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QUADRATIC PERTURBATIONS OF SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEMS FOR HYBRID ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: In this article, an existence theorem for quadratic perturbations of the second type for a periodic boundary value problem for second order ordinary differential equations is proved under mixed generalized Lipschitz and Carathéodory conditions. Existence results for extremal positive solutions are also proved for Carathéodory conditions as well as for discontinuous cases of nonlinearities under certain monotonicity conditions.

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1. Introduction

The study of nonlinear initial and boundary value problems for ordinary and partial differential equations is an important topic of nonlinear analysis that has been discussed widely by several nonlinear analysts over the course of time. The nonlinear differential equations are obtained from linear IVP or BVP by perturbing the unknown function nonlinearly. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, the real line, consider the BVP for first order linear ordinary differential equations,

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$$\left.\begin{aligned} &x'(t) = x(t) \text{ a.e. } t \in J, \\ &x(0) = x(T). \end{aligned}\right\} \tag{1}$$

Note that the unknown function x occurs on both sides of the first equation in (1) and it can be perturbed nonlinearly to obtain different nonlinear boundary value problems. A perturbation of the free unknown function on the right hand of BVP (1) is called a perturbation of the first type, whereas a perturbation of the unknown function under the derivative sign is called a perturbation of the BVP (1) of the second type.

Consider the following perturbation of the first type of the BVP (1)

$$\left.\begin{aligned} &x'(t) = f(t, x(t)) \text{ a.e. } t \in J, \\ &x(0) = x(T), \end{aligned}\right\} \tag{2}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$. The nonlinear BVP (2) is an implicit perturbation of the linear BVP (1) of first type. We mention that the BVP (1) is fundamental in the theory of first order nonlinear BVP and widely studied in the literature by several authors for different aspects of the solutions. Similarly, we have an implicit perturbation of the linear BVP (1) of second type,

$$\left.\begin{aligned} \frac{d}{dt}\big[f(t, x(t))\big] = x(t) \text{ a.e. } t \in J, \\ x(0) = x(T). \end{aligned}\right\} \tag{3}$$

Again, a mixed implicit perturbation of first and second type of the BVP (1) will be of the form

$$\left.\begin{aligned} \frac{d}{dt}\big[f(t, x(t))\big] = g(t, x(t)) \text{ a.e. } t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{4}$$

where $f, g : J \times \mathbb{R} \to \mathbb{R}$.

Details of different types of nonlinear perturbations for linear differential equations appear in Dhage [7]. In the same way, different types of perturbations for differential equations of different orders can be defined. In this paper, we prove existence of solutions as well as existence of extremal solutions for mixed quadratic perturbations of second order periodic boundary problems. The rest of the paper is organized as follows. In Section 2, we give the statement of the nonlinear second order differential equation. Section 3 deals with the auxiliary results that will be needed in the subsequent parts of the paper. The main existence result is given in Section 4, while a result on extremal solutions is given in Section 5.

2. Statement of the Problem

Given a closed and bounded interval $J = [0, 2\pi]$ in $\mathbb{R}$, consider the periodic boundary value problem (PBVP) for first order ordinary differential equations

$$\left.\begin{aligned} -\frac{d^2}{dt^2}\left[\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right] = g(t, x(t)) \quad \text{a.e. } t \in J, \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi), \end{aligned}\right\} \tag{5}$$

where $f : J \times \mathbb{R} \to \mathbb{R}_+ - \{0\}$, $k : J \times \mathbb{R} \to \mathbb{R}$, and $g : J \times \mathbb{R} \to \mathbb{R}$.

By a *solution* of PBVP (5) we mean a function $x \in AC^1(J, \mathbb{R})$ such that

(i) the function $t \mapsto \dfrac{d}{dt}\left[\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}\right]$ is absolutely continuous on J, and

(ii) x satisfies (5),

where $AC^1(J, \mathbb{R})$ is the space of continuous functions whose first derivative exists and is absolutely continuous on J.

When $f(t, x) = 1 = k(t, x)$ for all $t \in J$ and $x \in \mathbb{R}$, the PBVP (5) reduces to the PBVP

$$\left.\begin{aligned} -x''(t) = g(t, x(t)) \quad \text{a.e. } t \in J, \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi). \end{aligned}\right\} \tag{6}$$

where $g : J \times \mathbb{R} \to \mathbb{R}$. Note that PBVP (5) is a quadratic perturbation of second type for the PBVP (6) on the closed and bounded interval J. The PBVP (5) has been studied in several papers by many authors for different aspects of the solutions. See for example, Lakshmikantham and Leela [13], Leela [14], Nieto [15, 16], Yao [17], and the references therein. In this paper, we discuss the existence of solutions of PBVP (5) as well as the existence of extremal solutions. We do so under some suitable conditions on the nonlinearities f and g that yield generalizations of several results for PBVP (6) in the above mentioned papers. Our analysis relies on a nonlinear alternative of Leray-Schauder type (see Dhage [3, 5]) and an algebraic fixed point theorem in Banach algebras due to Dhage [3] . Our method of study is to convert the PBVP (5) into an equivalent integral equation and apply the hybrid fixed point theorems of Dhage [3, 4, 6, 7] under suitable conditions on the nonlinearities f and g.

