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REPRESENTATIONS THEOREMS FOR SKEW-SYMMETRIC TENSORIAL FUNCTIONS IN A 4-DIMENSIONAL EUCLIDEAN SPACE

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Abstract: In a 4-dimensional Euclidean space, representations theorems are here obtained for skew-symmetric tensor valued isotropic functions depending on an arbitrary number of scalars, skew-symmetric second order tensors and symmetric second order tensors; at least one of these last ones is assumed to have an eigenvalue with multiplicity 1. The case with at least a non null vector, among the independent variables, has already been treated in literature; so it is here not treated. The result is a finite, but long, set of skew-symmetric tensor valued isotropic functions such that every other skew-symmetric tensor valued function of the same variables can be expressed as a linear combination, through scalar coefficients, of the elements of this set. The methodology used to obtain this set is directed in trying to use similar representation theorems, already known in literature for the 3-dimensional case.

AMS Subject Classification: —???—
Key Words: —???—

1. Introduction

This paper completes a set of three other papers on this subject, [1], [2], [3]. They concern the representation theorems in a 4-dimensional Euclidean space with an arbitrary number of scalars, symmetric second order tensors and skew-symmetric second order tensors as independent variables. In particular, the paper [1] treats the case with at least a non null vector, among the independent variables, while [2], [3] and the present paper, treat the case where there are no vectors, but there is at least a symmetric second order tensors, among the independent variables, which has

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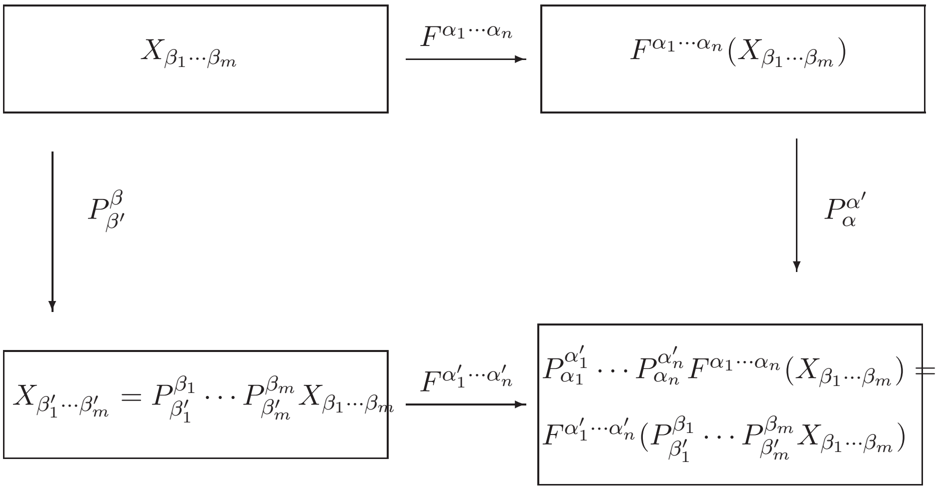

Figure 1: The condition defining isotropic functions.

an eigenvalue with multiplicity 1. The result of ref. [2] is a set S^0 of scalar valued isotropic functions, of the same variables, such that every other scalar function of these variables can be expressed as a function of the elements of the set S^0. The result of ref. [3] is a set $S^{2\,sy}$ of symmetric tensor valued isotropic functions, of the same variables, such that every other second order symmetric tensor valued function of these variables can be expressed as a linear combination of the elements of the set $S^{2\,sy}$, through scalar coefficients. A similar treatment for vector valued isotropic functions leads to the results that only the zero vectorial function is allowed. In the present paper we obtain a set $S^{2\,sk}$ of skew-symmetric tensor valued isotropic functions, of the same variables, such that every other second order skew-symmetric tensor valued function of these variables can be expressed as a linear combination of the elements of the set $S^{2\,sk}$, through scalar coefficients. This set is reported in Section 2.

To be more precise, representation theorems are useful to impose the physical principle requiring that the laws of physics don' t depend on the observer. Now these laws are expressed in terms of tensors $F^{\alpha_1\cdots\alpha_n}$ which, in turns, are functions of other tensors $X_{\beta_1\cdots\beta_m}$; moreover, we know the transformation laws of the components of a tensor, when the basis of the vectorial space is changed. Therefore, the above mentioned requirement amounts in imposing that the diagram in figure

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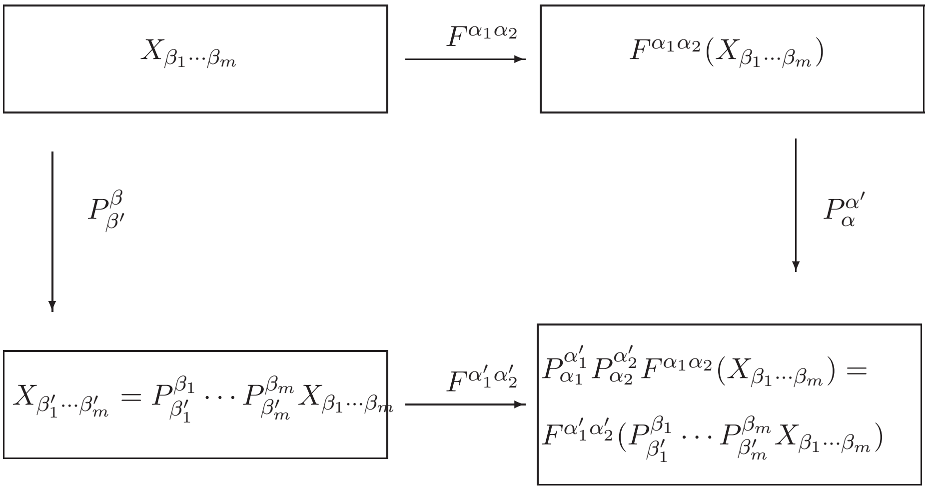

Figure 2: The condition defining second order isotropic tensorial functions.

1 is commutative, where $P^{\alpha'}_{\alpha}$ is the matrix of the change of basis. In other words, proceeding from the upper left corner of the figure and moving to the right hand side, we start from the independent variables $X_{\beta_1\cdots\beta_m}$ in the reference frame Σ, then we apply to them the function $F^{\alpha_1\cdots\alpha_n}$, in Σ; after that, proceeding to the lower side, we transform the result in the reference Σ'. Following the other side of the diagram, we transform the independent variables in Σ'; on the transformed variables, we apply the function $F^{\alpha'_1\cdots\alpha'_n}$, in Σ'. We require that the result is the same.

In particular, here we treat the case $n = 2$, so that the above diagram becomes that in figure 2 and, moreover, $F^{\alpha_1\alpha_2}$ is a skew-symmetric tensor. In [3] we have considered the case with $n = 2$ and $F^{\alpha_1\alpha_2}$ a symmetric tensor; also the case with $n = 1$ has been considered. In [2] we have considered the case with $n = 0$.

The methodology followed to obtain these sets uses similar representation theorems, already known in literature, for the 3-dimensional case. They can be found, for example, in [4]- [7]. In particular, in [6] Boheler proved that the representations exposed in [4] are not irreducible in the sense that some redundant elements of these representation can be eliminated, obtaining a subset with fewer elements, but satisfying the same properties. In [7] Pennisi and Trovato proved that once eliminated

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the elements indicated by Boheler, the remaining elements furnish a complete and irreducible representation. Finally, in [8], the case $n = 3$ has been considered still in the 3-dimensional case.

We report the result of the present paper, that is the set $S^{2\,sk}$, in Section 2 in order not to lose the thread of what one is saying, while we will describe now how it has been obtained. To this end, let us firstly recall some passages used in [2], [3], and let us call $\mathcal{A}$ the symmetric tensor endowed of at least an eigenvalue a with multiplicity 1. Let also $\vec{v}$ be a real eigenvector corresponding to the eigenvalue a; in the reference frames whose x_1 axis is directed as $\vec{v}$, the tensor $\mathcal{A}$ takes the form

$$\mathcal{A} = \begin{pmatrix} a & \vec{0}^T \\ \vec{0} & A \end{pmatrix} \tag{1}$$

with A symmetric second order tensor belonging to the 3-dimensional Euclidean space orthogonal to $\vec{v}$. Let

$$\lambda^4 - S_1\lambda^3 + S_2\lambda^2 - S_3\lambda^4 + S_4 = 0 \tag{2}$$

be the characteristic equation of $\mathcal{A}$, that is the equation whose roots are the eigenvalues. Obviously, the scalars S_i can be expressed in terms of the scalars $tr\mathcal{A}^i$ for $i = 1, .., 4$ belonging to the set S^0 of ref. [2] (the abbreviation tr denotes the trace of the subsequent tensor). If now we define

$$s_1 = S_1 - a \quad , \quad s_2 = S_2 - as_1 \quad , \quad s_3 = S_3 - as_2 \quad , \tag{3}$$

we have that $S_4 = as_3$ and $a^3 - s_1a^2 + s_2a - s_3 \neq 0$ because a is root of (2) with multiplicity 1; moreover, $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ is the characteristic equation of A, so that, thanks to the Hamilton-Kayley' s theorem, we have

$$A^3 = s_1A^2 - s_2A + s_3I \quad .$$

This fact suggests us to define

$$I_1 = \frac{\mathcal{A}^3 - s_1\mathcal{A}^2 + s_2\mathcal{A} - s_3I}{a^3 - s_1a^2 + s_2a - s_3} \quad \text{obtaining that} \quad I_1 = diag(1, 0, 0, 0). \tag{4}$$

This tensor, in turns, allows us to split every tensor into a part parallel to $\vec{v}$ and into a part belonging to the 3-dimensional Euclidean space orthogonal to $\vec{v}$. More precisely, every second order tensor $\mathcal{M}$ may be decomposed in

$$\mathcal{M} = \begin{pmatrix} m_{11} & \vec{m}^T \\ \vec{m} & M \end{pmatrix} \quad , \quad \mathcal{M} = \begin{pmatrix} 0 & -\vec{m}^T \\ \vec{m} & M \end{pmatrix} , \tag{5}$$

the first of which holds for symmetric tensors, and the second one for skew-symmetric tensors.

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Moreover, we can use the change of variables, from $\mathcal{M}$ to

$$\begin{aligned} & m_{11} = trI_1\mathcal{M}\,, \\ & \vec{M} = \mathcal{M}I_1 - m_{11}I_1 = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{m} & 0 \end{pmatrix}, \\ & M^* = \mathcal{M} - \vec{M} - \vec{M}^T - m_{11}I_1 = \\ & \qquad \mathcal{M} - \mathcal{M}I_1 - I_1\mathcal{M} + m_{11}I_1 = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & M \end{pmatrix} \text{ if } \mathcal{M} \text{ is symmetric}\,, \\ & M^* = \mathcal{M} - \vec{M} + \vec{M}^T = \\ & \qquad \mathcal{M} - \mathcal{M}I_1 - I_1\mathcal{M} = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & M \end{pmatrix}, \text{ if } \mathcal{M} \text{ is skew-symmetric}\,. \end{aligned} \tag{6}$$

Moreover, for every change of frames which leave unchanged the x_1 axis, we have that m_{11} behaves as a scalar, while $\vec{M}$ and M^* maintain their form except for substituting $\vec{m}$ and M with the expressions obtained for 3-dimensional vectors and second order tensors, with a 3-dimensional transformation of the reference frame. Can we apply the known representation theorems in a 3-dimensional space? We can say yes, for two reasons:

- First of all, we note that a consequence of the diagram in figure 2 is the following: if in the reference frames, whose x_1 axis is directed as $\vec{v}$, $F^{\alpha_1\alpha_2}$ is a linear combination, through scalar coefficients, of other functions of the same type, then this same property will be satisfied in every other frame. On the other hand, in these frames it is determined as an arbitrary function of the tensors in the 3-dimensional representation.

