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ON SOME RESULTS OF HK-CONVOLUTION

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Abstract: In this paper, Laplace convolution is treated as HK-integral. Basic elementary properties are discussed. Various conditions are imposed on the functions so that their convolution exists as HK-integral. The convolution operator is shown to be continuous on $\mathcal{HK}(\mathbb{R}^+) \times \mathcal{BV}(\mathbb{R}^+)$.

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1. Introduction

The Laplace convolution of two functions $f : [0, t] \to \mathbb{R}$, $g : [0, t] \to \mathbb{R}$ is the function $f * g$, if it exists, defined by the integral

$$\int_0^t f(u)g(t-u)du, \quad t \in \mathbb{R}^+. \tag{1}$$

If no assumptions are made about f and g, the convolution is clearly not defined. The convolutions plays an important role in pure and applied mathematics in Fourier analysis, approximation theory, differential equations, integral equation, and many other areas.

In the present paper, we consider the Laplace convolution as Henstock-Kurzweil integral. This integral extends the Riemann, Lebesgue, Perron and Denjoy integrals on the real line. Its definition is Riemann like, but its power is super Lebesgue. We have given some conditions so that the Laplace convolution exists as Henstock-Kurzweil integral. We call this as HK-convolution.

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The Laplace transform of convolution is the product of their Laplace transform is shown under different conditions. Again, convolution of two functions appears to be continuous in the Alexiewicz norm. HK-convolution as an operator becomes continuous on $\mathcal{HK}(\mathbb{R}^+) \times \mathcal{BV}(\mathbb{R}^+)$.

In this paper, following the line of Talvila [6], we obtain similar results for Laplace convolution.

2. Henstock-Kurzweil Integral

Let us begin by recalling the definition of Henstock-Kurzweil integral for bounded intervals.

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is said to be Henstock-Kurzweil (shortly HK-) integrable if there exists a real number I such that, for every positive ϵ, one can find a positive function δ_ϵ on $[a, b]$ (named a gauge) with the property that for any δ_ϵ-fine partition $(I_i, t_i)_{i=1}^n$ of $[a, b]$ (that is, to say that, for each i, $I_i \subseteq (t_i - \delta_\epsilon(t_i), t_i + \delta_\epsilon(t_i))$), one has $| \sum_{i=1}^n f(t_i)m(I_i) - I |< \epsilon$.

The real I is denoted by $(HK) \int_a^b f(x)dx$ or simply $\int_a^b f(x)dx$, if there is no confusion.

In the unbounded case, the Henstock-Kurzweil integral is defined as follows:

Definition 2. Let $f : [-\infty, \infty] \to \mathbb{R}$. A gauge is a (multi)function δ that associates to any $t \in [-\infty, \infty]$ an open interval (that is, an interval of the form (a, b), $[-\infty, b)$, $(a, \infty]$, $[-\infty, \infty]$) which contains t. A partition $(I_i, t_i)_{i=1}^n$ of $[-\infty, \infty]$ is δ-fine if $I_i \subseteq \delta(t_i)$ for each i, with the convention that the measure of an unbounded interval is 0. We say that f is Henstock-Kurzweil integrable if there exists a real number I such that, for every positive ϵ, one can find a gauge δ_ϵ such that, for any δ_ϵ-fine partition of $[-\infty, \infty]$, one has $| \sum_{i=1}^n f(t_i)m(I_i) - I |< \epsilon$.

One can prove that f is HK-integrable on each compact interval $[a, b] \subseteq [-\infty, \infty]$ and there exists the limit

$$\lim_{a\to-\infty,\, b\to\infty} \int_a^b f(x)dx = I.$$

This notion of integral is strictly more general than those of Riemann, of Lebesgue as well as the Riemann and Lebesgue improper integrals. Moreover, its natural definition(by means of integral sums) is much more easily to hand. A very simple example of HK-integrable function which is not Lebesgue(therefore not Riemann) integrable is given by the derivative of the function $F : [0, 1] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} x^2 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

It is not difficult to see, from the completeness of the space of real numbers, that the following Cauchy-type criterion holds:

Proposition. *A function* $f : [a, \infty] \to \mathbb{R}$ *is HK-integrable if and only if, for every* $\epsilon > 0$, *there exists* $A_\epsilon > a$ *such that, for every* $A_2 > A_1 > A_\epsilon, |\int_{A_1}^{A_2} f(x)dx |< \epsilon$.