In the following section we describe some basic tools from nonlinear functional analysis that will be used in subsequent parts of the paper.

3. Auxiliary Results

Let $B(J, \mathbb{R})$ denote the space of bounded real-valued functions defined on J and let $C(J, \mathbb{R})$ denote the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \quad \text{and} \quad (x \cdot y)(t) = x(t)y(t) \ \text{ for } \ t \in J.$$

Clearly, $C(J, \mathbb{R})$ becomes a Banach algebra with respect to above norm and multiplication. By $L^1(J, \mathbb{R})$, we denote the vector space of Lebesgue integrable functions defined on J, and the norm $\|\cdot\|_{L^1}$ in $L^1(J, \mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^{2\pi} |x(t)|\, ds.$$

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called $\mathcal{D}$-*Lipschitz* if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \tag{7}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ $(\alpha > 0)$, A is called *Lipschitz* with the Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with the contraction constant α. Furthermore, if $\psi(r) < r$ for all $r > 0$, then A is called a *nonlinear* $\mathcal{D}$*-contraction* on X. Sometimes we call the function ψ a $\mathcal{D}$*-function* of A on X for convenience.

An operator $B : X \to X$ is said to be *compact* if $\overline{B(S)}$ is a compact subset of X for any $S \subset X$. Similarly, $B : X \to X$ is *totally bounded* if B maps a bounded subset of X into a relatively compact subset of X. Finally, $B : X \to X$ is a *completely continuous* operator if it is continuous and totally bounded on X. It is clear that every compact operator is totally bounded, but the converse may not be true. However, both notions coincide on bounded subsets of X. The nonlinear alternative of Schaefer type recently proved by Dhage [7] is embodied in the following theorem (also see Dhage and Ntouyas [8], Dhage [3], and the references therein).

Theorem 3.1. (Dhage [7]) *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be, respectively, open and closed balls of radius r centered at origin 0 in a Banach algebra X. Let A, B, $C : \overline{\mathcal{B}_r(0)} \to X$ be three operators satisfying*

(a) A and C are Lipschitz with the Lipschitz constants α and β respectively,

(b) B is compact and continuous, and

(c) $\alpha M + \beta < 1$, where $M = \|B(\overline{\mathcal{B}_r(0)})\| := \sup\{\|Bx\| : x \in \overline{\mathcal{B}_r(0)}\}$.

Then either

(i) *the equation* $\lambda[Ax\,Bx + Cx] = x$ *has a solution for* $\lambda = 1$, *or*

(ii) *there exists an* $u \in X$ *with* $\|u\| = r$ *satisfying* $\lambda[Au\,Bu + Cu] = u$ *for some* $0 < \lambda < 1$.

The following lemma of Nieto [15] is useful in the study of second order periodic boundary value problems for ordinary differential equations.

Lemma 3.2. *For any real number* $m > 0$ *and* $\sigma \in L^1(J, \mathbb{R})$, x *is a solution to the differential equation*

$$\left.\begin{aligned} -x''(t) + m^2\,x(t) = \sigma(t) \ \text{a.e. } t \in J, \\ x(0) = x(2\pi),\, x'(0) = x'(2\pi), \end{aligned}\right\} \tag{8}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^{2\pi} G_m(t,s)\sigma(s)\,ds, \tag{9}$$

where

$$G_m(t,s) = \begin{cases} \dfrac{e^{m(t-s)} + e^{m(2\pi - t + s)}}{2m(e^{2m\pi} - 1)}, & 0 \le s \le t \le 2\pi, \\ \dfrac{e^{m(s-t)} + e^{m(2\pi - s + t)}}{2m(e^{2m\pi} - 1)}, & 0 \le t < s \le 2\pi. \end{cases} \tag{10}$$

Notice that the Green's function G_m is continuous and nonnegative on $J \times J$ and the numbers

$$\min\{\,|G_m(t,s)| \,:\, t,s \in [0,2\pi]\,\} = \frac{e^{m\pi}}{m(e^{2m\pi} - 1)}$$

and

$$\max\{\,|G_m(t,s)| \,:\, t,s \in [0,2\pi]\,\} = \frac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)}$$

exist for all positive real number m.

4. Existence Theory

We need the following definition in the sequel.

Definition 4.1. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is Carathéodory if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \mapsto \beta(t,x)$ is continuous almost everywhere for $t \in J$.

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A Carathéodory function $\beta(t,x)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h_r(t) \ \text{ a.e. } \ t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally, a Carathéodory function $\beta(t,x)$ is called $L^1_{\mathbb{R}}$-Carathéodory if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h(t) \ \text{ a.e. } \ t \in J$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a bound function of β.

We will use the following hypotheses in the sequel.

(A_1) The functions $t \mapsto f(t,x)$, $t \mapsto f_t(t,x)$, and $t \mapsto f_x(t,x)$ are periodic of period 2π for all $x \in \mathbb{R}$.

(A_2) The functions $t \mapsto k(t,x)$, $t \mapsto k_t(t,x)$, and $t \mapsto k_x(t,x)$ are periodic of period 2π for all $x \in \mathbb{R}$.

(A$_3$) The function $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is injective in $\mathbb{R}$.

