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SOLVABILITY CONDITIONS FOR SOME SYSTEMS WITH NON FREDHOLM OPERATORS

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Abstract: We derive solvability conditions in $H^2(\mathbb{R}^3; \mathbb{R}^2)$ for certain systems of nonhomogeneous elliptic partial differential equations involving Schrödinger type operators without Fredholm property using the technique developed in preceding works [10], [11], [12], [13], [14].

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1. Introduction

Solvability conditions for elliptic equations with non Fredholm operators were studied actively in recent years. Most of the works on the subject deal with a single equation of the second order (see [10], [11], [12], [13]), to the exception of the linearized Cahn-Hilliard problem studied in [14]. Such an equation of the fourth order can be easily related to the system of two nonhomogeneous equations of second order. The first one in it is just the Poisson equation which has an explicit solution decaying fast enough at infinity under the appropriate assumptions on its right side . The second one is the nonhomogeneous Schrödinger equation. In the present note we consider the system of two Schrödinger type equations

$$\begin{cases} \Delta u + a(x)u + \alpha u + b(x)v = f(x), \\ \Delta v + c(x)v + \beta v = g(x), \end{cases} \tag{1}$$

where $\alpha, \beta \geq 0$ are constants, $x \in \mathbb{R}^3$, the potential functions $a(x)$ and $c(x)$ are

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shallow and short-range and satisfy the conditions analogous to those used in works [10], [11], [12].

Assumption 1. The potential function $a(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate

$$|a(x)| \le \frac{C}{1+|x|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|a\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|a\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad and \quad \sqrt{c_{HLS}}\|a\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi$$

and the requirements for $c(x)$ here are exactly the same as for $a(x)$.

Here C denotes a finite positive constant and c_{HLS} given on p.98 of [6] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big| \le c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Here and further down the norm of a function $f_1 \in L^p(\mathbb{R}^3)$, $1 \le p \le \infty$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^3)}$. The homogeneous system corresponding to (1) will be given by

$$\Delta\theta + a(x)\theta + \alpha\theta = 0, \tag{2}$$

$$\Delta w + c(x)w + \beta w = 0. \tag{3}$$

We will be using $(f_1(x), f_2(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations when the functions involved in this inner product are not necessarily square integrable, like for instance $w(x)$ involved in the first relation of (6). The sphere of radius r in the space of three dimensions centered at the origin will be denoted by S_r^3. Due to the decay of the potential functions at infinity the essential spectra of the Schrödinger operators

$$h_\alpha = -\Delta - a(x) - \alpha \quad and \quad l_\beta = -\Delta - c(x) - \beta, \quad \alpha, \beta \ge 0 \tag{4}$$

on $L^2(\mathbb{R}^3)$ involved in the left sides of the equations of system (1) concide with the semi-axes $[-\alpha, \infty)$ and $[-\beta, \infty)$ respectively (see e.g. [4]) such that h_α and l_β do not have finite dimensional isolated kernels and the Fredholm alternative theorem fails to work for equations of system (1). We impose the conditions on the right side of (1) analogical to those used in the preceding works mentioned above and on the remaining coefficient of the first equation of the system.

Assumption 2. Let the function $f(x) : \mathbb{R}^3 \to \mathbb{R}$, such that $f(x) \in L^2(\mathbb{R}^3)$ and $|x|f(x) \in L^1(\mathbb{R}^3)$ and the requirements for $g(x)$ here are exactly the same ones as for $f(x)$.

Assumption 3. Let the function $b(x) : \mathbb{R}^3 \to \mathbb{R}$, such that $b(x) \in L^\infty(\mathbb{R}^3)$ and $|x|b(x) \in L^2(\mathbb{R}^3)$.

Let us introduce the functional space

$$\tilde{W}^{2,\,\infty}(\mathbb{R}^3) := \{w(x) : \mathbb{R}^3 \to \mathbb{C} \mid w, \nabla w, \Delta w \in L^\infty(\mathbb{R}^3)\} \tag{5}$$

used in establishing solvability conditions for the Laplacian problem with convection terms in [13] and for nonhomogeneous Schrödinger type equations in [12]. As distinct from the standard Sobolev space only the boundedness of the Laplacian of a function is required here, no explicit restrictions on its all second partial derivatives. Our main result is as follows.

Theorem 4. *Let Assumptions 1,2 and 3 hold. Then problem (1) admits a unique solution* $(u_0, v_0)^T \in H^2(\mathbb{R}^3; \mathbb{R}^2)$ *if and only if*

$$(g(x), w(x))_{L^2(\mathbb{R}^3)} = 0, \quad (f(x) - b(x)v_0(x), \theta(x))_{L^2(\mathbb{R}^3)} = 0 \tag{6}$$

for any $\theta(x),\ w(x) \in \tilde{W}^{2,\,\infty}(\mathbb{R}^3)$ *satisfying homogeneous equations (2) and (3) respectively with the space* $\tilde{W}^{2,\,\infty}(\mathbb{R}^3)$ *defined in (5).*

In the present work the solvability conditions for the system of equations are obtained as orthogonality conditions to the solutions of the corresponding homogeneous equations belonging to the appropriate functional space. The analogy with the standard Fredholm solvability conditions here is only formal because the operators involved do not satisfy the Fredholm property and their ranges are not closed.