Remark. One can provide the vector space of HK-integrable functions with the topology of the Alexiewicz norm $\| f \|= \sup_{a,b} | \int_a^b f(x)dx |$.

Throughout this paper we mainly follow the following terminology:

$\mathcal{HK}(\mathbb{R}^+)$ - The space of all HK-integrable functins over $\mathbb{R}^+$.

$\mathcal{BV}(\mathbb{R}^+)$ - The space of all bounded variation functins on $\mathbb{R}^+$.

$\mathcal{L}^1(\mathbb{R}^+)$ - The space of all Lebesgue integrable functins over $\mathbb{R}^+$.

$supp(g)$=support of g.

Since a function of bounded variation on any interval is measurable, we shall use following definition of support of function Gasquet et al [3].

Let $g : [0, t] \to \mathbb{R}$ be a function. Let θ_i, $i \in I$, be the family of open sets in $[0, t]$ for $t \in \mathbb{R}^+$ such that for all $i \in I$, $g = 0$ a.e. on θ_i. Let $\theta = \cup_{i\in I}\theta_i$ and then the support of g is defined to be the closed set $\mathbb{R}/\theta$, i.e. $supp(g) = \mathbb{R}/\theta$.

3. HK-Convolution

First we give two results about the existence of HK-convolution.

Proposition 1. *Let* $f \in \mathcal{HK}(\mathbb{R}^+)$ *and* $g \in \mathcal{BV}(\mathbb{R}^+)$, *then we have* $f * g$ *exists on* $\mathbb{R}^+$.

Proof. By definition, $f * g(t) = \int_0^t f(t - \tau)g(\tau)d\tau$ which exists as HK-integral by multiplier theorem, see [1]. By parts we get

$$|f * g(t)| \le \|f\| \left\{ \inf_{R^+} |g| + V_{[0,t]}\, g(\tau) \right\}. \qquad \square$$

Proposition 2. *Let* $f \in \mathcal{HK}_{loc}$ *and* $g \in \mathcal{BV}$ *such that the support of* g *is in the compact interval* $I = [a, b]$.

Then

$$|f * g(t)| \le \|f\|_{[t-b,t-a]} \left\{ \inf_{[a,b]} |g| + V_{[a,b]}g(\tau) \right\}.$$

Proof. Suppose $supp(g) \subset [a, b]$. Then we have $f * g(t) = \int_a^b f(t - \tau)g(\tau)d\tau$. Integrate by parts, we get

$$|f * g(t)| \le |g(\tau) \int_a^b f(t - \tau)d\tau| + | \int_a^b \int_a^b f(t - \tau)dg(\tau)|$$

and

$$|f * g(t)| \leq \|f\|_{[t-b,t-a]} \left\{ \inf_{[a,b]} |g| + V_{[a,b]} g(\tau) \right\}.$$

The following result shows that the HK-convolution is bounded.

Proposition 3. *Let* $f \in \mathcal{HK}(\mathbb{R}^+)$, $g \in \mathcal{L}^1(\mathbb{R}^+) \cap \mathcal{BV}(\mathbb{R}^+)$. *Then we have* $\|f * g\| \leq \|f\| \, \|g\|_1$.

Proof. For $0 \leq a < b \leq \infty$, we have $\int_a^b f * g(t)dt = \int_a^b \int_0^t f(t-\tau)g(\tau)d\tau dt$.

Let $I_1 = \int_a^b \int_0^t f(t-\tau)g(\tau)d\tau dt$ *and* $I_2 = \int_0^t \int_a^b f(t-\tau)g(\tau)dtd\tau$.

Observe that I_2 exists on $\mathbb{R}^+ \times \mathbb{R}^+$. Now as $g \in \mathcal{L}^1(\mathbb{R}^+)$ $\|g\|_1 \leq K_1$ and also $\int_{\mathbb{R}^+} V_{[0,t]} g \leq M_1$. By Lemma 25[6], we have

$$I_1 = I_2 \quad on \quad \mathbb{R}^+ \times \mathbb{R}^+,$$

$$|\int_a^b f * g(t)dt| \leq \|f\| \, \|g\|_1.$$

Therefore $\|f * g\| \leq \|f\| \, \|g\|_1$. □

Now we give some basic elementary properties of HK-convolution.

