_3, s_3] * [c_2, s_2]) * [c_1, s_1] &= [(c_3c_2, s_2s_3)] * [(c_1, s_1)] \\ &= [((c_3c_2)\, c_1, s_1\,(s_2s_3))] \\ &= [((c_3c_2)\, c_1, s_2\,(s_3s_1))]\,. \end{aligned}$$

Thus $([c_1, s_1] * [c_2, s_2]) * [c_3, s_3] = ([(c_3, s_3)] * [(c_2, s_2)]) * [(c_1, s_1)]$. Hence $(T, *)$ is an AG-groupoid. Let $[(c_1, s_1)] \in T$, then consider

$$\begin{aligned} [(c_1, s_1)] * [(c, c)] &= [(c_1c, cs_1)] \\ &= \{(c_2, s_2) \in A : (c_1c, cs_1) \sim (c_2, s_2)\} \\ &= \{(c_2, s_2) \in A : (c_1c)\, s_2 = c_2\,(cs_1)\} \\ &= \{(c_2, s_2) \in A : c\,(c_1s_2) = c\,(c_2s_1)\} \\ &= \{(c_2, s_2) \in A : c_1s_2 = c_2s_1\} \\ &= \{(c_2, s_2) \in A : (c_1, s_1) \sim (c_2, s_2)\} \\ &= [(c_1, s_1)]\,. \end{aligned}$$

Hence $[(c, c)]$ is a right identity in T for all $c \in C$. Now since T is an AG-groupoid therefore by [7, Theorem 2.4] it becomes a commutative monoid. Now define $\phi : S \to T$ by $\phi(s) = [(c, cs)]$ for all $s \in S$. Let $s_1, s_2 \in S$ such that $s_1 = s_2$. It is easy to verify that ϕ is well-defined. Let $s_1, s_2 \in S$.

$$\begin{aligned} \phi(s_1s_2) &= [(c, c\,(s_1s_2))] \\ &= [((c_2c_1), (c_2c_1)\,(s_1s_2))]\,, \text{ where } c = c_2c_1 \in C. \\ &= [((c_2c_1), (c_2s_1)\,(c_1s_2))] = [((c_2c_1), (c_2s_1)\,(c_1s_2))]\,[(e, e)] \\ &= [((c_2c_1)\, e, e((c_2s_1)\,(c_1s_2)))] = [((c_2c_1)\, e, e((c_2s_1)\,(c_1s_2)))] \\ &= [((ec_1)\, c_2), (c_2s_1)\,(c_1s_2)] = [(c_1, c_1s_2)][(c_2, c_2s_1)] \\ &= [(c_2, c_2s_1)][(c_1, c_1s_2)] = [(c_1s_1, c_1)]\,[(c_2s_2, c_2)] \\ &= \phi(s_1)\,\phi(s_2)\,. \end{aligned}$$

Consider

$$\begin{aligned} \mathrm{Ker}\phi &= \{s \in S : \phi(s) \text{ is the identity of } T\} \\ &= \{s \in S : \phi(s) = [(c, c)]\} \\ &= \{s \in S : [(c, cs)] = [(c, c)]\} \\ &= \{s \in S : (c, cs) \sim (c, c)\} = \{s \in S : cc = c(cs)\} \end{aligned}$$

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$$\begin{aligned} &= \{s \in S : cc = (cc)s\} = \{s \in S : e = s\} \\ &= \{s \in S : s = e\} = \{e\}. \end{aligned}$$

Hence ϕ is one-one. Thus $\phi : S \to T$ is an embedding. Now if $c \in C$, then $\phi(c) = \left[\left(c, c^2\right)\right]$ has an inverse $(\phi(c))^{-1} = \left[\left(c^2, c\right)\right] \in T$. Indeed, $\phi(c)(\phi(c))^{-1} = \left[c, c^2\right]\left[\left(c^2, c\right)\right] = \left[\left(c.c^2, c.c^2\right)\right] = [(c_1, c_1)]$, where $c_1 = c.c^2 \in C$ and $[(c_1, c_1)]$ is an identity in T. Now an arbitrary element $[(s, c)]$ in T can be written as

$$\begin{aligned} (\phi(c))^{-1}\phi(s) = [(c, cs)]\left[\left(c^2, c\right)\right] &= \left[\left(cc^2, c(cs)\right)\right] = \left[\left(cc^2, (cc)s\right)\right] \\ &= \left[\left(cc^2, c^2 s\right)\right] = [(c, s)]\left[\left(c^2, c^2\right)\right] = [(c, s)]. \end{aligned}$$