- We can convert the tensors of the 3-dimensional representation in the present 4-dimensional form; it suffices to observe that

$$M^*N^* = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & MN \end{pmatrix};\; N^*\vec{M} = \begin{pmatrix} 0 & \vec{0}^T \\ N\vec{m} & 0 \end{pmatrix}; \tag{7}$$

$$\vec{P}^T N^* \vec{M} = \begin{pmatrix} \vec{p}^T N\vec{m} & \vec{0}^T \\ \vec{0} & 0 \end{pmatrix};$$

from which $trM^*N^* = trMN$ and $tr\vec{P}^T N^*\vec{M} = \vec{p}^T N\vec{m}\,.$

We note now that the decomposition (5) and (6) holds also for the skew-symmetric tensorial functions, and not only for the independent variables. It follows that

$$\mathcal{M} = M^* + \vec{M} - \vec{M}^T\,, \tag{8}$$

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so that $\mathcal{M}$ is a linear combination of the representation for $\vec{M} - \vec{M}^T$ and of the representation for M^*. The first of these representations will be reported in sect. 3, while that for M^* will be found in sect. 4.

As it can be seen from the result, they are affected unfortunately by the tensor I_1 of eq. (4); to overcome this drawback, we have substituted I_1 from eq. (4) in the tensors of Sections 3, 4 and, for every term containing a linear combination of the variables $s_i/(a^3 - s_1a^2 + s_2a - s_3)$ we have taken the coefficients of this combination and inserted them in the final representation. In this way we have obtained the tensors listed in the following section, reporting firstly the elements coming from $\vec{M} - \vec{M}^T$ and, subsequently, those coming from M^*; we have also eliminated some terms already appearing in the list. In this way we are sure that every skew-symmetric second order tensor valued function, of our independent variables, can be expressed as a linear combination of the tensors of the set $S^{2\,sk}$ in Section 2; in other words, we can say that $S^{2\,sk}$ furnishes a complete representation. Obviously, we don' t know if $S^{2\,sk}$ is an irreducible representation, in the sense that no of its proper subsets furnishes a complete representation; we refrain from exploiting this aspect of irreducibility for the sake of brevity, because $S^{2\,sk}$ has many elements, as it can be seen in Section 2. On the other hand, in applications the numbers H, N and M are not generic, so that it will be easier to detect eventual redundant elements in $S^{2\,sk}$, and leave them out. In any case, we like here to note that irreducibility doesn' t mean linear independence; in fact, for every assigned value of the independent variables, all the elements of the set $S^{2\,sk}$ are linear combinations of at most 6 of them! But this subset of 6 elements changes for other values of the independent variables.

2. The Final Representation for 4-Dimensional Second Order Skew-Symmetric Tensor Valued Isotropic Functions

We will describe in this section the final result of the present paper. It is obtained, in the way described above, from the subsequent tables 1 and 2 ter. It amounts in the following

Representation Theorem. *Every second order skew- symmetric tensor valued isotropic function depending on the scalars λ_h, on the symmetric second order tensor $\mathcal{A}$ endowed at least with an eigenvalue of multiplicity 1, on the further symmetric second order tensors $\mathcal{A}_i$ and on the skew-symmetric second order tensors Ω_γ for $h = 1, ..., H$, $i = 1, ..., N$, $\gamma = 1, ..., M$, can be expressed as a linear combination, through scalar coefficients, of the elements of the following set $S^{2\,sk}$.*

We describe this set in terms of particular tensors depending on some indexes whose variability range is $h = 1, ..., H$; $i, j, k = 1, ..., N$ with $i < j < k$; $p, q = 1, ..., M$ with $p < q$; $\alpha, \beta = 1, ..., N$ with $\alpha < \beta$; $\gamma, \delta = 1, ..., M$; $\gamma < \delta$. The set $S^{2\,sk}$ obtained, in the above mentioned way, is constituted by the following tensors

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The set $S^{2\,sk}$

$\mathcal{A}\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}\,,\ \mathcal{A}^2\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^2\,,\ \mathcal{A}_i^2\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}_i^2\,,$

$(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\mathcal{A}_\alpha - \mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\,,$

$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha - \mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\,,\ \Omega_\gamma\,, \mathcal{A}\Omega_\gamma + \Omega_\gamma\mathcal{A}\,,$

$\mathcal{A}^2\Omega_\gamma + \Omega_\gamma\mathcal{A}^2\,,\ \mathcal{A}_i\Omega_\gamma + \Omega_\gamma\mathcal{A}_i\,,\ \mathcal{A}_i^2\Omega_\gamma + \Omega_\gamma\mathcal{A}_i^2\,,$

$(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\gamma + \Omega_\gamma(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\,,$

$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\gamma + \Omega_\gamma(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\,,\ \Omega_p^2\Omega_\gamma + \Omega_\gamma\Omega_p^2\,,$

$(\Omega_p\Omega_q - \Omega_q\Omega_p)\Omega_\gamma + \Omega_\gamma(\Omega_q\Omega_p - \Omega_p\Omega_q)\,,$

$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\Omega_\gamma + \Omega_\gamma(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\,,\ \Omega_p\mathcal{A}_\alpha + \mathcal{A}_\alpha\Omega_p\,,\ \Omega_p^2\mathcal{A}_\alpha - \mathcal{A}_\alpha\Omega_p^2$

$(\Omega_p\Omega_q - \Omega_q\Omega_p)\mathcal{A}_\alpha - \mathcal{A}_\alpha(\Omega_q\Omega_p - \Omega_p\Omega_q)\,,$

$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\mathcal{A}_\alpha - \mathcal{A}_\alpha(-\Omega_p\mathcal{A} + \mathcal{A}\Omega_p)\,,$

$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\mathcal{A}_\alpha - \mathcal{A}_\alpha(-\Omega_p\mathcal{A}_i + \mathcal{A}_i\Omega_p)\,,\ \mathcal{A}\mathcal{A}_i\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_i\mathcal{A},$

$\mathcal{A}_\alpha\mathcal{A}_\beta - \mathcal{A}_\beta\mathcal{A}_\alpha,\ \mathcal{A}_\alpha^2\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha^2,\ \mathcal{A}_\alpha^2\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha^2,$

$\mathcal{A}\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_\alpha,\ \mathcal{A}\mathcal{A}_\alpha^2\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha^2,$

$\mathcal{A}_i\mathcal{A}_\alpha^2 - \mathcal{A}_\alpha^2\mathcal{A}_i,\ \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}_\beta - \mathcal{A}_i\mathcal{A}_\beta\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}_\beta\mathcal{A}_i,$

$\Omega_\gamma\Omega_\delta - \Omega_\delta\Omega_\gamma,\ \mathcal{A}\Omega_\gamma^2 - \Omega_\gamma^2\mathcal{A},\ \mathcal{A}^2\Omega_\gamma^2 - \Omega_\gamma^2\mathcal{A}^2,$

$\mathcal{A}\Omega_\gamma^2\mathcal{A}^2 - \mathcal{A}^2\Omega_\gamma^2\mathcal{A},\ \Omega_p^2\Omega_\gamma^2 - \Omega_\gamma^2\Omega_p^2,$

$\Omega_p\Omega_\gamma\Omega_\delta - \Omega_p\Omega_\delta\Omega_\gamma + \Omega_\delta\Omega_\gamma\Omega_p - \Omega_\gamma\Omega_\delta\Omega_p,$

$\mathcal{A}\Omega_\gamma\Omega_\delta - \mathcal{A}\Omega_\delta\Omega_\gamma - \Omega_\delta\Omega_\gamma\mathcal{A} + \Omega_\gamma\Omega_\delta\mathcal{A},\ \mathcal{A}_\alpha\Omega_\gamma + \Omega_\gamma\mathcal{A}_\alpha,$

$\mathcal{A}\mathcal{A}_\alpha\Omega_\gamma + \mathcal{A}\Omega_\gamma\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\Omega_\gamma\mathcal{A},\ \mathcal{A}_\alpha^2\Omega_p + \Omega_p\mathcal{A}_\alpha^2,$

$\mathcal{A}_i\Omega_\gamma^2 - \Omega_\gamma^2\mathcal{A}_i,\ \mathcal{A}_i^2\Omega_\gamma^2 - \Omega_\gamma^2\mathcal{A}_i^2,$

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$$\Omega_p\mathcal{A}_\alpha\mathcal{A}_\beta - \Omega_p\mathcal{A}_\beta\mathcal{A}_\alpha + \mathcal{A}_\beta\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}_\beta\Omega_p,$$

$$\mathcal{A}_i\Omega_\gamma\Omega_\delta - \mathcal{A}_i\Omega_\delta\Omega_\gamma - \Omega_\delta\Omega_\gamma\mathcal{A}_i - \Omega_\gamma\Omega_\delta\mathcal{A}_i,$$

$$\Omega_p\mathcal{A}_\alpha\Omega_\gamma + \Omega_p\Omega_\gamma\mathcal{A}_\alpha - \Omega_\gamma\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\Omega_\gamma\Omega_p,$$

$$\mathcal{A}_i\mathcal{A}_\alpha\Omega_\gamma + \mathcal{A}_i\Omega_\gamma\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\Omega_\gamma\mathcal{A}_i,$$

$$\mathcal{A}\mathcal{A}_\alpha^2\mathcal{A}_i + \mathcal{A}_\alpha^2(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) - \mathcal{A}_i\mathcal{A}_\alpha^2\mathcal{A} - (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\mathcal{A}_\alpha^2,$$

$$\mathcal{A}\mathcal{A}_i\mathcal{A}_j + \mathcal{A}_j\mathcal{A}\mathcal{A}_i + \mathcal{A}_i\mathcal{A}_j\mathcal{A} - \mathcal{A}_j\mathcal{A}_i\mathcal{A} - \mathcal{A}_i\mathcal{A}\mathcal{A}_j - \mathcal{A}\mathcal{A}_j\mathcal{A}_i,$$

$$\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k + \mathcal{A}_j\mathcal{A}_k\mathcal{A}_i + \mathcal{A}_k\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_k\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_k\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i\mathcal{A}_k,$$

$$\mathcal{A}_i^2\mathcal{A}_\alpha^2 - \mathcal{A}_\alpha^2\mathcal{A}_i^2,\ \mathcal{A}_i\mathcal{A}_\alpha^2\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_\alpha^2\mathcal{A}_i$$

$$\mathcal{A}_i\mathcal{A}_\alpha^2\mathcal{A}_j + \mathcal{A}_\alpha^2(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j) - \mathcal{A}_j\mathcal{A}_\alpha^2\mathcal{A}_i - (\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha^2,$$

$$\mathcal{A}_i\mathcal{A}\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}\mathcal{A}_i,\ \mathcal{A}_i\mathcal{A}_j\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_j\mathcal{A}_i,\ \mathcal{A}_j\mathcal{A}_i\mathcal{A}_j^2 - \mathcal{A}_j^2\mathcal{A}_i\mathcal{A}_j,$$

$$\mathcal{A}\Omega_\gamma^2\mathcal{A}^2 - \Omega_\gamma^2(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) - \mathcal{A}^2\Omega_\gamma^2\mathcal{A} + (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\Omega_\gamma^2,$$