(A$_4$) $f(0,x)\big[1 - k_x(0,x)\big] \ne f_x(0,x)\big[x + k(0,x)\big]$ for all $x \in \mathbb{R}$, where $f_x(0,x) = \dfrac{\partial f(t,x)}{\partial x}\Big|_{t=0}$ and $k_x(0,x) = \dfrac{\partial k(t,x)}{\partial x}\Big|_{t=0}$.

(A$_5$) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell_1 \in C(J,\mathbb{R})$ such that

$$|f(t,x) - f(t,y)| \le \ell_1(t)\,|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$. Moreover, we set $\sup\limits_{t\in J} \ell_1(t) = L_1$.

(A$_6$) The function $k : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell_2 \in C(J,\mathbb{R})$ such that

$$|k(t,x) - k(t,y)| \le \ell_2(t)\,|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$. Moreover, we set $\sup\limits_{t\in J} \ell_2(t) = L_2$.

(A$_7$) The function g is Carathéodory.

Remark 4.2. Note that hypotheses (A_5) through (A_7) are common in the literature on the theory of nonlinear differential equations. Hypotheses (A_5) holds if the function $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is increasing in $\mathbb{R}$. Similarly, there do exist functions satisfying the hypotheses (A_0) through (A_4).

Now consider the linear perturbation of the PBVP (5) of first type,

$$\left.\begin{aligned} -\Big(\frac{x(t) - k(t,x(t))}{f(t,x(t))}\Big)'' + m^2\Big(\frac{x(t) - k(t,x(t))}{f(t,x(t))}\Big) \\ = g_m(t,x(t)) \text{ a.e. } t \in J, \\ x(0) = x(2\pi),\ x'(0) = x'(2\pi), \end{aligned}\right\} \tag{11}$$

where $m > 0$ is a real number and the function $g_m : J \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g_m(t,x) = g(t,x) + m^2\Big(\frac{x - k(t,x)}{f(t,x)}\Big). \tag{12}$$

Remark 4.3. Note that the PBVP (5) is equivalent to the PBVP (11) and a solution of the PBVP (5) is a solution for the PBVP (11) defined on J and vice versa.

Remark 4.4. If hypotheses (A_3) and (A_5) hold, then the function g_m defined by (12) is Carathéodory on $J \times \mathbb{R}$.

Lemma 4.5. *Assume that hypotheses (A_0)–(A_4) hold. Then for any real number $m > 0$ and $g_m(\cdot, x(\cdot)) \in L^1(J,\mathbb{R})$, x is a solution of the problem (11) if and only if it is a solution of the integral equation*

$$x(t) = k(t,x(t)) + \big[f(t,x(t))\big]\left(\int_0^{2\pi} G_m(t,s) g_m(s,x(s))\,ds\right), \tag{13}$$

where the Green's function $G_m(t,s)$ is defined by (10).

Proof. Set $y(t) = \dfrac{x(t) - k(t,x(t))}{f(t,x(t))}$. Since $f(t,x)$ and $k(t,x)$ are periodic in t of period 2π for all $x \in \mathbb{R}$, we have

$$y(0) = \frac{x(0) - k(0,x(0))}{f(0,x(0))} = \frac{x(2\pi) - k(2\pi,x(2\pi))}{f(2\pi,x(2\pi))} = y(2\pi).$$

Similarly, we have

$$\begin{aligned} y'(0) = &\frac{f(0,x(0))[x'(0) + k_t(0,x(0)) + k_x(0,x(0))x'(0)]}{[f(0,x(0))]^2} \\ &- \frac{[x(0) - k(0,x(0))]\big[f_t(0,x(0)) + f_x(0,x(0))x'(0)\big]}{[f(0,x(0))]^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{f(2\pi, x(2\pi))[x'(2\pi) + k_t(2\pi, x(2\pi)) + k_x(2\pi, x(2\pi))x'(2\pi)]}{[f(2\pi, x(2\pi))]^2} \\
&\quad - \frac{[x(2\pi) - k(2\pi, x(2\pi))]\big[f_t(2\pi, x(2\pi)) + f_x(2\pi, x(2\pi))x'(2\pi)\big]}{[f(2\pi, x(2\pi))]^2} \\
&= y'(2\pi).
\end{aligned}
$$

Now an application of Lemma 3.2 yields that the solution to differential equation (11) is the solution to integral equation (13). Conversely, suppose that x is any solution to the integral equation (13), then

$$
\begin{aligned}
y(0) = \frac{x(0) - k(0, x(0))}{f(0, x(0))} &= y(2\pi) \\
&= \frac{x(2\pi) - k(2\pi, x(2\pi))}{f(2\pi, x(2\pi))} = \frac{x(2\pi) - k(0, x(2\pi))}{f(0, x(2\pi))}.
\end{aligned}
$$