Proposition 4. *Suppose* $f \in \mathcal{HK}(\mathbb{R}^+)$ $g \in \mathcal{BV}(\mathbb{R}^+)$. *Then the following holds:*

i) $f * g(t) = g * f(t)$ *for all* $t \in \mathbb{R}$.

ii) $(\lambda f) * g = f * (\lambda g) = \lambda(f * g)$ *for some* $\lambda \in \mathbb{C}$.

iii) $(f * g)_x(t) = f_x * g(t) = f * g_x(t)$.

iv) $h * (f + g)(t) = h * f(t) + h * g(t)$ *for some* $h \in \mathcal{HK}(\mathbb{R}^+)$.

The associative property of HK-convolution is given in the next proposition.

Proposition 5. *If* $f \in \mathcal{HK}(\mathbb{R})$, $g \in \mathcal{BV}(\mathbb{R}^+)$ *and* $h \in \mathcal{L}^1(\mathbb{R}^+)$, *then we have*

$$(f * g) * h(t) = f * (g * h)(t) \quad \forall\, t \in \mathbb{R}^+.$$

Proof. Consider

$$(f * g) * h(t) = \int_0^t \int_0^\tau f(\xi)g(\tau - \xi)h(t-\tau)d\xi \, d\tau.$$

By changing the order of integration, we get

$$(f * g) * h(t) = \int_{\xi=0}^t f(\xi) \int_0^{t-\xi} g(u)h(t-u-\xi)du \, d\xi$$

$$= \int_{\xi=0}^t f(\xi) g * h(t-\xi)d\xi = f * (g * h)(t).$$

Now consider

$$I_1 = \int_{\tau=\xi}^{t} \int_{\xi=0}^{t} f(\xi) g(\tau - \xi) h(t - \tau) d\xi \, d\tau,$$

and

$$I_2 = \int_{\xi=0}^{t} f(\xi) \int_{\tau=\xi}^{t} g(\tau - \xi) h(t - \tau) d\tau \, d\xi.$$

Let

$$J = \int_{\tau=\xi}^{t} g(\tau - \xi) h(t - \tau) d\tau.$$

Integrating by parts, we get

$$|J| \leq \|h\|_1 \left\{ \inf_{[\xi,t]} |g| + V_{[\xi,t]} g(\tau - \xi) \right\}.$$

Therefore

$$I_2 \leq \|f\| \, \|h\|_1 \left\{ \inf_{[\xi,t]} |g| + V_{[\xi,t]} g(\tau - \xi) \right\}.$$

Thus, I_2 exists on $\mathbb{R}^+ \times \mathbb{R}^+$. And by Lemma 25 in [6] we are through.

Now we show that the Laplace transform of convolution is the product of Laplace transforms.

Proposition 6. *Let* $f_1 \in \mathcal{HK}(\mathbb{R}^+)$, $f_2 \in \mathcal{L}^1(\mathbb{R}^+) \cap \mathcal{BV}(\mathbb{R}^+)$. *Then we have* $f * g$ *exists. If* $\mathfrak{L}\{f_1(t)\}$ *and* $\mathfrak{L}\{f_2(t)\}$ *exist at* $s \in \mathbb{C}$ *then* $\mathfrak{L}\{f_1 * f_2(t)\}$ *exists at* $s \in \mathbb{C}$ *and*

$$\mathfrak{L}\{f_1 * f_2(t)\}(s) = \mathfrak{L}\{f_1(t)\}(s) \; \mathfrak{L}\{f_2(t)\}(s).$$

Proof. Consider,

$$\begin{aligned} \mathfrak{L}\{f_1 * f_2(t)\}(s) &= \int_0^\infty e^{-st} f_1 * f_2(t) dt \\ &= \int_0^\infty \int_0^t e^{-s\tau} f_1(\tau) e^{-s(t-\tau)} f_2(t-\tau) d\tau \, dt. \end{aligned}$$

By interchanging iterated integrals, we have

$$\begin{aligned} \mathfrak{L}\{f_1 * f_2(t)\}(s) &= \int_{\tau=0}^\infty e^{-s\tau} f_1(\tau) \int_{t=\tau}^\infty e^{-s(t-\tau)} f_2(t-\tau) dt \, d\tau \\ &= \int_0^\infty e^{-s\tau} f_1(\tau) d\tau \; \int_0^\infty e^{-su} f_2(u) du. \end{aligned}$$

Therefore $\mathfrak{L}\{f_1 * f_2(t)\}$ exists at $s \in \mathbb{C}$ and

$$\mathfrak{L}\{f_1 * f_2(t)\}(s) = \mathfrak{L}\{f_1(t)\}(s)\ \mathfrak{L}\{f_2(t)\}(s). \qquad \square$$

Now it remains to show the justification for interchanging the iterated integrals.

