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A NON-NOETHERIAN LASKERIAN DOMAIN IN WHICH EVERY PRIMARY IDEAL IS A VLUATION IDEAL

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Abstract: The purpose of this study is to characterize Laskerian domains with the property that every primary ideal is a valuation ideal.

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1. Introduction

Following [4, p. 275] an integral domain D is called Prufer domain if D_M is a valuation ring for each maximal ideal M of D. An ideal A of an integral domain D is said to be decomposable if A can be expressed as a finite intersection of primary ideals of D. A primary ideal Q of an integral domain D is said to be strongly primary if Q contains a power of its radical. An ideal A of D is strongly decomposable if A can be expressed as finite intersection of strongly primary ideals of D. Recall that an integral domain D is said to be Laskerian (respectively, strongly Laskerian) if each proper ideal of D is decomposable (respectively, strongly decomposable). The study of Laskerian rings originated from the work of Emanuel Lasker in 1905, where he introduced the notion of primary ideals and proved the primary decomposition theorem for an ideal of a polynomial ring in terms of primary ideals [8]. A Laskerian ring has a Noetherian spectrum, see [5, Theorem 4]. D has Noetherian spectrum if and only if every prime ideal is the radical of a finitely generated ideal, see [9, Corollary 2.4]. If D is Laskerian domain, then each ideal of D has only finitely many

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minimal prime divisors. Since a ring with later property has Noetherian spectrum if and only if the ascending chain condition for prime ideals is satisfied in D, see [9]. Whereas an integral domain D is Noetherian if every ascending chain of ideals terminates.

An ideal A of a domain D is a valuation ideal if there exist a valuation ring $D_v \supset D$ and an ideal A_v of D_v such that $A_v \cap D = A$. When we want to specify the particular valuation ring D_v, we shall say A is a v-ideal. If A is a v-ideal, then $AD_v \cap D = A$. By [4, p. 304, Exercise 6] if D is a Prufer domain then each primary ideal of D is a valuation ideal. Following [4, p. 305, Exercise 7] if D is Noetherian and each primary ideal of D is a valuation ideal, then D is a Prufer domain. In [10] authors have proved that a Laskerian integral domain in which every primary ideal is a valuation ideal is a Prufer domain.

The purpose of this study is to characterize the Laskerian domains with the property that every primary ideal is a valuation ideal. An integral domain D is said to satisfy the ascending chain condition for prime ideals provided any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset ...$ stabilizes. This is equivalent to saying that every non empty family of prime ideals contains a maximal element. If spec(D) is Noetherian, then radical ideals of D have ACC. Here we observe that a Laskerian domain with the property that each primary ideal is a valuation ideal forms a well behaved sub class of Prufer domains.

Lemma 1. *Let D be Laskerian integral domain. Then the following are equivalent:*

(1) Every primary ideal is a valuation ideal.

(2) D is Prufer and every proper prime ideal is maximal.

Proof. (2) $\Rightarrow$ (1) Holds even without hypothesis on D to be Laskerian, see [4, p. 304, Exercise 6].

(1) $\Rightarrow$ (2) Follows from [10, Lemma 1]. □

Definition 1. *If D is an integral domain with identity, D is almost Dedekind if D_M is a Noetherian valuation ring for each maximal ideal M of D. An almost Dedekind domain has dimension at most 1, and is, of course, a Prufer domain.*

Corollary 1. *Let D be Laskerian integral domain. If every primary ideal of D is a valuation ideal then every valuation ideal is primary.*

Proof. By [1] dim$D \leq 1$. By [4, p. 305, Exercise 9], every valuation ideal of D is a primary ideal. □

Corollary 2. *Let D be Laskerian domain. If every primary ideal of D is a valuation ideal, then prime ideals of D are linearly ordered.*

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Proof. Since D is Laskerian, then D has Noetherian spectrum. By [9] D satisfies ACC for prime ideals. Therefore, by [6, Theorem 3.4] the prime ideals of D are linearly ordered. □

Proposition 1. *Let D be a Laskerian domain such that each primary ideal of D is a valuation ideal then following are equivalent:*
(1) D is almost Dedekind.
(2) Primary ideals of D are prime powers.