Since the function $x \mapsto \dfrac{x - k(0, x)}{f(0, x)}$ is injective, we have $x(0) = x(2\pi)$. Again, assume that $y'(0) = y'(2\pi)$ and $x(0) = x(2\pi)$. Then, we obtain

$$
\begin{aligned}
&\frac{f(0, x(0))[x'(0) + k_t(0, x(0)) + k_x(0, x(0))x'(0)]}{[f(0, x(0))]^2} \\
&\quad - \frac{[x(0) - k(0, x(0))]\big[f_t(0, x(0)) + f_x(0, x(0))x'(0)\big]}{[f(0, x(0))]^2} \\
&= \frac{f(2\pi, x(2\pi))[x'(2\pi) + k_t(2\pi, x(2\pi)) + k_x(2\pi, x(2\pi))x'(2\pi)]}{[f(2\pi, x(2\pi))]^2} \\
&\quad - \frac{[x(2\pi) - k(2\pi, x(2\pi))]\big[f_t(2\pi, x(2\pi)) + f_x(2\pi, x(2\pi))x'(2\pi)\big]}{[f(2\pi, x(2\pi))]^2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\big[f(0, x(0)) - f(0, x(0))k_x(0, x(0)) - x(0)f_x(0, x(0)) + k(0, x(0))f_x(0, x(0))\big]x'(0) \\
&= \big[f(0, x(0)) - f(0, x(0))k_x(0, x(0)) - x(0)f_x(0, x(0)) + k(0, x(0))f_x(0, x(0))\big]x'(2\pi).
\end{aligned}
$$

Since $f(0, x) - f(0, x)k_x(0, x) - xf_x(0, x) + k(0, x)f_x(0, x) \neq 0$ for all $x \in \mathbb{R}$, we have $x'(0) = x'(2\pi)$. Therefore, x is a solution to PBVP (5). The proof is complete. □

We make use of the following hypothesis in the sequel.

(A_8) There exists a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $\gamma \in L^1(J, \mathbb{R})$ such that $\gamma(t) > 0$ a.e. $t \in J$ and

$$|g_m(t, x)| \leq \gamma(t)\psi\big(|x|\big) \text{ a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

Theorem 4.6. *Assume that the hypotheses* (A_0) *through* (A_7) *hold. Suppose there exists a real number* $r > 0$ *such that*

$$r > \frac{K_0 + F_0 \left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right] \|\gamma\|_{L^1}\psi(r)}{1 - L_1 \left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right] \|\gamma\|_{L^1}\psi(r) - L_2} \tag{14}$$

where

$$L_1 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right] \|\gamma\|_{L^1}\psi(r) + L_2 < 1,$$

$$F_0 = \sup_{t\in[0,2\pi]} |f(t,0)|, \quad \text{and} \quad K_0 = \sup_{t\in[0,2\pi]} |k(t,0)|.$$

Then the PBVP (5) has a solution defined on J.

Proof. Let $X = C(J,\mathbb{R})$. Define an open ball $\mathcal{B}_r(0)$ centered at origin 0 of radius r, where the real number r satisfies the inequality (14). Define three mappings A, B, and C on $\overline{\mathcal{B}_r(0)}$ by

$$Ax(t) = f(t,x(t)), \quad t \in J, \tag{15}$$

$$Bx(t) = \int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds, \quad t \in J, \tag{16}$$

and

$$Cx(t) = k(t,x(t)), \quad t \in J. \tag{17}$$

Clearly, A, B, and C define operators A, B, $C : \overline{\mathcal{B}_r(0)} \to X$. Now, the integral equation (13) is equivalent to the operator equation

$$Ax(t)\,Bx(t) + Cx = x(t), \quad t \in J. \tag{18}$$

We shall show that the operators A, B and C satisfy all the hypotheses of Theorem 3.1.

To show that A is Lipschitz on $\overline{\mathcal{B}_r(0)}$, let $x, y \in X$. Then by (A_5),

$$\begin{aligned} |Ax(t) - Ay(t)| &\le |f(t,x(t)) - f(t,y(t))| \\ &\le \ell_1(t)\,|x(t) - y(t)| \\ &\le L_1\,\|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \le L_1\|x - y\|$$

for all $x, y \in \overline{\mathcal{B}_r(0)}$. So A is Lipschitz on $\overline{\mathcal{B}_r(0)}$ with the Lipschitz constant L_1. Similarly, we can show that C is also Lipschitz on $\overline{\mathcal{B}_r(0)}$ with the Lipschitz constant L_2.

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Next we show that B is completely continuous on X. Using the standard arguments as in Granas et al. [11], it can be shown that B is a continuous operator on $\overline{\mathcal{B}_r(0)}$. We shall show that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equicontinuous set in X. Let $x \in \overline{\mathcal{B}_r(0)}$ be arbitrary. Since g is Carathéodory, we have

$$\begin{aligned}|Bx(t)| &\leq \Big| \int_0^{2\pi} G_m(t,s) g_m(s,x(s))\, ds \Big| \\ &\leq \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \int_0^{2\pi} \gamma(s)\psi(|x(s)|)\, ds \\ &\leq \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1}\psi(r).\end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \leq M$ for all $x \in \overline{\mathcal{B}_r(0)}$, where $M = \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1}\psi(r)$ This shows that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded set in X. Next, we show that $B(\overline{\mathcal{B}_r(0)})$ is an equi-continuous set in X. Let $x \in \overline{\mathcal{B}}_r(0)$ be arbitrary. Then for any $t_1, t_2 \in J$, we have

$$\begin{aligned}&|Bx(t_1) - Bx(t_1)| \\ &\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\, |g_m(s,x(s))|\, ds \\ &\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\, \gamma(s)\psi(|x(s)|)\, ds \\ &\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\, \gamma(s)\psi(r)\, ds \\ &\leq \left(\int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|^2\, ds \right)^{1/2} \left(\int_0^{2\pi} |\gamma(s)|^2\, ds \right)^{1/2} \psi(r). \qquad (19)\end{aligned}$$

