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n-PRIMARY MODULS

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Abstract: Let $n \in \{1, 2, 3, 4, 5\}$. The concept of n-primary modules is defined and the relationship among the families of such modules is considered.

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1. Introduction

Throughout this paper, R is a ring (not necessarily commutative) with identity and all modules are unital left modules. Let M be an R-module. For any submodules N, L of M, we set

$$(N :_R L) = \{r \in R : rL \subseteq N\}.$$

Note that $(N :_R L)$ is the annihilator in R of the R-module $(L+N)/N$ and $(N :_R L)$ is an ideal of R. For any submodule N of M and ideal A of R, we set

$$(N :_M A) = \{m \in M : Am \subseteq N\}.$$

Note that $(N :_M A)$ is a submodule of M.

Let R be any ring and let M be a non-zero R-module. By an associated prime ideal P of M we mean a prime ideal P of R such that $P = (0 :_R L)$ for some (non-zero) submodule L of M. We shall denote the (possibly empty) collection of associated prime ideals of M by $ass(M)$.

Let $n \in \{1, 2, 3, 4, 5\}$. We define the concept of n-primary modules and we investigate the relationship between n-primary modules and m-primary modules for distinct integers $n, m \in \{1, 2, 3, 4, 5\}$. We establish the following implications for a

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non-zero module M over a general ring R:

$$\begin{array}{ccccc} & & & & \text{1-primary} \\ & & & & \Downarrow \\ \text{4-primary} & \Rightarrow & \text{3-primary} & \Rightarrow & \text{2-primary} \\ \Downarrow & & & & \\ \text{5- primary} & & & & \end{array}$$

2. n-Primary Modules

We begin this section with the definition of n-primary modules for $n \in \{1, 2, 3, 4, 5\}$.

Definition 1. Let R be a ring and let M be a non-zero R-module. We shall call M 1-primary if, whenever $a \in R$ and $m \in M$ such that $am = 0$, then $m = 0$ or $a^n M = 0$ for some positive integer n. We shall call M 2-primary if, whenever $a \in R$ and $m \in M$ such that $aRm = 0$, then $m = 0$ or $a^n M = 0$ for some positive integer n. Next we shall call M 3-primary if, whenever A is an ideal of R and L a submodule of M such that $AL = 0$, then $L = 0$ or $A^n M = 0$ for some positive integer n. Moreover, M is called 4-primary if there exists a positive integer n such that $A^n M = 0$ for every ideal A of R such that $(0 :_M A) \neq 0$. Finally, M is called 5-primary if $|ass(M)| = 1$, i.e. M has precisely one associated prime ideal.

It is clear that every 1-primary module is 2-primary.

Lemma 2.1. *Let M be an R-module with annihilator $A = (0 :_R M)$. Then M is 2-primary if and only if $(B + A)/A$ is a nil ideal of the ring R/A for every ideal B of R such that $(0 :_M B) \neq 0$.*

Proof. ($\Rightarrow$) Let B be an ideal of R such that $BL = 0$ for some non-zero submodule L of M. Let $b \in B$. Then $bRL = 0$ and hence $b^n \in A$ for some positive integer n. Then $(B + A)/A$ is a nil ideal.

($\Leftarrow$) Suppose that $aRm = 0$ for some $a \in R$, $0 \neq m \in M$. Then $(RaR)m = 0$ implies that $a + A$ is nilpotent. The result follows.

Let R be a commutative ring. For any ideal A of R, we set

$$\sqrt{A} = \{r \in R : r^n \in A \text{ for some positive integer } n\}.$$

Then $\sqrt{A}$ is an ideal of R and, in case $A \neq R$, $\sqrt{A}$ is the intersection of all prime ideals of R containing A. For any R-module M we set

$$Z(M) = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}.$$

□

Lemma 2.2. *Let R be a commutative ring. Then the following statements are equivalent for a non-zero R-module M with annihilator A in R:*

(i) M is 1-primary.

(ii) M is 2-primary.

(iii) $Z(M) \subseteq \sqrt{A}$.

Moreover, in this case $\sqrt{A}$ is a prime ideal of R.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ Clear.

Now suppose that (i) holds. Let $P = \sqrt{A}$. Because $M \neq 0$, P is a proper ideal of R. Let $a, b \in R$ such that $ab \in P$. Then $(ab)^k M = 0$ for some positive integer k. It follows that $a^k \in Z(M)$ or $b^k \in Z(M)$. Hence $a \in P$ or $b \in P$. Thus P is a prime ideal of R. □

Lemma 2.3. *Let R be any ring. Then a non-zero R-module M is 4-primary if and only if there exists a prime ideal P of R such that:*

(i) $P^n \subseteq (0 :_R M) \subseteq P$ for some positive integer n, and

(ii) $(0 :_M B) = 0$ for every ideal B of R not contained in P.