Since the function $f_2 \in \mathcal{L}^1(\mathbb{R}^+)$, $\exists K_I$ such that $\|f\|_1 < K_I$, where $I = [a,b] \subset \mathbb{R}^+$ is any compact interval. Clearly

$$V_{[a,b]} e^{-s(t-\tau)} f_2(t-\tau) \leq e^{-\sigma u}\ \ V_{[t-b,t-a]} f_2(\tau) + |f_2(\tau)| 2e^{-\sigma r}$$

and

$$\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) d\tau \leq e^{-\sigma u} \int_0^\infty V_{[t-b,t-a]} f_2(\tau) d\tau \ + \ 2K_I e^{-\sigma r}.$$

Since $f_2 \in \mathcal{BV}(\mathbb{R}^+)$,we have $\exists$a constant M_I such that

$$\int_0^\infty V_{[t-b,t-a]} f_2(\tau) d\tau < M_I,$$

$$\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) d\tau \leq e^{-\sigma u}\ M_I \ + \ 2K_I e^{-\sigma r}.$$

Thus we have for each compact interval $I = [a,b] \subset \mathbb{R}^+$, $\int_0^\infty V_{[t-b,t-a]} e^{-s\tau} f_2(\tau) d\tau$ is finite and $\|f_2\|_1 \leq K_I$. Hence the integral $\int_{\tau=0}^\infty \int_{t=\tau}^\infty e^{-st} f_1(\tau) f_2(t-\tau) dt\, d\tau$ exists on $\mathbb{R}^+ \times \mathbb{R}^+$. And by lemma 25 [6], we can interchange the iterated integrals.

Proposition 7. *In addition to the hypothesis of the above result if $f_1 * f_2(t)$ is in $\mathcal{HK}(\mathbb{R}^+)$ and f_2 is continuous then $f_1 * f_2(t)$ is continuous w.r.t. Alexiewicz norm.*

Proof. Choose any $\delta > 0$ such that $0 < \delta \leq 1$ (the case for negative δ is analogus).

Define

$$D(t,\delta) = f_1 * f_2(t+\delta) - f_1 * f_2(t) = I_1 + I_2,$$

where

$$I_1 = \int_0^t f_1(\tau)\left[f_2(t+\delta-\tau) - f_2(t-\tau)\right] d\tau, \quad I_2 = \int_t^{t+\delta} f_1(\tau) f_2(t+\delta-\tau) d\tau.$$

Observe that $|I_2| \to 0$ as $\delta \to 0$. And integrating by parts we get

$$|I_1| \leq |f_2(t+\delta-\tau) - f_2(t-\tau)|\ \Big|\int_0^t f_1(\tau) d\tau\Big| + \sup_{u\in\mathbb{R}^+} \Big|\int_0^u f_1(\tau) d\tau\Big|\ |f_2(t+\delta-\tau) - f_2(u-\tau)|.$$

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Since f_2 is continuous on $\mathbb{R}^+$, we have $|I_1| \leq \epsilon$, $\forall \delta < \eta$. Thus $f_1 * f_2$ is continuous at $t \in \mathbb{R}^+$. □

Now, let $\alpha, \beta \in \mathbb{R}^+$.
Consider

$$\int_\alpha^\beta [f_1 * f_2(t+\delta) - f_1 * f_2(t)]\, dt = \int_\alpha^{\alpha+\delta} f_1 * f_2(t)dt - \int_\beta^{\beta+\delta} f_1 * f_2(t)dt.$$

We set $F_{1,2}(x) = \int_0^x f_1 * f_2(t)dt$. Then

$$\sup_{\alpha,\beta\in\mathbb{R}^+} |\int_\alpha^\beta [f_1 * f_2(t+\delta) - f_1 * f_2(t)]\, dt| \leq \sup_{\alpha\in\mathbb{R}^+} |F_{1,2}(\alpha+\delta) - F_{1,2}(\alpha)| + \sup_{\beta\in\mathbb{R}^+} |F_{1,2}(\beta+\delta) - F_{1,2}(\beta)|.$$