$$\mathcal{A}_i\Omega_\gamma^2\mathcal{A}_j - \Omega_\gamma^2(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i) - \mathcal{A}_j\Omega_\gamma^2\mathcal{A}_i + (\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\Omega_\gamma^2,$$

$$\mathcal{A}_i\Omega_\gamma^2\mathcal{A}_i^2 - \mathcal{A}_i^2\Omega_\gamma^2\mathcal{A}_i,\ \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i\,,$$

$$\mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i^2\,,\ (\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\,,$$

$$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\,,\ \mathcal{A}\Omega_\gamma\mathcal{A}\,,$$

$$\mathcal{A}^2\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma\mathcal{A}^2\,,\ \mathcal{A}_i\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma\mathcal{A}_i\,,\ \mathcal{A}_i^2\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma\mathcal{A}_i^2\,,$$

$$(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\,,$$

$$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\,,\ \Omega_p\Omega_\gamma\mathcal{A} - \mathcal{A}\Omega_\gamma\Omega_p\,,\ \Omega_p^2\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma\Omega_p^2\,,$$

$$(\Omega_p\Omega_q - \Omega_q\Omega_p)\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma(\Omega_q\Omega_p - \Omega_p\Omega_q)\,,$$

$$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\,,$$

$$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\Omega_\gamma\mathcal{A} + \mathcal{A}\Omega_\gamma(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\,,\ \Omega_p\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}\mathcal{A}_\alpha\Omega_p\,,\ \Omega_p^2\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha\Omega_p^2,$$

$$(\Omega_p\Omega_q - \Omega_q\Omega_p)\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha(\Omega_q\Omega_p - \Omega_p\Omega_q)\,,$$

$$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha(-\Omega_p\mathcal{A} + \mathcal{A}\Omega_p)\,,$$

$$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha(-\Omega_p\mathcal{A}_i + \mathcal{A}_i\Omega_p)\,,\ \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta - \mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha,$$

$$\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha,\ \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha,\ \mathcal{A}\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A},$$

$$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta - \mathcal{A}\mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta\mathcal{A},\ \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i,$$

$$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta - \mathcal{A}_i\mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta\mathcal{A}_i,$$

$$\Omega_\gamma\mathcal{A}\Omega_\delta - \Omega_\delta\mathcal{A}\Omega_\gamma,\ \mathcal{A}\Omega_\gamma\mathcal{A}\Omega_\gamma - \Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A},\ \mathcal{A}^2\Omega_\gamma\mathcal{A}\Omega_\gamma - \Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}^2,$$

$$\mathcal{A}\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}^2 - \mathcal{A}^2\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A},\ \Omega_p\Omega_\gamma\mathcal{A}\Omega_\gamma + \Omega_\gamma\mathcal{A}\Omega_\gamma\Omega_p,$$

$$\Omega_p\Omega_\gamma\mathcal{A}\Omega_\delta - \Omega_p\Omega_\delta\mathcal{A}\Omega_\gamma + \Omega_\delta\mathcal{A}\Omega_\gamma\Omega_p - \Omega_\gamma\mathcal{A}\Omega_\delta\Omega_p,\ \Omega_p^2\Omega_\gamma\mathcal{A}\Omega_\gamma - \Omega_\gamma\mathcal{A}\Omega_\gamma\Omega_p^2,$$

$$\mathcal{A}\Omega_\gamma\mathcal{A}\Omega_\delta - \mathcal{A}\Omega_\delta\mathcal{A}\Omega_\gamma - \Omega_\delta\mathcal{A}\Omega_\gamma\mathcal{A} + \Omega_\gamma\mathcal{A}\Omega_\delta\mathcal{A},\ \mathcal{A}_\alpha\mathcal{A}\Omega_\gamma + \Omega_\gamma\mathcal{A}\mathcal{A}_\alpha,$$

$$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma + \mathcal{A}\Omega_\gamma\mathcal{A}\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\mathcal{A}\Omega_\gamma\mathcal{A},$$

$$\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\Omega_p + \Omega_p\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha,\ \mathcal{A}_i\Omega_\gamma\mathcal{A}\Omega_\gamma - \Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}_i,\ \mathcal{A}_i^2\Omega_\gamma\mathcal{A}\Omega_\gamma - \Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}_i^2,$$

$$\Omega_p\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta - \Omega_p\mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha + \mathcal{A}_\beta\mathcal{A}\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta\Omega_p,$$

$$\mathcal{A}_i\Omega_\gamma\mathcal{A}\Omega_\delta - \mathcal{A}_i\Omega_\delta\mathcal{A}\Omega_\gamma - \Omega_\delta\mathcal{A}\Omega_\gamma\mathcal{A}_i - \Omega_\gamma\mathcal{A}\Omega_\delta\mathcal{A}_i,$$

$$\Omega_p\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma + \Omega_p\Omega_\gamma\mathcal{A}\mathcal{A}_\alpha - \Omega_\gamma\mathcal{A}\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}\Omega_\gamma\Omega_p,$$

$$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma + \mathcal{A}_i\Omega_\gamma\mathcal{A}\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}\Omega_\gamma\mathcal{A}_i,$$

$$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) - \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A} - (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha,$$

$$\mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i^2,\ \Omega_p^2\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\Omega_p^2,\ \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i,$$

$$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_j + \mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j) - \mathcal{A}_j\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha\mathcal{A}_i - (\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha,$$

$$\mathcal{A}\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}^2 - \Omega_\gamma\mathcal{A}\Omega_\gamma(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) - \mathcal{A}^2\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A} + (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\Omega_\gamma\mathcal{A}\Omega_\gamma,$$

$$\mathcal{A}_i\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}_j - \Omega_\gamma\mathcal{A}\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i) - \mathcal{A}_j\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}_i + (\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\Omega_\gamma\mathcal{A}\Omega_\gamma,$$

$\mathcal{A}_i\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}_i^2 - \mathcal{A}_i^2\Omega_\gamma\mathcal{A}\Omega_\gamma\mathcal{A}_i$, $\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i$,

$\mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2$, $(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)$,

$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)$,

$\mathcal{A}^2\Omega_\gamma\mathcal{A}^2$, $\mathcal{A}_i\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma\mathcal{A}_i$, $\mathcal{A}_i^2\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma\mathcal{A}_i^2$,

$(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)$,

$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)$, $\Omega_p\Omega_\gamma\mathcal{A}^2 - \mathcal{A}^2\Omega_\gamma\Omega_p$,

$\Omega_p^2\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma\Omega_p^2$, $(\Omega_p\Omega_q - \Omega_q\Omega_p)\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma(\Omega_q\Omega_p - \Omega_p\Omega_q)$,

$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})$,

$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\Omega_\gamma\mathcal{A}^2 + \mathcal{A}^2\Omega_\gamma(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)$, $\Omega_p\mathcal{A}_\alpha\mathcal{A}^2 + \mathcal{A}^2\mathcal{A}_\alpha\Omega_p$,

$\Omega_p^2\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\Omega_p^2$, $(\Omega_p\Omega_q - \Omega_q\Omega_p)\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha(\Omega_q\Omega_p - \Omega_p\Omega_q)$,

$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha(-\Omega_p\mathcal{A} + \mathcal{A}\Omega_p)$,

$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha(-\Omega_p\mathcal{A}_i + \mathcal{A}_i\Omega_p)$, $\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta - \mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha$,

$\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$, $\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$,

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}$,

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta - \mathcal{A}\mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta\mathcal{A}$,

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i$,

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta - \mathcal{A}_i\mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta\mathcal{A}_i$,

$\Omega_\gamma\mathcal{A}^2\Omega_\delta - \Omega_\delta\mathcal{A}^2\Omega_\gamma$, $\mathcal{A}\Omega_\gamma\mathcal{A}^2\Omega_\gamma - \Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}$, $\mathcal{A}^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma - \Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}^2$,

$\mathcal{A}\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}^2 - \mathcal{A}^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}$, $\Omega_p\Omega_\gamma\mathcal{A}^2\Omega_\gamma + \Omega_\gamma\mathcal{A}^2\Omega_\gamma\Omega_p$,

$\Omega_p^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma - \Omega_\gamma\mathcal{A}^2\Omega_\gamma\Omega_p^2$,

$\Omega_p\Omega_\gamma\mathcal{A}^2\Omega_\delta - \Omega_p\Omega_\delta\mathcal{A}^2\Omega_\gamma + \Omega_\delta\mathcal{A}^2\Omega_\gamma\Omega_p - \Omega_\gamma\mathcal{A}^2\Omega_\delta\Omega_p$,

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$\mathcal{A}\Omega_\gamma\mathcal{A}^2\Omega_\delta - \mathcal{A}\Omega_\delta\mathcal{A}^2\Omega_\gamma - \Omega_\delta\mathcal{A}^2\Omega_\gamma\mathcal{A} + \Omega_\gamma\mathcal{A}^2\Omega_\delta\mathcal{A}$, $\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma + \Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha$,

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma + \mathcal{A}\Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma\mathcal{A}$,

$\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\Omega_p + \Omega_p\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$,

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^2\Omega_\gamma - \Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}_i$, $\mathcal{A}_i^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma - \Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}_i^2$,

$\Omega_p\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta - \Omega_p\mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha + \mathcal{A}_\beta\mathcal{A}^2\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta\Omega_p$,

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^2\Omega_\delta - \mathcal{A}_i\Omega_\delta\mathcal{A}^2\Omega_\gamma - \Omega_\delta\mathcal{A}^2\Omega_\gamma\mathcal{A}_i - \Omega_\gamma\mathcal{A}^2\Omega_\delta\mathcal{A}_i$,

$\Omega_p\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma + \Omega_p\Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha - \Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma\Omega_p$,

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma + \mathcal{A}_i\Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma\mathcal{A}_i$,

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) - \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A} - (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$,

$\mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2$, $\Omega_p^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\Omega_p^2$,

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i$,

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_j + \mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j) +$

$\qquad - \mathcal{A}_j\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i - (\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$,

$\mathcal{A}\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}^2 - \Omega_\gamma\mathcal{A}^2\Omega_\gamma(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) +$

$\qquad - \mathcal{A}^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A} + (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\Omega_\gamma\mathcal{A}^2\Omega_\gamma$,

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}_j - \Omega_\gamma\mathcal{A}^2\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i) +$

$\qquad - \mathcal{A}_j\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}_i + (\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\Omega_\gamma\mathcal{A}^2\Omega_\gamma$,

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}_i^2 - \mathcal{A}_i^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma\mathcal{A}_i$,

$\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha$, $\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}$, $\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2$, $\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i$,

$\mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2$, $(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)$,

$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)$, $\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma$, $\mathcal{A}\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma\mathcal{A}$,

$\mathcal{A}^2\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma\mathcal{A}^2\,,\ \mathcal{A}_i\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma\mathcal{A}_i\,,\ \mathcal{A}_i^2\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma\mathcal{A}_i^2\,,$

$(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma(\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\,,$

$(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\,,\ \Omega_p\Omega_\gamma\mathcal{A}^3 - \mathcal{A}^3\Omega_\gamma\Omega_p\,,$

$\Omega_p^2\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma\Omega_p^2\,,\ (\Omega_p\Omega_q - \Omega_q\Omega_p)\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma(\Omega_q\Omega_p - \Omega_p\Omega_q)\,,$

$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\,,$

$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\Omega_\gamma\mathcal{A}^3 + \mathcal{A}^3\Omega_\gamma(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\,,\ \Omega_p\mathcal{A}_\alpha\mathcal{A}^3 + \mathcal{A}^3\mathcal{A}_\alpha\Omega_p\,,$