Hence, for all $t_1, t_2 \in J$,

$$|Bx(t_1) - Bx(t_1)| \to 0 \ \text{ as } \ t_1 \to t_2$$

uniformly for all $x \in \overline{\mathcal{B}_r(0)}$. Thus, $B(\overline{\mathcal{B}_r(0)})$ is an equi-continuous set in X. Now $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equi-continuous set in X, so it is compact by the Arzelà-Ascoli theorem. As a result, B is a compact and continuous operator on $\overline{\mathcal{B}_r(0)}$. Therefore, all the conditions of Theorem 3.1 are satisfied and so either conclusion (i) or conclusion (ii) holds. We will show that the conclusion (ii) is not possible.

Let $u \in X$ with $\|u\| = r$ be a solution to the operator equation $\lambda[Au\,Bu + Cu] = u$ for some $0 < \lambda < 1$. Then we have

$$u(t) = \lambda\, k(t,u(t)) + \lambda\left[f(t,u(t))\right] \left(\int_0^{2\pi} G_m(t,s) g_m(s,u(s))\, ds \right)$$

for $t \in J$. Therefore,

$$\begin{aligned}
|u(t)| &\leq \lambda\,|k(t,u(t))| + \lambda\,|f(t,u(t))| \left(\left| \int_0^{2\pi} G_m(t,s) g_m(s,u(s))\,ds \right| \right) \\
&\leq \lambda\Big(\big|k(t,u(t)) - k(t,0)\big| + \big|k(t,0)\big| \Big) \\
&\quad + \lambda\Big(\big|f(t,u(t)) - f(t,0)\big| + \big|f(t,0)\big| \Big) \\
&\quad \times \left(\int_0^{2\pi} G_m(t,s) |g_m(s,u(s))|\,ds \right) \\
&\leq [\ell_2(t)\,|u(t)| + K_0] + [\ell_1(t)\,|u(t)| + F_0] \\
&\quad \times \left(\int_0^{2\pi} \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] |g_m(s,u(s))|\,ds \right) \\
&\leq L_1 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] |u(t)| \left(\int_0^{2\pi} \gamma(s)\psi(|u(s)|)\,ds \right) \\
&\quad + F_0 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \left(\int_0^{2\pi} \gamma(s)\psi(|u(s)|)\,ds \right) \\
&\quad + L_2\,|u| + K_0 \\
&\leq L_1 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1} \psi(\|u\|) |u(t)| \\
&\quad + F_0 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1} \psi(\|u\|) + L_2 \|u\| + K_0. \qquad (20)
\end{aligned}$$

Taking the supremum in the above inequality (20),

$$\|u\| \leq \frac{F_0 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1} \psi(\|u\|) + K_0}{1 - L_1 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1} \psi(\|u\|) - L_2}.$$

Since $\|u\| = r$, we have

$$r \leq \frac{F_0 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1} \psi(r) + K_0}{1 - L_1 \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \right] \|\gamma\|_{L^1} \psi(r) - L_2}.$$

This contradicts (14), so conclusion (ii) of Theorem 3.1 does not hold. Therefore, the operator equation $Ax\,Bx + Cx = x$ and consequently the PBVP (5) has a solution defined on J. This completes the proof. □

Remark 4.7. We note that in Theorem 4.6, we only need the hypotheses (A_1)–(A_3) to hold in the interval $[-r, r]$.

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Often we may be interested in knowing the behavior of the solutions for a given dynamical system in question. Therefore, in the following section we prove a result on the existence of extremal positive solutions of the PBVP (5).

5. Existence of Extremal Positive Solutions

To exploit the monotonic nature of the nonlinearities involved in a quadratic nonlinear problem, we need some hybrid fixed point theorems from ordered Banach algebras. In the following we state some useful fixed point theorems for the purpose of our study.

A non-empty closed set K in a Banach algebra X is called a *cone* if (i) $K+K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is *positive* if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is *normal* if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. Additional details on cones and their properties appear in Heikkilä and Lakshmikantham [12].

We equip the space $C(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0 \text{ for all } t \in J\}. \tag{21}$$

It is well known that the cone K is positive and normal in $C(J, \mathbb{R})$.

Lemma 5.1. (Dhage [3]) *Let K be a positive cone in a real Banach algebra X and let u_1, u_2, v_1, $v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.*

For any a, $b \in X$, $a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

We use the following fixed point theorems of Dhage [3, 4, 6] for proving the existence of extremal solutions for the BVP (5) under certain monotonicity conditions.

Theorem 5.2. (Dhage [3]) *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that A, $B : [a, b] \to K$ and $C : [a, b] \to X$ are three operators such that*

(a) A and C are Lipschitz with the Lipschitz constants α and β, respectively,

(b) B is completely continuous, and

(c) $Ax\,Bx + Cx \in [a, b]$ for each $x \in [a, b]$.