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ Suppose that M is 4-primary. Let n be a positive integer such that $B^n M = 0$ for every ideal B of R such that $(0 :_M B) \neq 0$. Let $A = (0 :_R M)$ and let P denote the sum of all ideals C of R such that $(0 :_M C) \neq 0$. Note that $A \subseteq P$. Let $p_i \in P$ $(1 \leq i \leq n)$. There exist a positive integer k and ideals B_j $(1 \leq j \leq k)$ of R such that $(0 :_M B_j) \neq 0$ $(1 \leq j \leq k)$ and $p_i \in B_1 + \cdots + B_k$ $(1 \leq i \leq n)$. Note that $B_j^n M = 0$ $(1 \leq j \leq k)$. If $B = B_1 + \cdots + B_k$, then $B^{k(n-1)+1} M = 0$. It follows that $(0 :_M B) \neq 0$ and hence that $B^n M = 0$. Thus $p_1 \cdots p_n M = 0$. This implies that $P^n M = 0$. Hence P is a proper ideal of R. Note that we have proved (i) and (ii).

It remains to show that P is a prime ideal of R. Let G and H be ideals of R none of them is contained in P. Then $(0 :_M G) = 0$ and $(0 :_M H) = 0$. Let $m \in (0 :_M GH)$. Then $GHm = 0$, so that $Hm = 0$ and hence $m = 0$. Thus $(0 :_M GH) = 0$ and GH is not contained in P. It follows that P is a prime ideal of R. □

Corollary 2. *Let R be any ring. Then every non-zero 4-primary R-module is 5-primary.*

Proof. Let M be any non-zero 4-primary R-module. Let P be the prime ideal considered in Lemma 2.3. Then $P^k M = 0$ and $P^{k-1} M \neq 0$ for some positive integer k. Let $L = P^{k-1} M$. If $B = (0 :_R L)$, then B is an ideal of R and, by Lemma 2.3, $B \subseteq P$. It follows that $P = (0 :_R L)$ and hence $P \in ass(M)$. Let $Q \in ass(M)$. Then $Q = (0 :_R N)$ for some (non-zero) submodule N of M. By Lemma 2.3, $Q \subseteq P$. But $P^k \subseteq (0 :_R M) \subseteq Q$ and hence $P \subseteq Q$. Thus $P = Q$. If follows that $ass(M) = \{P\}$ and hence M is 5-primary. □

Lemma 2.5. *Let R be any ring. Consider the following statements for an R-module M:*

(i) M is 4-primary.

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(ii) M is 3-primary.
(iii) M is 2-primary.
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).
Proof. Clear.

In [3] and [4], rings R are considered satisfying the following two properties:

(P1) for every proper ideal I of R, there exist a positive integer n such prime ideals $P_i (1 \leq i \leq n)$ such that $P_1 \cdots P_n \subseteq I \subseteq P_1 \cap \cdots \cap P_n$, and

(P2) for every ascending chain $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots$ of prime ideals of R there exists a positive integer n such that $Q_n = Q_{n+1} = Q_{n+2} = \cdots$.

For such rings we have the following result. □

Proposition 2.6. *Let R be a ring which satisfies (P1) and (P2). Then a non-zero R-module M is 3-primary if and only if M is 4-primary.*

Proof. Suppose that M is 3-primary. Let $A = (0 :_R M)$. By hypothesis, there exist a positive integer n and prime ideals $P_i (1 \leq i \leq n)$ of R such that $P_1 \cdots P_n \subseteq A \subseteq P_1 \cap \cdots \cap P_n$. Then $P_1 \cdots P_n M = 0$. Hence there exists $1 \leq i \leq n$ such that $(0 :_M P_i) \neq 0$. Let P be a prime ideal chosen maximal in the collection of prime ideals Q of R such that $A \subseteq Q$ and $(0 :_M Q) \neq 0$. Let $L = (0 :_M P) \neq 0$. Then $PL = 0$ implies that $P^k M = 0$ for some positive integer k. Let B be an ideal of R such that $BN = 0$ for some non-zero submodule N of M. Because M is 3-primary, $B^t \subseteq A \subseteq P$ for some positive integer t and hence $B \subseteq P$ so that $B^k M = 0$. It follows that M is 4-primary.