$\Omega_p^2\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha\Omega_p^2\,,\ (\Omega_p\Omega_q - \Omega_q\Omega_p)\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha(\Omega_q\Omega_p - \Omega_p\Omega_q)\,,$

$(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha(-\Omega_p\mathcal{A} + \mathcal{A}\Omega_p)\,,$

$(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^3 - \mathcal{A}^3\mathcal{A}_\alpha(-\Omega_p\mathcal{A}_i + \mathcal{A}_i\Omega_p)\,,$

$\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta - \mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha,\ \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha,\ \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha,$

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A},$

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta - \mathcal{A}\mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta\mathcal{A},$

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i,$

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta - \mathcal{A}_i\mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha - \mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta\mathcal{A}_i,$

$\Omega_\gamma\mathcal{A}^3\Omega_\delta - \Omega_\delta\mathcal{A}^3\Omega_\gamma,\ \mathcal{A}\Omega_\gamma\mathcal{A}^3\Omega_\gamma - \Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A},\ \mathcal{A}^2\Omega_\gamma\mathcal{A}^3\Omega_\gamma - \Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}^2,$

$\mathcal{A}\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}^2 - \mathcal{A}^2\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A},\ \Omega_p\Omega_\gamma\mathcal{A}^3\Omega_\gamma + \Omega_\gamma\mathcal{A}^3\Omega_\gamma\Omega_p,$

$\Omega_p^2\Omega_\gamma\mathcal{A}^3\Omega_\gamma - \Omega_\gamma\mathcal{A}^3\Omega_\gamma\Omega_p^2,$

$\Omega_p\Omega_\gamma\mathcal{A}^3\Omega_\delta - \Omega_p\Omega_\delta\mathcal{A}^3\Omega_\gamma + \Omega_\delta\mathcal{A}^3\Omega_\gamma\Omega_p - \Omega_\gamma\mathcal{A}^3\Omega_\delta\Omega_p,$

$\mathcal{A}\Omega_\gamma\mathcal{A}^3\Omega_\delta - \mathcal{A}\Omega_\delta\mathcal{A}^3\Omega_\gamma - \Omega_\delta\mathcal{A}^3\Omega_\gamma\mathcal{A} + \Omega_\gamma\mathcal{A}^3\Omega_\delta\mathcal{A},$

$\mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma + \Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha,$

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma + \mathcal{A}\Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma\mathcal{A},$

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$\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\Omega_p + \Omega_p\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha,$

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^3\Omega_\gamma - \Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}_i,\ \mathcal{A}_i^2\Omega_\gamma\mathcal{A}^3\Omega_\gamma - \Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}_i^2,$

$\Omega_p\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta - \Omega_p\mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha + \mathcal{A}_\beta\mathcal{A}^3\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\beta\Omega_p,$

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^3\Omega_\delta - \mathcal{A}_i\Omega_\delta\mathcal{A}^3\Omega_\gamma - \Omega_\delta\mathcal{A}^3\Omega_\gamma\mathcal{A}_i - \Omega_\gamma\mathcal{A}^3\Omega_\delta\mathcal{A}_i,$

$\Omega_p\mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma + \Omega_p\Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha - \Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma\Omega_p,$

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma + \mathcal{A}_i\Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha + \Omega_\gamma\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}^3\Omega_\gamma\mathcal{A}_i,$

$\mathcal{A}\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) +$

$- \mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A} - (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha,$

$\mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2,\ \Omega_p^2\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha - \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\Omega_p^2,$

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i,$

$\mathcal{A}_i\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_j + \mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j) +$

$- \mathcal{A}_j\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i - (\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha\mathcal{A}^3\mathcal{A}_\alpha,$

$\mathcal{A}\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}^2 - \Omega_\gamma\mathcal{A}^3\Omega_\gamma(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}) +$

$- \mathcal{A}^2\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A} + (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\Omega_\gamma\mathcal{A}^3\Omega_\gamma,$

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}_j - \Omega_\gamma\mathcal{A}^3\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i) +$

$- \mathcal{A}_j\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}_i + (\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\Omega_\gamma\mathcal{A}^3\Omega_\gamma,$

$\mathcal{A}_i\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}_i^2 - \mathcal{A}_i^2\Omega_\gamma\mathcal{A}^3\Omega_\gamma\mathcal{A}_i.$

3. The Representation for $\vec{M} - \vec{M}^T$

In ref. [3] we have found the representation for $\vec{M}$, and we report here the result, that $\vec{M}$ is a linear combination of the tensors in the following table

Table 1

$I_1\,,\ \mathcal{A}_\alpha I_1\,,\ \mathcal{A}\mathcal{A}_\alpha I_1\,,\ \mathcal{A}^2\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_i\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_i^2\mathcal{A}_\alpha I_1\,,$

$(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\mathcal{A}_\alpha I_1\,,\ (\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha I_1\,,$

$\Omega_\gamma I_1\,,\ \mathcal{A}\Omega_\gamma I_1\,,\ \mathcal{A}^2\Omega_\gamma I_1\,,\ \mathcal{A}_i\Omega_\gamma I_1\,,\ \mathcal{A}_i^2\Omega_\gamma I_1\,,\ (\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\Omega_\gamma I_1\,,\ (\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\Omega_\gamma I_1\,,$

$\Omega_p\Omega_\gamma I_1\,,\ \Omega_p^2\Omega_\gamma I_1\,,\ (\Omega_p\Omega_q-\Omega_q\Omega_p)\Omega_\gamma I_1\,,\ (\mathcal{A}\Omega_p-\Omega_p\mathcal{A})\Omega_\gamma I_1\,,\ (\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\Omega_\gamma I_1\,,$

$\Omega_p\mathcal{A}_\alpha I_1\,,\ \Omega_p^2\mathcal{A}_\alpha I_1\,,\ (\Omega_p\Omega_q-\Omega_q\Omega_p)\mathcal{A}_\alpha I_1\,,\ (\mathcal{A}\Omega_p-\Omega_p\mathcal{A})\mathcal{A}_\alpha I_1\,,\ (\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\mathcal{A}_\alpha I_1\,,$

with $i,j=1,...,N$ with $i<j$; $p,q=1,...,M$ with $p<q$; $\alpha=1,...,N$; $\gamma=1,...,M$.

In this way we have obtained the representation for $\vec{M}$, while we needed that of $\vec{M}-\vec{M}^T$. To this end, it is obvious, it will suffice to detract, from each element of Table 1, its transposed, that is the product of the transposed of its factors, in the inverse order. We omit the result for the sake of brevity.
In sect. 3 of ref. [3] it has also been proved that each element of the following set (9) is a linear combination of the elements of the Table 1.

$$\begin{aligned}
&\mathcal{A}\mathcal{A}_i\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha I_1\,,\ \Omega_p\Omega_q\mathcal{A}_\alpha I_1\,,\ \mathcal{A}\Omega_p\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_i\Omega_p\mathcal{A}_\alpha I_1\,,\\
&\mathcal{A}\mathcal{A}_i\Omega_\gamma I_1\,,\ \mathcal{A}_i\mathcal{A}_j\Omega_\gamma I_1\,,\ \Omega_p\Omega_q\Omega_\gamma I_1\,,\ \mathcal{A}\Omega_p\Omega_\gamma I_1\,,\ \mathcal{A}_i\Omega_p\Omega_\gamma I_1\,,\\
&\Omega_p\mathcal{A}\Omega_p\Omega_\gamma I_1\,,\ \Omega_p^2\mathcal{A}\Omega_\gamma I_1\,,\ \Omega_p\mathcal{A}_i\Omega_p\Omega_\gamma I_1\,,\ \Omega_p^2\mathcal{A}_i\Omega_\gamma I_1\,,\\
&\mathcal{A}_i\mathcal{A}\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_j\mathcal{A}_i\mathcal{A}_\alpha I_1\,,\ \Omega_q\Omega_p\mathcal{A}_\alpha I_1\,,\ \Omega_p\mathcal{A}\mathcal{A}_\alpha I_1\,,\ \Omega_p\mathcal{A}_i\mathcal{A}_\alpha I_1\,,\\
&\mathcal{A}_i\mathcal{A}\Omega_\gamma I_1\,,\ \mathcal{A}_j\mathcal{A}_i\Omega_\gamma I_1\,,\ \Omega_q\Omega_p\Omega_\gamma I_1\,,\ \Omega_p\mathcal{A}\Omega_\gamma I_1\,,\ \Omega_p\mathcal{A}_i\Omega_\gamma I_1\,.
\end{aligned}\tag{9}$$

We report this property because it will be used also in the present paper. But here we will need also the following property:
Each element of the following set (10) is a linear combination of the elements of the Table 1.

$$\begin{aligned}
&\mathcal{A}_i\mathcal{A}_j\mathcal{A}_i\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_j\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_i\mathcal{A}\mathcal{A}_i\mathcal{A}_\alpha I_1\,,\\
&\mathcal{A}_i^2\mathcal{A}_j\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_j^2\mathcal{A}_i\mathcal{A}_\alpha I_1\,,\ \mathcal{A}_i^2\mathcal{A}\mathcal{A}_\alpha I_1\,.
\end{aligned}\tag{10}$$

It can be proved as the previous ones. In particular, we note that the vectors $A_iA_jA_i\vec{a}_\alpha$, $A_jA_iA_j\vec{a}_\alpha$, $A_iAA_i\vec{a}_\alpha$, $A_i^2A_j\vec{a}_\alpha$, $A_j^2A_i\vec{a}_\alpha$, $A_i^2A\vec{a}_\alpha$ are not present in table 1 of ref. [3]. The reason is obvious: they have not been inserted because are linear combinations of the remainder; let us now apply to both sides, of each of these linear combinations, the passages used to obtain (in ref. [3]) Table 1 ter from Table 1; we find, obviously, that the right hand side of this linear combination becomes a new linear combination, but of the elements in table 1 ter of ref. [3]. Instead, for the left hand side, we have to perform the some calculations used in ref. [3] to obtain the table 1 ter.. We obtain the above mentioned result.
The properties (9) and (10) will be used below, because sometimes each of these tensors, detracted by its transposed, will appear; well, in these cases it can be omitted because its contribute has already been considered trough the contribute of the tensors in Table 1.