If the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has a least and a greatest positive solution in $[a, b]$ provided $\alpha M + \beta < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 5.3. (Dhage [6]) *Let K be a cone in a Banach algebra X and let a, $b \in X$. Suppose that A, $B : [a, b] \to K$ and $C : X \to X$ are three nondecreasing operators such that*

(a) A is completely continuous,

(b) B is totally bounded,

(c) C is totally bounded, and

(d) $Ax\,By + Cz \in [a, b]$ for each $x, y, z \in [a, b]$.

If the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has a least and a greatest positive solution in $[a, b]$.

Theorem 5.4. (Dhage [6]) *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that A, $B : [a, b] \to K$ and $C : X \to X$ are three nondecreasing operators such that*

(a) A is Lipschitz with a Lipschitz constant α,

(b) B is totally bounded,

(c) C is totally bounded,

(d) $Ax\,By + Cz \in [a, b]$ for each $x, y, z \in [a, b]$.

If the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has a least and a greatest positive solution in $[a, b]$ provided $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Remark 5.5. Note that condition (c) of Theorem 5.2 and condition (d) of Theorems 5.3 and 5.4 hold if the operators A, B, and C are positive, monotone increasing and there exist elements a and b in X such that $a \le Aa\,Ba + Ca$ and $Ab\,Bb + Cb \le b$. Again, each of these hybrid fixed point theorems has some advantages and disadvantages over the others.

We need the following definitions in the sequel.

Definition 5.6. A function $a \in AC^1(J, \mathbb{R})$ is called a lower solution of the PBVP (5) defined on J if the function

$$t \mapsto \frac{d}{dt}\left[\frac{a(t) - k(t, a(t))}{f(t, a(t))}\right]$$

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is absolutely continuous and satisfies

$$\left.\begin{aligned} -\frac{d^2}{dt^2}\left[\frac{a(t)-k(t,a(t))}{f(t,a(t))}\right] &\leq g(t,a(t)) \text{ a.e. } t\in J,\\ a(0)\leq a(2\pi),\ a'(0) &= a'(2\pi). \end{aligned}\right\}$$

Again, a function $b \in AC^1(J,\mathbb{R})$ is called an upper solution of the PBVP (5) defined on J if the function

$$t \mapsto \frac{d}{dt}\left[\frac{b(t)-k(t,b(t))}{f(t,b(t))}\right]$$

is absolutely continuous and satisfies

$$\left.\begin{aligned} -\frac{d^2}{dt^2}\left[\frac{b(t)-k(t,b(t))}{f(t,b(t))}\right] &\geq g(t,b(t)) \text{ a.e. } t\in J,\\ b(0)\geq b(2\pi),\ b'(0) &= b'(2\pi). \end{aligned}\right\}$$

A solution to the PBVP (5) is a lower as well as an upper solution of the PBVP (5) defined on J.

Definition 5.7. A solution x_M of the PBVP (5) is said to be maximal if for any other solution x of PBVP (5) existing on J we have $x(t) \leq x_M(t)$ for all $t \in J$. Again, a solution x_m of the PBVP (5) is said to be minimal if for any other solution x of PBVP (5) existing on J we have $x_m(t) \leq x(t)$.

Remark 5.8. Upper and lower solutions of the PBVP (5) are respectively upper and lower solutions of the PBVP (14), and vice-versa. Similarly, the maximal and minimal solutions of the PBVP (5) are respectively the maximal and minimal solutions of the PBVP (14), and vice-versa.

5.1. Carathéodory Case

We consider the following set of assumptions:

(B_0) The function g_m is nonnegative, i.e., $g_m : J \times \mathbb{R} \to \mathbb{R}^+$.

(B_1) The PBVP (5) has a lower solution a and an upper solution b defined on J with $a \leq b$.

(B_2) The function $x \mapsto \dfrac{x-k(0,x)}{f(0,x)}$ is increasing and the hypothesis (A_2) holds in the interval $\big[\min_{t\in J} a(t), \max_{t\in J} b(t)\big]$.

(B_3) The functions $f(t,x)$, $k(t,x)$ and $g_m(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.

(B_4) The function $h : J \to \mathbb{R}$ defined by

$$h(t) = g_m(t, b(t)),$$

is Lebesgue integrable.

We remark that hypothesis (B_4) holds in particular if b is continuous and g_m is L^1-Carathéodory on $J \times \mathbb{R}$.

Remark 5.9. If the hypotheses (B_1)–(B_3) hold, then the map $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is injective and

$$\frac{a(0) - k(0, a(0))}{f(0, a(0))} \leq \frac{a(2\pi) - k(0, a(2\pi))}{f(2\pi, a(2\pi))}$$

and

$$\frac{b(0) - k(0, b(0))}{f(0, b(0))} \geq \frac{b(2\pi) - k(0, b(2\pi))}{f(2\pi, b(2\pi))}$$

which guarantees that $a \leq Aa\,Ba + Ca$ and $Ab\,Ba + Cb \leq b$.

Remark 5.10. Assume that hypotheses (B_0) through (B_4) hold. Then the function $t \mapsto g_m(t, x(t))$ is Lebesgue integrable on J and

$$|g_m(t, x(t))| = g_m(t, x(t)) \leq g_m(t, b(t)) = h(t), \quad \text{a.e. } t \in J,$$

for all $x \in [a, b]$.