By [1, Theorems 7.1 and 7.4], any ring with left (or right) Krull dimension satisfies (P1) and (P2). For such rings we have the following result. □

Proposition 2.7. *Let R be a ring with left (or right) Krull dimension. Then a non-zero R-module M is 2-primary if and only if M is 3-primary.*

Proof. Suppose that M is 3-primary. Then M is 2-primary by Lemma 2.5. Conversely, suppose that M is 2-primary. Let $A = (0 :_R M)$. Let B be an ideal of R such that $Bm = 0$ for some non-zero element m of M. By Lemma 2.1, $(B + A)/A$ is a nil ideal of the ring R/A. But the ring R/A has left (or right) Krull dimension. Hence $(B + A)/A$ is a nilpotent ideal of R/A by [2, Theorem 6.3.7]. Hence $B^k \subseteq A$ and $B^k M = 0$ for some positive integer k. It follows that M is 3-primary. □

Summary. We have established the following implications for a module M over a general ring R:

$$\begin{array}{ccccc} & & & & \text{1-primary} \\ & & & & \Downarrow \\ \text{4-primary} & \Rightarrow & \text{3-primary} & \Rightarrow & \text{2-primary} \\ \Downarrow & & & & \\ \text{5- primary} & & & & \end{array}$$

Moreover, if R is a commutative ring, then

$$\text{2-primary} \Rightarrow \text{1-primary}.$$

If R satisfies (P1) and (P2), then

$$\text{3-primary} \Rightarrow \text{4-primary}.$$

Next if R has left (or right) Krull dimension, then

$$\text{2-primary} \Rightarrow \text{3-primary} \Rightarrow \text{4-primary}.$$

The examples below show that even for commutative rings none of the following implications is true:

$$\text{2-primary} \Rightarrow \text{3-primary},$$
$$\text{3-primary} \Rightarrow \text{4-primary},$$
$$\text{3-primary} \Rightarrow \text{5-primary, and}$$
$$\text{5-primary} \Rightarrow \text{4-primary}.$$

Example 2.8. Let p be any prime integer, let K be a field of characteristic p, let G be the prufer p-group, and let R be the group algebra $K[G]$. Then R is a commutative ring with unique maximal ideal $J = \sum_{g \in G}(g-1)R$ and J is a nil ideal of R such that $J = J^2$. Moreover

(i) the R-module $R \oplus (R/J)$ is 2-primary and 5-primary but not 3-primary, and

(ii) the R-module R is 3-primary but neither 4-primary nor 5-primary.

Proof. It is clear that R is a commutative ring and it is well known that J has the stated properties.

(i) Let $M = R \oplus (R/J)$. Then $(0 :_R M) = 0$ and $J = \sqrt{0}$. If B is any ideal of R such that $B \not\subseteq J$, then $B = R$ and hence $(0 :_M B) = 0$. By Lemma 2.1, M is a 2-primary R-module. Next $J(0 \oplus (R/J)) = 0$ gives that $(0 :_M J) \neq 0$ but J is not a nilpotent ideal of R. Thus M is not 3-primary. Note also that $ass(M) = \{J\}$ so that M is 5-primary.

(ii) Let M' denote the R-module R. Let C be any ideal of R such that $(0 :_{M'} C) \neq 0$. Then $C \subset J$ and hence $C = (x-1)R$ for some $x \in G$. It follows that C is nilpotent. Hence M' is 3-primary. By Lemma 2.3, M' is not 4-primary. Moreover, $(0 :_{M'} J) = 0$, so that $ass(M')$ is the empty set. Thus M' is not 5-primary.

The following example shows that there exists a (necessarily non-commutative) ring R and an R-module M such that M is 2-primary but not 1-primary. □

Example 2.9. Let K be a field of characteristic zero and let R be the ring of all 2×2 matrices over K. Then R is a non-commutative ring and the R-module R is 2-primary but not 1-primary.

Proof. It is clear that R is a non-commutative ring. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ be any elements in R such that $ARB = 0$. In particular,

$A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}B = 0$, $A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}B = 0$, $A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}B = 0$, and $A\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}B = 0$ which implies that $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence the R-module R is 2-primary.

Now, let $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. Then $CD = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But $D \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $C^nR \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all positive integers n. Hence the R-module R is not 1-primary. □

References

[1] R. Gordon, J.C. Robson, *Krull Dimension*, Mem. Amer. Math. Soc., **133** (1973).

[2] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester (1987).

[3] P.F. Smith, Injective modules and prime ideals, *Comm. Algebra*, **9** (1981), 989-999.

[4] P.F. Smith, The injective test lemma in fully bounded rings, *Comm. Algebra*, **9** (1981), 1701-1708.