4. The Representation for M^*

As usual, let us begin by using for M^* the 3-dimensional representation theorems reported in [7]; but, we observe that in [7] the second order tensors are indicated in correspondence with the indexes i, j, k going from 1 to N. But here we have to explicitate the terms containing the tensor A which comes from $\mathcal{A}$. Consequently, we can consider the indexes i, j, k going from 0 to N; after that we can write explicitly the terms with the index 0 and substitute A_0 with A. In this way we obtain a set of tensors S; we don' t report them for the sake of brevity, but effectively they can be identified with those reported in the next Table 2, raws from 1 to 18, and with $\vec{a}_\alpha$, $\vec{a}_\beta$, instead of the original $\vec{v}_m$, $\vec{v}_n$. We have now to take into account of the fact that the vectors $\vec{v}_\alpha$, $\vec{v}_\beta$ may originate from the 4-dimensional symmetric second order tensors, or from the skew-symmetric ones. Therefore, let us write 3 copies of the set S: In the first one we substitute $\vec{v}_\alpha$, $\vec{v}_\beta$ with $\vec{a}_\alpha$, $\vec{a}_\beta$ (which originate from the 4-dimensional symmetric second order tensors) obtaining just the raws from 1 to 18 of the table 2. In the second copy we leave the terms not depending on vectors (because already considered above) and, in the remaining ones, we substitute $\vec{v}_\alpha$, $\vec{v}_\beta$ with $\vec{\omega}_\gamma$, $\vec{\omega}_\delta$ (which originate from the 4-dimensional skew-symmetric second order tensors), so obtaining the raws from 19 to 28 of table 2. From the third copy we take only the terms where both $\vec{v}_\alpha$, $\vec{v}_\beta$ appear (because the remaining ones have already been considered above) and substitute $\vec{v}_\alpha$ with $\vec{a}_\alpha$ and $\vec{v}_\beta$ with $\vec{\omega}_\gamma$ so obtaining the raws from 29 to 32 of table 2.
We obtain that M is a linear combination of the following 3-dimensional tensors

Table 2

1. W_p, $W_pW_q - W_qW_p$,

2. $AA_i - A_iA$, $A^2A_i - A_iA^2$, $AA_i^2 - A_i^2A$, $AA_iA^2 - A^2A_iA$,

3. $A_iAA_i^2 - A_i^2AA_i$, $AA_iA_j + A_iA_jA + A_jAA_i - A_iAA_j - AA_jA_i - A_jA_iA$,

4. $\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha$, $\vec{a}_\alpha \bigotimes A\vec{a}_\alpha - A\vec{a}_\alpha \bigotimes \vec{a}_\alpha$, $\vec{a}_\alpha \bigotimes A^2\vec{a}_\alpha - A^2\vec{a}_\alpha \bigotimes \vec{a}_\alpha$,

5. $A\vec{a}_\alpha \bigotimes A^2\vec{a}_\alpha - A^2\vec{a}_\alpha \bigotimes A\vec{a}_\alpha$,

6. $A\vec{a}_\alpha \bigotimes A_i\vec{a}_\alpha - A_i\vec{a}_\alpha \bigotimes A\vec{a}_\alpha + \vec{a}_\alpha \bigotimes (AA_i - A_iA)\vec{a}_\alpha - (AA_i - A_iA)\vec{a}_\alpha \bigotimes \vec{a}_\alpha$,

7. $A(\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha) + (\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha)A$,

8. $AW_p + W_pA$, $AW_p^2 - W_p^2A$,

9. $\vec{a}_\alpha \bigotimes W_p\vec{a}_\alpha - W_p\vec{a}_\alpha \bigotimes \vec{a}_\alpha$, $\vec{a}_\alpha \bigotimes W_p^2\vec{a}_\alpha - W_p^2\vec{a}_\alpha \bigotimes \vec{a}_\alpha$,

10. $W_p(\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha) - (\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha)W_p$,

11. $A_iA_j - A_jA_i$, $A_i^2A_j - A_jA_i^2$, $A_iA_j^2 - A_j^2A_i$, $A_iA_jA_i^2 - A_i^2A_jA_i$,

12. $A_jA_iA_j^2 - A_j^2A_iA_j$,

13. $A_iA_jA_k + A_jA_kA_i + A_kA_iA_j - A_jA_iA_k - A_iA_kA_j - A_kA_jA_i$,

14. $\vec{a}_\alpha \bigotimes A_i\vec{a}_\alpha - A_i\vec{a}_\alpha \bigotimes \vec{a}_\alpha$, $\vec{a}_\alpha \bigotimes A_i^2\vec{a}_\alpha - A_i^2\vec{a}_\alpha \bigotimes \vec{a}_\alpha$,

15. $A_i\vec{a}_\alpha \bigotimes A_i^2\vec{a}_\alpha - A_i^2\vec{a}_\alpha \bigotimes A_i\vec{a}_\alpha$,

16. $A_i\vec{a}_\alpha \bigotimes A_j\vec{a}_\alpha - A_j\vec{a}_\alpha \bigotimes A_i\vec{a}_\alpha + \vec{a}_\alpha \bigotimes (A_iA_j - A_jA_i)\vec{a}_\alpha - (A_iA_j - A_jA_i)\vec{a}_\alpha \bigotimes \vec{a}_\alpha$,

17. $A_i(\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha) + (\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha)A_i$,

18. $A_iW_p + W_pA_i$, $A_iW_p^2 - W_p^2A_i$,

19. $\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma$, $\vec{\omega}_\gamma \bigotimes A\vec{\omega}_\gamma - A\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$, $\vec{\omega}_\gamma \bigotimes A^2\vec{\omega}_\gamma - A^2\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$,

20. $A\vec{\omega}_\gamma \bigotimes A^2\vec{\omega}_\gamma - A^2\vec{\omega}_\gamma \bigotimes A\vec{\omega}_\gamma$,

21. $A\vec{\omega}_\gamma \bigotimes A_i\vec{\omega}_\gamma - A_i\vec{\omega}_\gamma \bigotimes A\vec{\omega}_\gamma + \vec{\omega}_\gamma \bigotimes (AA_i - A_iA)\vec{\omega}_\gamma - (AA_i - A_iA)\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$,

22. $A(\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma) + (\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma)A$,

23. $\vec{\omega}_\gamma \bigotimes W_p\vec{\omega}_\gamma - W_p\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$, $\vec{\omega}_\gamma \bigotimes W_p^2\vec{\omega}_\gamma - W_p^2\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$,

24. $W_p(\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma) - (\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma)W_p$,

25. $\vec{\omega}_\gamma \bigotimes A_i\vec{\omega}_\gamma - A_i\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$, $\vec{\omega}_\gamma \bigotimes A_i^2\vec{\omega}_\gamma - A_i^2\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$,

26. $A_i\vec{\omega}_\gamma \bigotimes A_i^2\vec{\omega}_\gamma - A_i^2\vec{\omega}_\gamma \bigotimes A_i\vec{\omega}_\gamma$,

27. $A_i\vec{\omega}_\gamma \bigotimes A_j\vec{\omega}_\gamma - A_j\vec{\omega}_\gamma \bigotimes A_i\vec{\omega}_\gamma + \vec{\omega}_\gamma \bigotimes (A_iA_j - A_jA_i)\vec{\omega}_\gamma - (A_iA_j - A_jA_i)\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma$,

28. $A_i(\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma) + (\vec{\omega}_\gamma \bigotimes \vec{\omega}_\delta - \vec{\omega}_\delta \bigotimes \vec{\omega}_\gamma)A_i$,

29. $\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha$,

30. $A(\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha) + (\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha)A$,

31. $W_p(\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha) - (\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha)W_p$,

32. $A_i(\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha) + (\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha)A_i$,

with $i, j, k = 1, ..., N$ with $i < j < k$; $p, q = 1, ..., M$ with $p < q$; $\alpha, \beta = 1, ..., N$ with $\alpha < \beta$; $\gamma, \delta = 1, ..., M$; $\gamma < \delta$.

This is the representation for M; but we needed that for M^*. Well, as we have done for $\vec{M}$, we find now that M^* is a linear combination of the matrices which can be obtained from the tensors in the previous table, in the following way:

- First of all, we append an apex * to each second order tensor.
- After that we substitute each vector with its expression in capital letters.
- Moreover, for the tensors like $\vec{m} \bigotimes \vec{m'}$, we use the property

$$\begin{aligned} \vec{M}\vec{M}'^T &= \begin{pmatrix} 0 & \vec{0}^T \\ \vec{m} & 0 \end{pmatrix} \begin{pmatrix} 0 & \vec{m}'^T \\ \vec{0} & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 \\ m_2 & 0 & 0 & 0 \\ m_3 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & m_1' & m_2' & m_3' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & m_1 m_1' & m_1 m_2' & m_1 m_3' \\ 0 & m_2 m_1' & m_2 m_2' & m_2 m_3' \\ 0 & m_3 m_1' & m_3 m_2' & m_3 m_3' \end{pmatrix} = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{m} \bigotimes \vec{m}' \end{pmatrix} .$$

So we obtain the new table

Table 2 bis

1. W_p^*, $W_p^* W_q^* - W_q^* W_p^*$,

2. $A^* A_i^* - A_i^* A^*$, $A^{*2} A_i^* - A_i^* A^{*2}$, $A^* A_i^{*2} - A_i^{*2} A^*$, $A^* A_i^* A^{*2} - A^{*2} A_i^* A^*$,

3. $A_i^* A^* A_i^{*2} - A_i^{*2} A^* A_i^*$,

   $A^* A_i^* A_j^* + A_i^* A_j^* A^* + A_j^* A^* A_i^* - A_i^* A^* A_j^* - A^* A_j^* A_i^* - A_j^* A_i^* A^*$,

4. $\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha$, $\vec{A}_\alpha \bigotimes A^* \vec{A}_\alpha - A^* \vec{A}_\alpha \bigotimes \vec{A}_\alpha$, $\vec{A}_\alpha \bigotimes A^{*2} \vec{A}_\alpha - A^{*2} \vec{A}_\alpha \bigotimes \vec{A}_\alpha$,

5. $A^* \vec{A}_\alpha \bigotimes A^{*2} \vec{A}_\alpha - A^{*2} \vec{A}_\alpha \bigotimes A^* \vec{A}_\alpha$,

6. $A^* \vec{A}_\alpha \bigotimes A_i^* \vec{A}_\alpha - A_i^* \vec{A}_\alpha \bigotimes A^* \vec{A}_\alpha +$

   $+ \vec{A}_\alpha \bigotimes (A^* A_i^* - A_i^* A^*) \vec{A}_\alpha - (A^* A_i^* - A_i^* A^*) \vec{A}_\alpha \bigotimes \vec{A}_\alpha$,

7. $A^* (\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha) + (\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha) A^*$,

8. $A^* W_p^* + W_p^* A^*$, $A^* W_p^{*2} - W_p^{*2} A^*$,

9. $\vec{A}_\alpha \bigotimes W_p^* \vec{A}_\alpha - W_p^* \vec{A}_\alpha \bigotimes \vec{A}_\alpha$, $\vec{A}_\alpha \bigotimes W_p^{*2} \vec{A}_\alpha - W_p^{*2} \vec{A}_\alpha \bigotimes \vec{A}_\alpha$,

10. $W_p^* (\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha) - (\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha) W_p^*$,

11. $A_i^* A_j^* - A_j^* A_i^*$, $A_i^{*2} A_j^* - A_j^* A_i^{*2}$, $A_i^* A_j^{*2} - A_j^{*2} A_i^*$, $A_i^* A_j^* A_i^{*2} - A_i^{*2} A_j^* A_i^*$,

12. $A_j^* A_i^* A_j^{*2} - A_j^{*2} A_i^* A_j^*$,

13. $A_i^* A_j^* A_k^* + A_j^* A_k^* A_i^* + A_k^* A_i^* A_j^* - A_j^* A_i^* A_k^* - A_i^* A_k^* A_j^* - A_k^* A_j^* A_i^*$,

14. $\vec{A}_\alpha \bigotimes A_i^* \vec{A}_\alpha - A_i^* \vec{A}_\alpha \bigotimes \vec{A}_\alpha$, $\vec{A}_\alpha \bigotimes A_i^{*2} \vec{A}_\alpha - A_i^{*2} \vec{A}_\alpha \bigotimes \vec{A}_\alpha$,

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15. $A_i^*\vec{A}_\alpha \bigotimes A_i^{*2}\vec{A}_\alpha - A_i^{*2}\vec{A}_\alpha \bigotimes A_i^*\vec{A}_\alpha\,,$