Theorem 5.11. *Suppose that the assumptions (A_0) through (A_4) and (B_0) through (B_4) hold. If* $L_1 \left[\dfrac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)} \right] \|h\|_{L^1} + L_2 < 1$, *where h is given in Remark 5.10, $L_1 = \max_{t \in J} \ell_1(t)$, and $L_2 = \max_{t \in J} \ell_2(t)$, then the PBVP (5) has a minimal and a maximal positive solution defined on J.*

Proof. Now the PBVP (5) is equivalent to integral equation (13) defined on J. Let $X = C(J, \mathbb{R})$. Define three operators A, B, and C on X by (15), (16), and (17), respectively. Then, the integral equation (13) is transformed into the operator equation $Ax(t)Bx(t) + Cx(t) = x(t)$ in a Banach algebra X. Notice that (B_1) implies $A, B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm-bounded set in X. It can be shown, as in the proof of Theorem 3.1, that A and C are Lipschitz with Lipschitz constants L_1 and L_2, respectively. Similarly, B is a completely continuous operator on $[a, b]$. Again, the hypothesis (B_2) implies that A, B, and C are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \leq y$. Then by (B_3),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$Bx(t) = \int_0^{2\pi} G_m(t, s) g_m(s, x(s))\, ds$$

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$$\begin{aligned}&\le \int_0^{2\pi} G_m(t,s) g_m(s,y(s))\,ds\\ &= By(t)\end{aligned}$$

and

$$Cx(t) = k(t,x(t)) \le k(t,y(t)) = Cy(t)$$

for all $t \in J$. So A, B, and C are nondecreasing operators on $[a,b]$. Lemma 4.5, Remark 5.9, and hypothesis (B_3) together imply that

$$\begin{aligned}a(t) &\le k(t,a(t)) + [f(t,a(t))]\left(\int_0^{2\pi} G_m(t,s) g_m(s,a(s)))\,ds\right)\\ &\le k(t,x(t)) + [f(t,x(t))]\left(\int_0^{2\pi} G_m(t,s) g_m(s,x(s))\,ds\right)\\ &\le k(t,b(t)) + [f(t,b(t))]\left(\int_0^{2\pi} G_m(t,s) g_m(s,b(s)))\,ds\right)\\ &\le b(t),\end{aligned}$$

for all $t \in J$ and $x \in [a,b]$. As a result, $a(t) \le Ax(t)Bx(t) + Cx(t) \le b(t)$ for all $t \in J$ and $x \in [a,b]$. Hence, $Ax\,Bx + Cx \in [a,b]$ for all $x \in [a,b]$. Again,

$$\begin{aligned}M &= \|B([a,b])\|\\ &= \sup\{\|Bx\| : x \in [a,b]\}\\ &\le \sup\left\{\sup_{t\in J}\int_0^{2\pi} G_m(t,s)|g_m(s,x(s))|\,ds \;\Big|\; x \in [a,b]\right\}\\ &\le \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\int_0^{2\pi} h(s)\,ds\\ &= \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1}.\end{aligned}$$

Since $\alpha M + \beta \le L_1\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1} + L_2 < 1$, we apply Theorem 5.2 to the operator equation $Ax\,Bx + Cx = x$ to yield that the PBVP (5) has a minimal and a maximal positive solution defined on J. This completes the proof. □

5.2. Discontinuous Case

We need the following definition in our discussion of the discontinuous case.

Definition 5.12. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be Chandrabhan if

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(i) $t \mapsto \beta(t, x(t))$ is measurable for each $x \in C(J, \mathbb{R})$, and

(ii) $x \mapsto \beta(t, x)$ is nondecreasing almost everywhere for $t \in J$.

A Chandrabhan function $\beta(t, x)$ is called L^1-Chandrabhan if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x)| \leq h_r(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Finally, a Chandrabhan function $\beta(t, x)$ is called $L^1_{\mathbb{R}}$-Chandrabhan if

(iv) there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x)| \leq h(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a bound function of β.

We will make use of the following hypotheses.

(C_1) The function $f(t, x)$ and $k(t, x)$ are nondecreasing in x almost everywhere for $t \in J$.

(C_2) The function g_m defined by (12) is $L^1_{\mathbb{R}}$-Chandrabhan.

Theorem 5.13. *Suppose that the assumptions (A_1)–(A_2), (B_0)–(B_2) and (C_1)–(C_2) hold. Then the PBVP (5) has a minimal and a maximal positive solution defined on J.*

Proof. Now the PBVP (5) is equivalent to integral equation (13) defined on J. Let $X = C(J, \mathbb{R})$. Define three operators A, B, and C on X by (15), (16), and (17), respectively. Then the integral equation (13) is transformed into the operator equation $Ax(t)\, Bx(t) + Cx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A,\ B : [a, b] \to K$. Also, conditions (B_0) and (B_2) guarantee that $a \leq Aa\, Ba + Ca$ and $Ab\, Bb + Cb \leq b$.