Proof. (1) $\Rightarrow$ (2) Trivial.
(2) $\Rightarrow$ (1) By [1] nonzero proper prime ideals of D are maximal. Apply [7, Theorem 1]. □

Corollary 3. *Let D be a Laskerian domain, and suppose every primary ideal of D is a valuation ideal. Then D is a valuation ring.*

Proof. By Lemma 1, D is a Prufer domain with one maximal ideal M and hence $D = D_M$. □

Lemma 2. *Let D be a Laskerian domain. If every primary ideal is a valuation ideal. Then D is completely integrally closed.*

Proof. By Lemma 1, D is Prufer of dimension at most 1. Therefore by [4, p. 333, Exercise 23] D is completely integrally closed. □

Following [2, p. 1] a fractional ideal F of an integral domain D is a $D-$submodule of K where there exist $0 \neq d \in D$ such that $dF \subseteq D$. A fractional ideal F of D is a divisorial or v-ideal of D in case $F = F_v$; and F is an invertible ideal of D provided $FF^{-1} = D$.

Proposition 2. *Let D be a Laskerian domain, and suppose every primary ideal of D is a valuation ideal, then each non zero ideal of D is divisorial if and only if D is a Dedekind domain.*

Proof. Suppose that each nonzero ideal of D is divisorial. By Lemma 1 D is 1-dimensional Prufer, consequently D is CIC. Therefore by [4, Theorem 34.3] divisorial fractional ideals of D form a group under the operation $A * B = (AB)_v$ with D as an identity element. For every ideal A of D there exist a fractional ideal B of D such that $(AB)_v = D$. But $AB = (AB)_v$, so A is invertible. Hence D is Dedekind.

Conversely, in a Dedekind domain each nonzero ideal is invertible therefore divisorial. □

Remark 1. Combining [10, Proposition 2] and Proposition 2, see that for a Laskerian domain D, with the property that every primary ideal of D is a valuation ideal, each non zero ideal of D is divisorial even if D is an almost Dedekind domain.

It is well known that overrings of Prufer (almost Dedekind, Dedekind) domains are prufer (respectively almost Dedekind, Dedekind). If D is 1-dimensional Noetherian Prufer domain (Dedekind domain) then each overring of D is 1-dimensional Noetherian Prufer domain (Dedekind domain). We observe that if D is Laskerian Prufer domain then each overring of D is 1-dimensional Laskerian integrally closed (Laskerian Prufer).

Proposition 3. *Let D be Laskerian integral domain. If every primary ideal of D is a valuation ideal then each overring of D is 1-dimensional Laskerian integrally closed.*

Proof. By Lemma 1, D is 1-dimensional Prufer domain. Therefore each overring of D is 1-dimensional integrally closed Prufer. Furthermore by Corollary 3, D is a valuation ring. This ensures that each overring of D is a valuation ring. Therefore by [4, Exercise 9, p. 456] overrings of D are Laskerian. □

Remark 2. *Since D is one dimensional Prufer domain, D is almost Dedekind if and only if each non trivial valuation overring of D is strongly Laskerian.*

Theorem 1. *Assume every primary ideal of a domain D is a valuation ideal, then following are equivalent:*

(1) D is Laskerian.

(2) D is strongly Laskerian.

(3) D is Noetherian.

Proof. (1) $\Rightarrow$ (2) By Corollary 3, D is a valuation ring. To prove D strongly Laskerian it is sufficient to show that every primary ideal A of D is strongly primary. Let $a, b \in K^*$, where K^* is quotient field of D and $ab \in A$. Further suppose that $a \notin A$. Since D is a valuation domain, if $a \notin D$ then $a^{-1} \in D$ and we have $b = a^{-1}ab \in A$. Hence we may as well assume that $a \in D$. Since $a = b^{-1}ab \notin A$, it follows that $b \in D$. Now since $a, b \in D$ with A primary, we have $b^n \in A$ for some $n \geq 1$, hence A is strongly primary. Hence D is a strongly Laskerian domain.

(2) $\Rightarrow$ (3) By [4, Exercise 8, p. 456] in case of valuation ring strongly Laskerian and Noetherian properties coincide. Since D is a valuation ring – obvious.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1) Trivial. □

Following [2, p. 33] for an integral domain D with quotient field K and for nonzero ideal I we define $(I : I) = \{x \in K : xI \subseteq I\}$ and $I^{-1} = (D : I) = \{x \in K : xI \subseteq D\}$

Theorem 2. *Let D be Laskerian domain in which every primary ideal is a valuation ideal, and let I be non zero ideal of D. Assume I^{-1} is a ring. The following conditions are equivalent:*

(1) $I^{-1} = (I : I)$.