16. $A_i^*\vec{A}_\alpha \bigotimes A_j^*\vec{A}_\alpha - A_j^*\vec{A}_\alpha \bigotimes A_i^*\vec{A}_\alpha +$

$+\vec{A}_\alpha \bigotimes (A_i^*A_j^* - A_j^*A_i^*)\vec{A}_\alpha - (A_i^*A_j^* - A_j^*A_i^*)\vec{A}_\alpha \bigotimes \vec{A}_\alpha\,,$

17. $A_i^*(\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha) + (\vec{A}_\alpha \bigotimes \vec{A}_\beta - \vec{A}_\beta \bigotimes \vec{A}_\alpha)A_i^*\,,$

18. $A_i^*W_p^* + W_p^*A_i^*\,,\ A_i^*W_p^{*2} - W_p^{*2}A_i\,,$

19. $\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma\,,\ \vec{\Omega}_\gamma \bigotimes A^*\vec{\Omega}_\gamma - A^*\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,\ \vec{\Omega}_\gamma \bigotimes A^{*2}\vec{\Omega}_\gamma - A^{*2}\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,$

20. $A^*\vec{\Omega}_\gamma \bigotimes A^{*2}\vec{\Omega}_\gamma - A^{*2}\vec{\Omega}_\gamma \bigotimes A^*\vec{\Omega}_\gamma\,,$

21. $A^*\vec{\Omega}_\gamma \bigotimes A_i^*\vec{\Omega}_\gamma - A_i^*\vec{\Omega}_\gamma \bigotimes A^*\vec{\Omega}_\gamma +$

$+\vec{\Omega}_\gamma \bigotimes (A^*A_i^* - A_i^*A^*)\vec{\Omega}_\gamma - (A^*A_i^* - A_i^*A^*)\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,$

22. $A^*(\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma) + (\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma)A^*\,,$

23. $\vec{\Omega}_\gamma \bigotimes W_p^*\vec{\Omega}_\gamma - W_p^*\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,\ \vec{\Omega}_\gamma \bigotimes W_p^{*2}\vec{\Omega}_\gamma - W_p^{*2}\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,$

24. $W_p^*(\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma) - (\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma)W_p^*\,,$

25. $\vec{\Omega}_\gamma \bigotimes A_i^*\vec{\Omega}_\gamma - A_i^*\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,\ \vec{\Omega}_\gamma \bigotimes A_i^{*2}\vec{\Omega}_\gamma - A_i^{*2}\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,$

26. $A_i^*\vec{\Omega}_\gamma \bigotimes A_i^{*2}\vec{\Omega}_\gamma - A_i^{*2}\vec{\Omega}_\gamma \bigotimes A_i^*\vec{\Omega}_\gamma\,,$

27. $A_i^*\vec{\Omega}_\gamma \bigotimes A_j^*\vec{\Omega}_\gamma - A_j^*\vec{\Omega}_\gamma \bigotimes A_i^*\vec{\Omega}_\gamma +$

$+\vec{\Omega}_\gamma \bigotimes (A_i^*A_j^* - A_j^*A_i^*)\vec{\Omega}_\gamma - (A_i^*A_j^* - A_j^*A_i^*)\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\gamma\,,$

28. $A_i^*(\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma) + (\vec{\Omega}_\gamma \bigotimes \vec{\Omega}_\delta - \vec{\Omega}_\delta \bigotimes \vec{\Omega}_\gamma)A_i^*\,,$

29. $\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha\,,$

30. $A^*(\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha) + (\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha)A^*\,,$

31. $W_p^*(\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha) - (\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha)W_p^*\,,$

32. $A_i^*(\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha) + (\vec{A}_\alpha \bigotimes \vec{\Omega}_\gamma - \vec{\Omega}_\gamma \bigotimes \vec{A}_\alpha)A_i^*\,,$

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with $i, j, k = 1, ..., N$ with $i < j < k$; $p, q = 1, ..., M$ with $p < q$; $\alpha, \beta = 1, ..., N$ with $\alpha < \beta$; $\gamma, \delta = 1, ..., M$; $\gamma < \delta$.

Let us now use eqs. (6) to convert these tensors in their original notation. We find that M^* is a linear combination of the tensors in the following table (The first number at the beginning of each raw of the table denotes the number of raw in the table 2 bis, from which we are now obtaining the corresponding one; the index denotes the position of that tensor in its raw). Obviously, we leave out eventual added terms which are linear combinations of tensors previously obtained in this table or originating from Table 1 ter, or from eqs. (9) and (10).

Table 2 ter

$(1)_{1,2}$, $(2)_1$	Ω_p, $\Omega_p\Omega_q - \Omega_q\Omega_p$, $\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A}$,
$(2)_{2,4}$, $(4)_1$	$\mathcal{A}^2\mathcal{A}_i - \mathcal{A}_i\mathcal{A}^2$, $\mathcal{A}\mathcal{A}_i\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_i\mathcal{A}$, $\mathcal{A}_\alpha I_1\mathcal{A}_\beta - \mathcal{A}_\beta I_1\mathcal{A}_\alpha$,
$(4)_{2,3}$	$\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A} - \mathcal{A}\mathcal{A}_\alpha I_1\mathcal{A}_\alpha$, $\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha I_1\mathcal{A}_\alpha$,
$(2)_3$, (5)	$\mathcal{A}\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}$, $\mathcal{A}\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}^2 - \mathcal{A}^2\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}$,
(7)	$\mathcal{A}\mathcal{A}_\alpha I_1\mathcal{A}_\beta - \mathcal{A}\mathcal{A}_\beta I_1\mathcal{A}_\alpha - \mathcal{A}_\beta I_1\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha I_1\mathcal{A}_\beta\mathcal{A}$,
$(11)_1$, $(14)_1$	$\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$, $\mathcal{A}_i\mathcal{A}_\alpha I_1\mathcal{A}_\alpha - \mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_i$,
(17)	$\mathcal{A}_i\mathcal{A}_\alpha I_1\mathcal{A}_\beta - \mathcal{A}_i\mathcal{A}_\beta I_1\mathcal{A}_\alpha - \mathcal{A}_\beta I_1\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha I_1\mathcal{A}_\beta\mathcal{A}_i$,
$(19)_{1,2,3}$	$\Omega_\gamma I_1\Omega_\delta - \Omega_\delta I_1\Omega_\gamma$, $\mathcal{A}\Omega_\gamma I_1\Omega_\gamma - \Omega_\gamma I_1\Omega_\gamma\mathcal{A}$, $\mathcal{A}^2\Omega_\gamma I_1\Omega_\gamma - \Omega_\gamma I_1\Omega_\gamma\mathcal{A}^2$,
20, $(23)_1$	$\mathcal{A}\Omega_\gamma I_1\Omega_\gamma\mathcal{A}^2 - \mathcal{A}^2\Omega_\gamma I_1\Omega_\gamma\mathcal{A}$, $\Omega_p\Omega_\gamma I_1\Omega_\gamma + \Omega_\gamma I_1\Omega_\gamma\Omega_p$,
$(23)_2$	$\Omega_p^2\Omega_\gamma I_1\Omega_\gamma - \Omega_\gamma I_1\Omega_\gamma\Omega_p^2$,
(24)	$\Omega_p\Omega_\gamma I_1\Omega_\delta - \Omega_p\Omega_\delta I_1\Omega_\gamma + \Omega_\delta I_1\Omega_\gamma\Omega_p - \Omega_\gamma I_1\Omega_\delta\Omega_p$,
$(8)_{1,2}$	$\mathcal{A}\Omega_p + \Omega_p\mathcal{A}$, $\mathcal{A}\Omega_p^2 - \Omega_p^2\mathcal{A}$,
(22)	$\mathcal{A}\Omega_\gamma I_1\Omega_\delta - \mathcal{A}\Omega_\delta I_1\Omega_\gamma - \Omega_\delta I_1\Omega_\gamma\mathcal{A} + \Omega_\gamma I_1\Omega_\delta\mathcal{A}$,
(29)	$\mathcal{A}_\alpha I_1\Omega_\gamma + \Omega_\gamma I_1\mathcal{A}_\alpha$,
(30)	$\mathcal{A}\mathcal{A}_\alpha I_1\Omega_\gamma + \mathcal{A}\Omega_\gamma I_1\mathcal{A}_\alpha + \Omega_\gamma I_1\mathcal{A}_\alpha\mathcal{A} + \mathcal{A}_\alpha I_1\Omega_\gamma\mathcal{A}$,
$(9)_1$, $(18)_1$	$\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\Omega_p + \Omega_p\mathcal{A}_\alpha I_1\mathcal{A}_\alpha$, $\mathcal{A}_i\Omega_p + \Omega_p\mathcal{A}_i$,
$(25)_{1,2}$	$\mathcal{A}_i\Omega_\gamma I_1\Omega_\gamma - \Omega_\gamma I_1\Omega_\gamma\mathcal{A}_i$, $\mathcal{A}_i^2\Omega_\gamma I_1\Omega_\gamma - \Omega_\gamma I_1\Omega_\gamma\mathcal{A}_i^2$,
(10)	$\Omega_p\mathcal{A}_\alpha I_1\mathcal{A}_\beta - \Omega_p\mathcal{A}_\beta I_1\mathcal{A}_\alpha + \mathcal{A}_\beta I_1\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha I_1\mathcal{A}_\beta\Omega_p$,
(28)	$\mathcal{A}_i\Omega_\gamma I_1\Omega_\delta - \mathcal{A}_i\Omega_\delta I_1\Omega_\gamma - \Omega_\delta I_1\Omega_\gamma\mathcal{A}_i - \Omega_\gamma I_1\Omega_\delta\mathcal{A}_i$,
(31)	$\Omega_p\mathcal{A}_\alpha I_1\Omega_\gamma + \Omega_p\Omega_\gamma I_1\mathcal{A}_\alpha - \Omega_\gamma I_1\mathcal{A}_\alpha\Omega_p - \mathcal{A}_\alpha I_1\Omega_\gamma\Omega_p$,
(32)	$\mathcal{A}_i\mathcal{A}_\alpha I_1\Omega_\gamma + \mathcal{A}_i\Omega_\gamma I_1\mathcal{A}_\alpha + \Omega_\gamma I_1\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha I_1\Omega_\gamma\mathcal{A}_i$.

Before proceeding with the other tensors, it is now useful to introduce some theorems in the 3-dimensional context; after that, we will see their implications for the 4-dimensional case.