Step I. First we show that A is completely continuous on $[a, b]$. Now the cone K in X is normal, so the order interval $[a, b]$ is norm-bounded in X. Hence, there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. Since f is continuous on the compact set $J \times [-r, r]$, it attains its maximum, say M. Therefore, for any subset S of $[a, b]$ we have

$$\begin{aligned} \|A(S)\|_{\mathcal{P}} &= \sup\{\|Ax\| : x \in S\} \\ &= \sup\Big\{ \sup_{t \in J} |f(t, x(t))| : x \in S \Big\} \end{aligned}$$

$$\leq \sup\left\{\sup_{t\in J}|f(t,x)| : x\in[-r,r]\right\}$$
$$\leq M.$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next, we note that the function $f(t,x)$ is uniformly continuous on $[0,2\pi]\times[-r,r]$. Therefore, for any $t,\tau\in[0,2\pi]$ we have

$$|f(t,x)-f(\tau,x)|\to 0 \text{ as } t\to\tau$$

uniformly for all $x\in[-r,r]$. Similarly, for any x, $y\in[-r,r]$

$$|f(t,x)-f(t,y)|\to 0 \text{ as } x\to y$$

uniformly for all $t\in[0,2\pi]$. Hence, for any $t,\tau\in[0,2\pi]$ and for any $x\in S$, we have

$$\begin{aligned}|Ax(t)-Ax(\tau)| &= |f(t,x(t))-f(\tau,x(\tau))|\\ &\leq |f(t,x(t))-f(\tau,x(t))|\\ &\quad + |f(\tau,x(t))-f(\tau,x(\tau))|\to 0 \quad \text{as} \quad t\to\tau\end{aligned}$$

uniformly for all $x\in S$. This shows that $A(S)$ is an equi-continuous set in X. Now an application of the Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[a,b]$.

Step II. Next we show that B is totally bounded operator on $[a,b]$. To do this, we shall show that $B(S)$ is uniformly bounded and equi-continuous set in X for any subset S of $[a,b]$. Let $y\in B(S)$ be arbitrary. Then,

$$y(t)=\int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds$$

for some $x\in S$. By hypothesis (C_2), we have

$$\begin{aligned}|y(t)| &= \int_0^{2\pi} G_m(t,s)|g_m(s,x(s))|\,ds\\ &\leq \int_0^{2\pi} G_m(t,s)|g_m(s,b(s))|\,ds\\ &\leq \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\int_0^{2\pi} h(s)\,ds\\ &\leq \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1},\end{aligned}$$

where h may be the same as that given in hypothesis (B_4). Taking the supremum over t,

$$\|y\|\leq\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Again, for any $t_1, t_2 \in J$,

$$\begin{aligned}
&|Bx(t_1) - Bx(t_2)| \\
&\quad \leq \int_0^{2\pi} |G_m(t_1, s) - G_m(t_2, s)|\, |g_m(s, x(s))|\, ds \\
&\quad \leq \int_0^{2\pi} |G_m(t_1, s) - G_m(t_2, s)|\, h(s)\, ds \\
&\quad \leq \left(\int_0^{2\pi} |G_m(t_1, s) - G_m(t_2, s)|^2\, ds\right)^{1/2} \left(\int_0^{2\pi} |h(s)|^2\, ds\right)^{1/2}. \qquad (22)
\end{aligned}$$

Hence, for all t_1, $t_2 \in J$,

$$|Bx(t_1) - Bx(t_1)| \to 0 \text{ as } t_1 \to t_2,$$

uniformly for all $x \in S$. This shows that $B(S)$ is a equi-continuous set of functions in $[a, b]$ for all $S \subset [a, b]$. Now $B(S)$ is a uniformly bounded and equi-continuous, so it is totally bounded by the Arzelà-Ascoli theorem. Similarly, it can be shown as in above Step I that C is also a totally bounded operator in $[a, b]$. Thus, all the conditions of Theorem 5.3 are satisfied and so the PBVP (5) has a maximal and a minimal positive solution defined on J. □

Our final result in this paper is the following.

Theorem 5.14. *Suppose that the assumptions* (A_1)*–*(A_2)*,* (A_4)*,* (B_0)*–*(B_2) *and* (C_1)*–*(C_2) *hold. If*

$$L_1 \left[\frac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)}\right] \|h\|_{L^1} < 1,$$

where h is given in Remark 5.8 and $L_1 = \max_{t \in J} \ell_1(t)$, then the PBVP (5) has a minimal and a maximal positive solution defined on J.

Proof. As before, the PBVP (5) is equivalent to integral equation (13) defined on J. Let $X = C(J, \mathbb{R})$ and define three operators A, B, and C on X by (15), (16), and (17), respectively. Then the integral equation (13) is transformed into the operator equation $Ax(t)\, Bx(t) + Cx(t) = x(t)$ in a Banach algebra X. Notice that hypothesis (B_0) implies A, $B : [a, b] \to K$, and conditions (B_0) and (B_2) show that $a \leq Aa\, Ba + Ca$ and $Ab\, Bb + Cb \leq b$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 3.1 and Theorem 5.2 that the operator A is Lipschitz with the Lipschitz constants $\alpha = L_1$. Again, B is totally bounded with bound $M = \|B([a, b])\| = \left[\frac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)}\right] \|h\|_{L^1}$. Furthermore, $\alpha M = L_1 \left[\frac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)}\right] \|h\|_{L^1} < 1$. Finally, C is totally bounded, so the desired conclusion follows by an application of Theorem 5.4. □

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