Understanding the spectral properties of the operators without Fredholm property is very important, for instance when establishing the existence in certain functional spaces of stationary and travelling wave solutions of reaction-diffusion equations (see e.g. [2], [3], [9], [13]).

2. Proof of the Solvability Conditions

Proof of Theorem 4. Under Assumption 1 the Schrödinger operators h_α and l_β given in (4) are self-adjoint and unitarily equivalent to $-\Delta - \alpha$ and $-\Delta - \beta$ on $L^2(\mathbb{R}^3)$ respectively via the wave operators (see [1], [5], [8], [10]) and their functions of the continuous spectra satisfying

$$h_0\varphi_k(x) = k^2\varphi_k(x),\ k \in \mathbb{R}^3, \quad l_0\eta_q(x) = q^2\eta_q(x),\ q \in \mathbb{R}^3, \tag{7}$$

the Lippmann-Schwinger equations for the perturbed plane waves (see e.g. [7] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(a\varphi_k)(y)dy,$$

$$\eta_q(x) = \frac{e^{iqx}}{(2\pi)^{\frac{3}{2}}} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||x-y|}}{|x-y|}(c\eta_q)(y)dy$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad (\eta_k(x), \eta_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3$$

form the complete systems in $L^2(\mathbb{R}^3)$. The functions $\varphi_0(x)$ and $\eta_0(x)$ correspond to the case when the wave vectors vanish. By means of Theorem 3 of [12] the second equation of (1) admits a unique solution $v_0 \in H^2(\mathbb{R}^3)$ if and only if the first orthogonality condition of (6) holds. When $\beta > 0$ we consider $\eta_q(x)$, $q \in S^3_{\sqrt{\beta}}$. These functions belong to $\tilde{W}^{2,\infty}(\mathbb{R}^3)$, which was proven in Lemma A3 of [13] and solve the homogeneous equation (3) via (7). When the parameter β vanishes we consider similarly $\eta_0(x)$. Hence it remains to solve the first equation of (1) given by

$$\Delta u + a(x)u + \alpha u = f(x) - b(x)v_0. \tag{8}$$

Since $b(x)$ is bounded due to Assumption 3, the second term in the right side of the equation above is square integrable as well. Via the Schwarz inequality and Assumption 3

$$\||x|b(x)v_0(x)\|_{L^1(\mathbb{R}^3)} \leq \||x|b(x)\|_{L^2(\mathbb{R}^3)}\|v_0\|_{L^2(\mathbb{R}^3)} < \infty.$$

Therefore, by means of Theorem 3 of [12] equation (8) possesses a unique solution $u_0 \in H^2(\mathbb{R}^3)$ if and only if the second orthogonality condition of (6) holds. When $\alpha > 0$ one considers the functions of the continuous spectrum $\varphi_k(x)$, $k \in S^3_{\sqrt{\alpha}}$. They are contained in $\tilde{W}^{2,\infty}(\mathbb{R}^3)$ due to Lemma A3 of [13] and satisfy the homogeneous equation (2) by means of (7). When the parameter $\alpha = 0$, we take into consideration analogously $\varphi_0(x)$. □

References

[1] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin (1987).

[2] A. Ducrot, M. Marion, V. Volpert, Systemes de réaction-diffusion sans propriété de Fredholm, CRAS, **340** (2005), 659–664.

[3] A. Ducrot, M. Marion, V. Volpert, Reaction-diffusion problems with non Fredholm operators, Advances Diff. Equations, **13**, No. 11-12 (2008), 1151–1192.

[4] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting, Applied Analysis, In preparation.

[5] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Math. Ann., **162** (1965/1966), 258–279.

[6] E. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997).

[7] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press (1979).

[8] I. Rodnianski, W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, Invent. Math., **155**, No. 3 (2004), 451–513.

[9] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski, Solvability conditions for elliptic problems with non-Fredholm operators, Appl.Math., **29**, No. 2 (2002), 219–238.

[10] V. Vougalter, V. Volpert, Solvability conditions for some non Fredholm operators, To appear in: Proc. Edinb. Math. Soc., http://hal.archives-ouvertes.fr/hal-00362446/fr/

[11] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators*, Int. J. Pure Appl. Math., **60**, No. 2 (2010), 169–191.

[12] V. Vougalter, V. Volpert. *Solvability relations for some non Fredholm operators*, Int. Electron. J. Pure Appl. Math., **2**, No. 1 (2010), 75–83

[13] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms*, To appear in: Commun. Pure Appl. Anal.

[14] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation*, Preprint (2010).