As T is commutative, so $(\phi(c))^{-1}\phi(s) = \phi(s)(\phi(c))^{-1} = [(c, s)]$. If $S = C$, then every element of T is invertible. Consider $[(c, s)][(s, c)] = [(cs, cs)] = \left[\left(c^2, c^2\right)\right] = [(c_1.c_1)]$, which is an identity in T. Hence T is an Abelian group. □

By [8], a semigroup S is said to be M-torsion free if for all $x, y \in S$ there exists $1 \leq m \in M \subseteq \mathbb{Z}^+$ with $x^m = y^m$, then $x = y$ (see [8, p. 332]).

Now in the following we extend [8, p. 332] for an AG*-groupoid with left identity e.

Definition 1. Let $(S, *)$ be an AG*-groupoid with left identity e, then S is said to be M-torsion free if for all $x, y \in S$ there exist $1 \leq m \in M \subseteq \mathbb{Z}^+$ with $x^m = y^m$, then $x = y$.

Example 1. Take AG*-groupoid $(Q^+, *)$, with left identity 1 in which the binary operation $*$ defined as $a * b = b.a^{-1}$. $(Q^+, *)$ is an O-torsion free, where O is the set of odd positive integers. In particular for $m = 3$, take $x^3 = y^3$ and by locally associative property we have $x^2 * x = y^2 * y$. Now as for all $x \in Q^+$, $x^2 = 1$, so $1 * x = 1 * y$. This implies $x = y$. Hence $(Q^+, *)$ is O-torsion free AG*-groupoid. Similarly $(\mathbb{Z}, \circ)$ is an O-torsion free, where O is the set of odd positive integers, AG*-groupoid with left identity 0 defined as $a \circ b = b - a$.

Lemma 1. *Let $(S, *)$ be an AG*-groupoid with left identity e. If $\leq$ is total order on S compatible with $*$, then S is M-torsion free and cancellative.*

Proof. Let $a, b \in S$ and say $a < b$ (that is $a \leq b$ and $a \neq b$). If $a < b$, this implies $a * x < b * x$ for all $x \in S$. Since $\leq$ is compatible with respect to $*$, this implies S is cancellative.

Now if $a < b$, then $a * a < a * b ... (1)$ and $a * b < b * b ... (2)$. It further implies that $a * a < a * b < b * b$. From (1), we have $(a * a) * a < (a * b) * a$ and from (2), we can say $(a * b) * b < (b * b) * b$. Now for $a < b$, the compatibility of $*$ implies that $(a * b) * a < (a * b) * b$ and hence $(a * a) * a < (a * b) * b < (b * b) * b$.

Continuing this process for m-times, where m is minimal in the set M, we have $a^m < ... < b^m$. This implies $a^m < b^m$ for some $m \in M$. Hence $(S, *)$ is M-torsion free. □

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The following theorem establishes a relation between an AG*-groupoid and its quotient group.

Theorem 2. *Let T be the quotient group of a cancellative AG*-groupoid S with left identity e. Then T is M-torsion free if and only if for all $x, y \in S$, $x^n = y^n$ implies $x = y$, where $n \in M \subseteq Z^+$.*

Suppose $T = C \times S/\sim$ is torsion free. This implies $[(x, x)]$ is only element of $C \times S/\sim$ of finite order. So, $[(x, x)]^n = [(x, x)]$. Assume that $x^n = y^n$, where $n \in M \subseteq Z^+$. Then $x.x^n = x.y^n$. So by power associativity of S, we have $x^{1+n} = x.y^n$ or $x^n.x = x.y^n$. This implies $(x^n, y^n) \sim (x, x)$ or $[(x, y)]^n = [(x, x)]$ and hence it implies $x = y$. Now conversely suppose that for all $x, y \in S$, $x^n = y^n$ implies $x = y$. Let $[x, y] \in C \times S/\sim$ such that $[(x, y)]^n = [(x, x)]$. This implies $(x^n, y^n) \sim (x, x)$ and therefore $x^n.x = x.y^n$. So by power associativity in S, $x^{n+1} = x.y^n$ or $x.x^n = x.y^n$. This implies $x^n = y^n$ and so $x = y$. Thus $[(x, x)]^n = [(x, x)]$ and hence $T = C \times S/\sim$ is M-torsion free.