(2) $I = \text{Rad}\,(I)$.

The minimal prime ideals of I in $(I : I)$ are all maximal ideals.

Proof. By Lemma 1, D is Laskerian prufer domain. The rest follows from [2, Theorem 3.1.12, p. 42] □

Proposition 4. *Let D be Laskerian domain such that every primary ideal is a valuation ideal. Let I be non zero ideal of D. If I is a primary ideal that is not prime, then I^{-1} is not a ring.*

Proof. On contrary suppose I^{-1} is a ring. By [1] and [2, Lemma 3.1.13] $I^{-1} = (I : I)$. By [2, Theorem 3.1.12] $I =$Rad(I), which is a contradiction to the supposition that every primary ideal is prime. Hence I^{-1} is not a ring. □

2. The Transform Formula for Ideals

Following [2, p. 33] $T(A) = \cup_{n\geq 1}(D : A^n)$. An integral domain D is said to satisfy trace formula for ideals if for all ideals A and B if

$$T(AB) = T(A) + T(B).$$

It is well known that transform formula for finitely generated ideals hold in Prufer domains and holds for all ideals in Dedekind domains. We extend the previous result by weakening the Noetherian hypothesis as:

If D is a Laskerian domain in which every primary ideal is a valuation ideal (of course not Dedekind), then the transform formula holds for all ideals of D.

Before proving this we provide alternate proofs of the following well known lemmas.

Lemma 3. *Let D be Laskerian domain. Then each prime ideal of D is the radical of a finitely generated ideal.*

Proof. Let P be non zero prime ideal of D. Since D is Laskerian domain, D has ACC on prime ideals. This implies, there exists a prime ideal $Q \subset P$ such that each prime ideal of D properly contained in P is contained in Q. Let $p \in P \backslash Q$ and note that P is minimal over (p). Being Laskerian (p) has only finitely many primes. It follows from [2, Lemma 4.2.29] that P is the radical of a finitely generated ideal. □

Lemma 4. *Let D be Laskerian commutative ring, then radical of each ideal of D is equal to radical of some finitely generated ideal.*

Proof. On contrary assume there exist an ideal that is not radical of a finitely generated ideal. Let Γ be set of all such ideals in D, then Γ is inductive. Let P be maximal ideal in Γ. We show that P is a prime ideal of D. If P is not prime ideal of D, then there exist $x, y \in D \backslash P$ such that $xy \in P$. Take $A = (P, x)$ and $B = (P, y)$ and see that $A, B \notin \Gamma$. As Rad(AB) =Rad(A)Rad(B). On the other hand Rad(AB) =Rad(P), therefore $P \notin \Gamma$. This contradiction shows that P is a prime ideal. But by Lemma 3 each prime ideal is radical of some finitely generated ideal. This completes the proof. □

For the sake of definiteness we state and prove the following theorem.

Theorem 3. *If D is a Laskerian domain in which every primary ideal is a valuation ideal, then the transform formula holds for all ideals of D.*

Proof. Let A be an ideal of D. Since D is Laskerian, D has Noetherian spectrum. Therefore every ideal of D has finitely many minimal prime divisors, and hence $T(A) = T(\text{Rad}(A))$. Also by Lemma 4, for ideals A and B of D, there exist finitely generated ideals A_0 and B_0 such that Rad(A) =Rad(A_0) and Rad(B) =Rad(B_0).

$$\begin{aligned} T(AB) &= T(\text{Rad}(AB)) = T(\text{Rad}(A) \cap \text{Rad}(B)) \quad \text{(see [1, Exercise 1.13 (iii)])} \\ &= T(\text{Rad}(A_0) \cap \text{Rad}(B_0)) = T(\text{Rad}(A_0 B_0)) \\ &= T(A_0 B_0) = T(A_0) + T(B_0) \quad \text{(see Lemma 1 and [2, Theorem 4.5.4])} \\ &= T(A) + T(B). \end{aligned}$$

The proof is complete. □

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