Theorem 1. *The tensor*

$$A\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes (A\vec{a}_\alpha)\,, \tag{11}$$

is a linear combination of

$$\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha\,,\ \vec{a}_\alpha \bigotimes (A\vec{a}_\alpha) - A\vec{a}_\alpha \bigotimes \vec{a}_\alpha\,, \tag{12}$$
$$A\vec{a}_\alpha \bigotimes \vec{a}_\beta - A\vec{a}_\beta \bigotimes \vec{a}_\alpha + \vec{a}_\alpha \bigotimes (A\vec{a}_\beta) - \vec{a}_\beta \bigotimes (A\vec{a}_\alpha).$$

Proof. In fact, if $\vec{a}_\alpha = \vec{0}$, the theorem holds because the tensor (11) is zero.
If $\vec{a}_\alpha \neq \vec{0}$, we choose the reference frame with x_1 axis directed as $\vec{a}_\alpha$ and x_2 axis directed as the component of $A\vec{a}_\alpha$ orthogonal to $\vec{a}_\alpha$; in this frame we have $\vec{a}_\alpha \equiv (u,0,0)$, $A^{13} = 0$ and $\vec{a}_i \equiv (v_1, v_2, v_3)$ with $u > 0$ and $A^{12} \geq 0$. After that, we note that
If $A_i^{12} = 0$, then $A\vec{a}_\alpha = A^{11}\vec{a}_\alpha$ so that the tensor (11) is equal to $(12)_1$ multiplied by A^{11}.
If $A_i^{12} \neq 0$, $v_3 = 0$, then $\vec{a}_\beta = h\vec{a}_\alpha + kA\vec{a}_\alpha$ for suitable values of h, k. Consequently, the tensor (11) is equal to $(12)_2$ multiplied by $-h$.
If $A_i^{12} \neq 0$, $v_3 \neq 0$, we write (11) equal to a generic linear combination of $(12)_{1-3}$; after that, the component 23 gives the coefficient of $(12)_3$, the component 13 gives the coefficient of $(12)_1$, the component 12 gives the coefficient of $(12)_2$.
Now we note that the contribute of the tensors (12) has already been inserted in table 2 ter; by applying the same procedure to the tensor (11), we find that

$$\mathcal{A}\mathcal{A}_\alpha I_1 \mathcal{A}_\beta - \mathcal{A}_\beta I_1 \mathcal{A}_\alpha \mathcal{A} \tag{13}$$

is a linear combination of the tensors in Table 2 ter. By comparing this result with that in the fifth raw of table 2 ter, we find that also

$$\mathcal{A}\mathcal{A}_\beta I_1 \mathcal{A}_\alpha - \mathcal{A}_\alpha I_1 \mathcal{A}_\beta \mathcal{A}$$

is a linear combination of the tensors in Table 2 ter. This last result shows that (13) holds also with α and β exchanged; consequently, (13) holds without the restriction $\alpha < \beta$. The case $\alpha = \beta$ is already inserted in the third raw of table 2 ter. These conclusions will be used later to complete the table 2 ter, which was stopped before. But other results are necessary, such as the next

Theorem 2. *The tensor*

$$A_i\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes (A_i\vec{a}_\alpha)\,, \tag{14}$$

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is a linear combination of

$$\vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\beta \bigotimes \vec{a}_\alpha \,,\ \vec{a}_\alpha \bigotimes (A_i \vec{a}_\alpha) - A_i \vec{a}_\alpha \bigotimes \vec{a}_\alpha \,, \tag{15}$$
$$A_i \vec{a}_\alpha \bigotimes \vec{a}_\beta - A_i \vec{a}_\beta \bigotimes \vec{a}_\alpha + \vec{a}_\alpha \bigotimes (A_i \vec{a}_\beta) - \vec{a}_\beta \bigotimes (A_i \vec{a}_\alpha).$$

Proof. It is the same of the previous theorem, with A_i instead of A.
Now we note that the contribute of the tensors (15) has already been inserted in table 2 ter; by applying the same procedure to the tensor (14), we find that

$$\mathcal{A}_i \mathcal{A}_\alpha I_1 \mathcal{A}_\beta - \mathcal{A}_\beta I_1 \mathcal{A}_\alpha \mathcal{A}_i \tag{16}$$

is a linear combination of the tensors in Table 2 ter. By comparing this result with that in the seventh raw of table 2 ter, we find that also

$$\mathcal{A}_i \mathcal{A}_\beta I_1 \mathcal{A}_\alpha - \mathcal{A}_\alpha I_1 \mathcal{A}_\beta \mathcal{A}_i$$

is a linear combination of the tensors in Table 2 ter. This last result shows that (16) holds also with α and β exchanged; consequently, (16) holds without the restriction $\alpha < \beta$. The case $\alpha = \beta$ is already inserted in the sixth raw of table 2 ter.

Theorem 3. *The tensor*

$$W_p \vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes (W_p \vec{a}_\alpha)\,, \tag{17}$$

is a linear combination of

$$\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha \,,\ \vec{\omega}_\gamma \bigotimes (W_p \vec{\omega}_\gamma) - W_p \vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma \,, \tag{18}$$
$$W_p \vec{a}_\alpha \bigotimes \vec{\omega}_\gamma - \vec{\omega}_\gamma \bigotimes (W_p \vec{a}_\alpha) - W_p \vec{\omega}_\gamma \bigotimes \vec{a}_\alpha + \vec{a}_\alpha \bigotimes (W_p \vec{\omega}_\gamma).$$

Proof. In fact, if $\vec{\omega}_\gamma = \vec{0}$, the theorem holds because the tensor (17) is zero.
If $\vec{\omega}_\gamma \neq \vec{0}$ and $\vec{a}_\alpha$ is parallel to $\vec{\omega}_\gamma$, that is $\vec{a}_\alpha = \mu \vec{\omega}_\gamma$, then we have that the tensor (17) is equal to the tensor $(18)_2$ multiplied by $-\mu$ and the theorem is proved also in this case.
If $\vec{\omega}_\gamma$ and $\vec{a}_\alpha$ are linearly independent, we can perform the calculations in the reference frame where $\vec{\omega}_\gamma \equiv (u, 0, 0)$ and $\vec{a}_\alpha \equiv (v_1, v_2, 0)$ with $u > 0$ and $v_2 > 0$. After that, we note that
If $W_p^{13} \neq 0$, then the tensor (17) is a linear combination of $(18)_{1,2}$.
If $W_p^{13} = 0$, $W_p^{23} = 0$ then $W_p \vec{a}_\alpha = h \vec{a}_\alpha + k \vec{\omega}_\gamma$ for suitable values of h, k. Consequently, the tensor (17) is equal to $(18)_1$ multiplied by h.
If $W_p^{13} = 0$, $W_p^{23} \neq 0$, then the tensor (17) is a linear combination of $(18)_{1,3}$.
Now we note that the contribute of the tensors (18) has already been inserted in table 2 ter; by applying the same procedure to the tensor (17), we find that

$$\Omega_p \mathcal{A}_\alpha I_1 \Omega_\gamma - \Omega_\gamma I_1 \mathcal{A}_\alpha \Omega_p$$

is a linear combination of the tensors in Table 2 ter.
By comparing this result with that in the raw -1 of table 2 ter, we find that also

$$\Omega_p\Omega_\gamma I_1\mathcal{A}_\alpha - \mathcal{A}_\alpha I_1\Omega_\gamma\Omega_p$$

is a linear combination of the tensors in Table 2 ter.

Theorem 4. *The tensor*

$$A_i\vec{a}_\alpha\bigotimes(A_i\vec{a}_\beta) - A_i\vec{a}_\beta\bigotimes(A_i\vec{a}_\alpha)\,, \tag{19}$$

is a linear combination of

$$\begin{aligned}&\vec{a}_\alpha\bigotimes\vec{a}_\beta - \vec{a}_\beta\bigotimes\vec{a}_\alpha\,,\ \vec{a}_\alpha\bigotimes(A_i\vec{a}_\alpha) - A_i\vec{a}_\alpha\bigotimes\vec{a}_\alpha\,,\ \vec{a}_\beta\bigotimes(A_i\vec{a}_\beta) - A_i\vec{a}_\beta\bigotimes\vec{a}_\beta\,,\\ &\vec{a}_\beta\bigotimes(A_i\vec{a}_\alpha) - A_i\vec{a}_\alpha\bigotimes\vec{a}_\beta - \vec{a}_\alpha\bigotimes(A_i\vec{a}_\beta) + A_i\vec{a}_\beta\bigotimes\vec{a}_\alpha).\end{aligned} \tag{20}$$

Proof. In fact, if $\vec{a}_\alpha = \vec{0}$, the theorem holds because the tensor (19) is zero.
If $\vec{a}_\alpha \neq \vec{0}$ and $\vec{a}_\beta$ is parallel to $\vec{a}_\alpha$, that is $\vec{a}_\beta = \mu\vec{a}_\alpha$, then we have that the tensor (19) is zero.
If $\vec{a}_\alpha$ and $\vec{a}_\beta$ are linearly independent, we can perform the calculations in the reference frame where $\vec{a}_\alpha \equiv (u, 0, 0)$ and $\vec{a}_\beta \equiv (v_1, v_2, 0)$ with $u > 0$ and $v_2 > 0$. After that, we note that
If $A_i^{13} \neq 0$, then the tensor (19) is a linear combination of $(20)_{1,2,4}$.
If $A_i^{13} = 0$, $A_i^{23} = 0$ then the tensor (19) is equal to $(20)_1$ multiplied by $A_i^{11}A_i^{22} - (A_i^{12})^2$.
If $A_i^{13} = 0$, $A_i^{23} \neq 0$, then the tensor (19) is a linear combination of $(20)_{1,3,4}$.
Now we note that the contribute of the tensors (20) has already been inserted in table 2 ter; by applying the same procedure to the tensor (19), we find that

$$\mathcal{A}_i\mathcal{A}_\alpha I_1\mathcal{A}_\beta\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_\beta I_1\mathcal{A}_\alpha\mathcal{A}_i \tag{21}$$

is a linear combination of the tensors in Table 2 ter. Now, if we exchange α and β, we see that (19) changes sign so that it still remains a linear combination of the tensors in Table 2 ter (with coefficient the opposite of the previous ones). Consequently, the statement (21) holds without the restriction $\alpha < \beta$. In the case $\alpha = \beta$ the tensor (21) is zero.

Theorem 5. *The tensor*

$$\vec{a}_\beta\bigotimes(A_i^2\vec{a}_\alpha) - A_i^2\vec{a}_\alpha\bigotimes\vec{a}_\beta\,, \tag{22}$$

is a linear combination of

$$\begin{aligned}&\vec{a}_\beta\bigotimes(A_i^2\vec{a}_\beta) - A_i^2\vec{a}_\beta\bigotimes\vec{a}_\beta\,,\ \vec{a}_\alpha\bigotimes\vec{a}_\beta - \vec{a}_\beta\bigotimes\vec{a}_\alpha\,,\\ &\vec{a}_\beta\bigotimes(A_i\vec{a}_\beta) - A_i\vec{a}_\beta\bigotimes\vec{a}_\beta\,,\end{aligned} \tag{23}$$

$$\vec{a}_\beta \bigotimes (A_i \vec{a}_\alpha) - A_i \vec{a}_\alpha \bigotimes \vec{a}_\beta - \vec{a}_\alpha \bigotimes (A_i \vec{a}_\beta) + A_i \vec{a}_\beta \bigotimes \vec{a}_\alpha).$$

Proof. In fact, if $\vec{a}_\beta = \vec{0}$, the theorem holds because the tensor (22) is zero.
If $\vec{a}_\beta \neq \vec{0}$ and $\vec{a}_\alpha$ is parallel to $\vec{a}_\beta$, that is $\vec{a}_\alpha = \mu \vec{a}_\beta$, then we have that the tensor (22) is equal to $(22)_1$ multiplied by μ.
If $\vec{a}_\alpha$ and $\vec{a}_\beta$ are linearly independent, we can perform the calculations in the reference frame where $\vec{a}_\beta \equiv (u, 0, 0)$ and $\vec{a}_\alpha \equiv (v_1, v_2, 0)$ with $u > 0$ and $v_2 > 0$. After that, we note that
If $A_i^{13} \neq 0$, then the tensor (22) is a linear combination of $(23)_{2,3}$.
If $A_i^{13} = 0$, $A_i^{23} = 0$ then the tensor (22) is the linear combination of $(23)_{1,2}$ trough the coefficients v_1/u and $-[(A_i^{12})^2 + (A_i^{22})^2]$ respectively.
If $A_i^{13} = 0$, $A_i^{23} \neq 0$, then the tensor (22) is a linear combination of $(20)_{2,4}$.
Now we note that the contribute of the tensors (23) has already been inserted in table 2 ter; by applying the same procedure to the tensor (22) and using the result of Theorem 2, we find that

$$\mathcal{A}_i^2 \mathcal{A}_\alpha I_1 \mathcal{A}_\beta - \mathcal{A}_\beta I_1 \mathcal{A}_\alpha \mathcal{A}_i^2 \tag{24}$$

is a linear combination of the tensors in Table 2 ter. We note that in the proof of this theorem, the condition $\alpha < \beta$ played no role; moreover, also (23) with α and β exchanged have been already inserted in Table 2 ter (In fact, only $(23)_{2,4}$ depend on both α and β; moreover, they change simply sign by exchanging α and β). Consequently, the statement (24) holds without the restriction $\alpha < \beta$. In the case $\alpha = \beta$ the tensor (22) becomes $(23)_2$.