Theorem 3. *Let S be a M-torsion free cancellative AG*-groupoid with left identity e with quotient group T. Then S admits a total order compatible with its operation if and only if T has a total order.*

Proof. If T is totally ordered under $\leq$, then the relation $\leq$ induces a total order on S compatible with the AG*-groupoid operation. Conversely, if S is totally ordered under $\leq$, then we define a relation $\sim$ on T as follows:

each element of T is expressible in the form $c^{'}s$ for some $c, s \in S$ and $c^{'}$ is inverse of c. Now for $t_1 = c_1^{'}s_1$ and $t_2 = c_3^{'}s_3$ in T, we define $t_1 \sim t_2$ by $c_1^{'}s_1 \leq c_3^{'}s_3$. Now $c_1^{'}s_1 \leq c_3^{'}s_3$ and by def. of AG*-groupoid, $c_1(c_1^{'}s_1) \leq c_1(c_3^{'}s_3) \Longrightarrow (c_1^{'}c_1)s_1 \leq (c_3^{'}c_1)s_3$. It follows that $s_1 \leq (c_3^{'}c_1)s_3$ and $c_3s_1 \leq c_3((c_3^{'}c_1)s_3))$, so by [6, Lemma 4] $c_3s_1 \leq (c_3^{'}c_1)(c_3s_3) \Longrightarrow c_3s_1 \leq (c_3^{'}c_3)(c_1s_3)$ or $c_3s_1 \leq e(c_1s_3)$ or $c_3s_1 \leq c_1s_3$.

Then $\sim$ is a well defined relation of total order on T that is consistent with the group operation on T and for the restriction of the relation $\leq$ on S, we just to verify that $\sim$ is well defined and that it agrees with the relation $\leq$ on S.

Thus, if $t_1 = c_1^{'}s_1 = c_2^{'}s_2$ and $t_2 = c_3^{'}s_3 = c_4^{'}s_4$, where $c_3s_1 \leq c_1s_3$, then

$$(c_3s_1)(c_2s_4) \leq (c_1s_3)(c_2s_4). \tag{1}$$

Now for the values of c_2 and s_4, we consider $c_1^{'}s_1 = c_2^{'}s_2$, then by cancellativity we have $(c_1^{'}s_1)s_2^{'} = (c_2^{'}s_2)s_2^{'} = (s_2^{'}s_2)c_2^{'} = ec_2^{'}$. So, $(c_1^{'}s_1)s_2^{'} = c_2^{'}$ and $((c_1^{'}s_1)s_2^{'})^{'} = (c_2^{'})^{'}$ implies that $(c_1s_1^{'})s_2 = c_2$. Now for s_4, consider $c_3^{'}s_3 = c_4^{'}s_4$. Then $c_4(c_3^{'}s_3) = c_4(c_4^{'}s_4)$ and by [6, Lemma 4], we have $c_4(c_3^{'}s_3) = (c_4^{'}c_4)s_4 = s_4$. Now by repeated use of definitions of AG-groupoid, AG*-groupoid and medial law in (1), it can be easily verified that if $(c_3s_1)(c_2s_4) \leq (c_1s_3)(c_2s_4)$, then $(c_4s_2) \leq (c_2s_4)$ and hence $\sim$ is well defined. Define $\phi : S \to T$ by $\phi(s) = c^{'}(cs)$, where $c^{'}$ is inverse of $c \in S$. Then ϕ is an embedding. Indeed

$$\phi(s_1s_2) = c^{'}(c(s_1s_2)) = (cc^{'})(s_1s_2) = e(s_1s_2) = s_1s_2$$
$$= [(cc^{'})s_1][(cc^{'})s_2] = [c^{'}(cs_1)][c^{'}(cs_2)] = \phi(s_1)\phi(s_2).$$

Now let $\phi(s_1) = \phi(s_2)$. This implies that $[c^{'}(cs_1)] = [c^{'}(cs_2)]$ or $(cc^{'})s_1 = (cc^{'})s_2$ and hence $s_1 = s_2$.

Hence for $s, t \in S$, we have $s \sim t$ if and only if $c(cs) \leq c(ct)$ if and only if $s \leq t$. □

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