Theorem 6. *The tensor*

$$A_i A_j \vec{a}_\alpha \bigotimes \vec{a}_\alpha - \vec{a}_\alpha \bigotimes (A_i A_j \vec{a}_\alpha)\,, \tag{25}$$

is a linear combination of

$$\begin{aligned} & A_i \vec{a}_\alpha \bigotimes \vec{a}_\alpha - \vec{a}_\alpha \bigotimes (A_i \vec{a}_\alpha)\,,\; A_j \vec{a}_\alpha \bigotimes \vec{a}_\alpha - \vec{a}_\alpha \bigotimes (A_j \vec{a}_\alpha)\,, \\ & \vec{a}_\alpha \bigotimes (A_j^2 \vec{a}_\alpha) - A_j^2 \vec{a}_\alpha \bigotimes \vec{a}_\alpha\,,\; \vec{a}_\alpha \bigotimes (A_i^2 \vec{a}_\alpha) - A_i^2 \vec{a}_\alpha \bigotimes \vec{a}_\alpha\,, \\ & A_i \vec{a}_\alpha \bigotimes (A_j \vec{a}_\alpha) - A_j \vec{a}_\alpha \bigotimes (A_i \vec{a}_\alpha) + \\ & \quad + \vec{a}_\alpha \bigotimes (A_i A_j - A_j A_i) \vec{a}_\alpha - (A_i A_j - A_j A_i) \vec{a}_\alpha \bigotimes \vec{a}_\alpha. \end{aligned} \tag{26}$$

Proof. In fact, if $\vec{a}_\alpha = \vec{0}$, the theorem holds because the tensor (25) is zero.
If $\vec{a}_\alpha \neq \vec{0}$ and $A_j \vec{a}_\alpha$ is parallel to $\vec{a}_\alpha$, that is $A_j \vec{a}_\alpha = \mu \vec{a}_\alpha$, then we have that the tensor (25) is equal to $(26)_1$ multiplied by μ.
If $\vec{a}_\alpha$ and $A_j \vec{a}_\alpha$ are linearly independent, we can perform the calculations in the reference frame where $\vec{a}_\alpha \equiv (u, 0, 0)$ and $A_j^{13} = 0$ with $u > 0$ and $A_j^{12} > 0$. After that, we note that

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If $A_i^{13} \neq 0$, then the tensor (25) is a linear combination of $(26)_{1,2}$.
If $A_i^{13} = 0$, $A_j^{23} \neq 0$ then the tensor (25) is a linear combination of $(26)_{1,3}$.
If $A_i^{13} = 0$, $A_j^{23} = 0$, $A_i^{23} = 0$ then the components 13 and 23 of the tensor (25) are zero and (25) is proportional to $(26)_2$.
If $A_i^{13} = 0$, $A_j^{23} = 0$, $A_i^{23} \neq 0$, $A_i^{12} \neq 0$ then (25) is a linear combination of $(26)_{2,4}$.
If $A_i^{13} = 0$, $A_j^{23} = 0$, $A_i^{23} \neq 0$, $A_i^{12} = 0$ then (25) is a linear combination of $(26)_{2,5}$.
Now we note that the contribute of the tensors (26) has already been inserted in table 2 ter or its "partial" continuation ("partial", means until this theorem has not been used); by applying the same procedure to the tensor (25), we find that

$$\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha I_1\mathcal{A}_\alpha - \mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_j\mathcal{A}_i \tag{27}$$

is a linear combination of the tensors in Table 2 ter and its "partial" continuation. We note that in the proof of this theorem, the condition $i < j$ played no role, so that (25) with i and j exchanged, is a linear combination of (26) with i and j exchanged. Also these tensors have been inserted in Table 2 ter (In fact, only $(26)_5$ depends on both i and j and it changes simply sign by exchanging i and j). The same thing can be said if there is A instead of A_j. Consequently, the statement (27) holds without the restriction $i < j$ and also with A instead of A_j.

Theorem 7. *The tensor*

$$A_i\vec{\omega}_\gamma \bigotimes \vec{a}_\alpha - \vec{a}_\alpha \bigotimes (A_i\vec{\omega}_\gamma)\,, \tag{28}$$

is a linear combination of

$$\begin{aligned} &\vec{\omega}_\gamma \bigotimes \vec{a}_\alpha - \vec{a}_\alpha \bigotimes \vec{\omega}_\gamma\,, \\ &\vec{\omega}_\gamma \bigotimes (A_i\vec{\omega}_\gamma) - A_i\vec{\omega}_\gamma \bigotimes \vec{\omega}_\gamma\,, \\ &A_i\vec{\omega}_\gamma \bigotimes \vec{a}_\alpha - \vec{a}_\alpha \bigotimes (A_i\vec{\omega}_\gamma) - A_i\vec{a}_\alpha \bigotimes \vec{\omega}_\gamma + \vec{\omega}_\gamma \bigotimes (A_i\vec{a}_\alpha)\text{"}. \end{aligned} \tag{29}$$

Proof. In fact, if $\vec{\omega}_\gamma = \vec{0}$, the theorem holds because the tensor (28) is zero.
If $\vec{\omega}_\gamma \neq \vec{0}$ and $A_i\vec{\omega}_\gamma$ is parallel to $\vec{\omega}_\gamma$, that is $A_i\vec{\omega}_\gamma = \mu\vec{\omega}_\gamma$, then we have that the tensor (28) is equal to the tensor $(29)_1$ multiplied by μ and the theorem is proved also in this case.
If $A_i\vec{\omega}_\gamma$ and $\vec{\omega}_\gamma$ are linearly independent, we can perform the calculations in the reference frame with x_1 axis directed as $\vec{\omega}_\gamma$ and x_2 axis directed as the component of $A_i\vec{\omega}_\gamma$ orthogonal to $\vec{\omega}_\gamma$; in this frame we have $\vec{\omega}_\gamma \equiv (u, 0, 0)$ and $A_i^{13} = 0$ with $u > 0$ and $A_i^{12} > 0$. After that, it is easy to verify that the tensor (28) is a linear combination (29); in particular, the component 23 shows that the coefficient of $(29)_3$ is 1.
The components 13 and 12 give the other coefficients, except for the case in which the third component of $\vec{a}_\alpha$ is zero. But, in this last case, we have $\vec{a}_\alpha = h\vec{\omega}_\gamma + kA_i\vec{\omega}_\gamma$ and the tensor (28) is equal to $(28)_2$ multiplied by $-h$.

Now we note that the contribute of the tensors (29) has already been inserted in table 2 ter; by applying the same procedure to the tensor (28), we find that

$$\mathcal{A}_i\Omega_\gamma I_1\mathcal{A}_\alpha - \mathcal{A}_\alpha I_1\Omega_\gamma\mathcal{A}_i$$

is a linear combination of the tensors in Table 2 ter.
Now, with aid from these theorems we can complete the Table 2 ter and obtain

Table 2 ter (Continuation)

(6)	$\mathcal{A}\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_i + \mathcal{A}_\alpha I_1\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})+$ $-\mathcal{A}_i\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A} - (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\mathcal{A}_\alpha I_1\mathcal{A}_\alpha,$
$(11)_{2,3}$	$\mathcal{A}_i^2\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i^2$, $\mathcal{A}_i\mathcal{A}_j^2 - \mathcal{A}_j^2\mathcal{A}_i$
$(3)_2$	$\mathcal{A}\mathcal{A}_i\mathcal{A}_j + \mathcal{A}_j\mathcal{A}\mathcal{A}_i + \mathcal{A}_i\mathcal{A}_j\mathcal{A}+$ $-\mathcal{A}_j\mathcal{A}_i\mathcal{A} - \mathcal{A}_i\mathcal{A}\mathcal{A}_j - \mathcal{A}\mathcal{A}_j\mathcal{A}_i$
(13)	$\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k + \mathcal{A}_j\mathcal{A}_k\mathcal{A}_i + \mathcal{A}_k\mathcal{A}_i\mathcal{A}_j+$ $-\mathcal{A}_k\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_k\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i\mathcal{A}_k$
$(14)_2$, $(9)_2$	$\mathcal{A}_i^2\mathcal{A}_\alpha I_1\mathcal{A}_\alpha - \mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_i^2$, $\Omega_p^2\mathcal{A}_\alpha I_1\mathcal{A}_\alpha - \mathcal{A}_\alpha I_1\mathcal{A}_\alpha\Omega_p^2,$
$(8)_2$, (15)	$\mathcal{A}_i\Omega_p^2 - \Omega_p^2\mathcal{A}_i$, $\mathcal{A}_i\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_i,$
(16)	$\mathcal{A}_i\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_j + \mathcal{A}_\alpha I_1\mathcal{A}_\alpha(\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)+$ $-\mathcal{A}_j\mathcal{A}_\alpha I_1\mathcal{A}_\alpha\mathcal{A}_i - (\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\alpha I_1\mathcal{A}_\alpha,$
(3), $(11)_4$	$\mathcal{A}_i\mathcal{A}\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}\mathcal{A}_i$, $\mathcal{A}_i\mathcal{A}_j\mathcal{A}_i^2 - \mathcal{A}_i^2\mathcal{A}_j\mathcal{A}_i,$
(12)	$\mathcal{A}_j\mathcal{A}_i\mathcal{A}_j^2 - \mathcal{A}_j^2\mathcal{A}_i\mathcal{A}_j,$
(21)	$\mathcal{A}\Omega_\gamma I_1\Omega_\gamma\mathcal{A}^2 - \Omega_\gamma I_1\Omega_\gamma(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})+$ $-\mathcal{A}^2\Omega_\gamma I_1\Omega_\gamma\mathcal{A} + (\mathcal{A}_i\mathcal{A} - \mathcal{A}\mathcal{A}_i)\Omega_\gamma I_1\Omega_\gamma,$
(27)	$\mathcal{A}_i\Omega_\gamma I_1\Omega_\gamma\mathcal{A}_j - \Omega_\gamma I_1\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)+$ $-\mathcal{A}_j\Omega_\gamma I_1\Omega_\gamma\mathcal{A}_i + (\mathcal{A}_j\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j)\Omega_\gamma I_1\Omega_\gamma,$
(26)	$\mathcal{A}_i\Omega_\gamma I_1\Omega_\gamma\mathcal{A}_i^2 - \mathcal{A}_i^2\Omega_\gamma I_1\Omega_\gamma\mathcal{A}_i.$

Aim of the next passage is to eliminate the tensor I_1 from the above representations, expressing it in terms of $\mathcal{A}$ and its powers; it has already been described in the introduction. The result is reported in Section 2, so that our work has reached its end.

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