$F_{1,2}$ is an indefinite integral of $f_1 * f_2$ which is HK-integrable on $\mathbb{R}^+$. Hence continuous on $\mathbb{R}^+$, see [4]. Therefore we have for given $\epsilon > 0$, $\exists\, \eta > 0$ such that

$$|F_{1,2}(\xi+\delta) - F_{1,2}(\xi)| < \frac{\epsilon}{2}, \quad \forall \delta < \eta, \forall \xi \in \mathbb{R}^+.$$

Therefore we have

$$\sup_{\alpha,\beta\in\mathbb{R}^+} |\int_\alpha^\beta [f_1 * f_2(t+\delta) - f_1 * f_2(t)]\, dt| < \epsilon, \quad \forall \delta < \eta.$$

Hence for given $\epsilon > 0$, $\exists \eta > 0$ such that

$$\|f_1 * f_2(t+\delta) - f_1 * f_2(t)\| < \epsilon, \quad \forall \delta < \eta.$$

The following proposition concerns with neccesary and sufficient conditions on f_n and $g_n : [0,t] \to \mathbb{R}$ so that the HK-convolution as an operator is continuous. It involves either uniform boundedness or uniform convergence of the indefinite integral of f_n.

Proposition 8. *Let $\{f_n\}$ be a sequence of HK-integrable functions such that $f_n : [0,t] \to \mathbb{R}$,$\forall$ n and $\int_0^t f_n = \int_0^t f$ as $n \to \infty$ for some HK-integrable function $f : [0,t] \to \mathbb{R}$. Define $F_n(x) = \int_0^x f_n$, $F(x) = \int_0^x f$.*

If $\{g_n\}$ be a sequence of uniform bounded variation functions, $g_n : [0,t] \to \mathbb{R}$,$\forall$ n, such that $g_n \longrightarrow g$ pointwise on $[0,t]$, where $g : [0,t] \to \mathbb{R}$. Then $f_n * g_n(t) \longrightarrow f * g(t)$ as $n \to \infty$ if and only if i) $F_n \to F$ uniformly on $[0,t]$ as $n \to \infty$.

ii) $F_n \to F$ pointwise on $[0,t]$ and $\{F_n\}$ is uniformly bounded and $V(g_n - g) \to 0$.

Proof. i) Suppose $F_n \to F$ uniformly on $[0,t]$ as $n \to \infty$.
Consider $f_n * g_n - f * g(t) = I_1 + I_2$, where

$$I_1 = \int_0^t (f_n(u) - f(u))\, g_n(t-u)du, \quad I_2 = \int_0^t (g_n(t-u) - g(t-u))\, f(u)du.$$

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By integrating parts we get,

$$I_1 = g_n(t-u)\{F_n - F\} - \int_0^t \{F_n - F\}\, dg_n,$$

$$I_1 = Max_{0\le x\le t}|F_n - F| \int_0^t dg_n + M\,|F_n - F|.$$

By assumption $F_n \to F$ uniformly, Hence $I_1 \to 0$ as $n \to \infty$. Moreover

$$I_2 = \{g_n(t-u) - g(t-u)\} \int_0^t f - \int_0^t F dg_n + \int_0^t F dg.$$

By assumption and by dominated convergence theorem for Riemann-Stieltjes integral it also tends to 0. Hence $I_2 \to 0$ as $n \to \infty$. Therefore $f_n * g_n(t) \to f * g(t)$ as $n \to \infty$.

Now suppose that either $F_n \nrightarrow F$ on $[0,t]$ or $F_n - F \to 0$ not uniformly on $[0,t]$. Then there is a sequence $\{x_n\}$ in $[0,t]$ on which $F_n \nrightarrow F$ uniformly and there is a subsequence $\{y_n\}\,,\ n \in I\ \ of\ \ \{x_n\}\,,\ n \in \Lambda,\ \ I \subset \Lambda$, such that $F_n(y_n) \nrightarrow F(y_n), \forall\, n$ and $y_n \to y$ as $n \to \infty$ i.e. $F_n(y_n) - F(y_n) \nrightarrow 0$.

Let us assume without loss of generality that $0 < y_n \le y \le t$. Let H be Heaviside step function.

$$g_n(t-u) = \begin{cases} H(u-y_n), & n \in I, \\ H(u-y), & \text{otherwise}, \end{cases}$$

and $g(t-u) = H(u-y)$. Then we have, for $n \in I$ the following

$$f_n * g_n(t) = F_n(t) - F_n(y_n) \quad \text{and} \quad f * g(t) = F(t) - F(y).$$

But $F_n(t) \to F(t)$ and since $F_n(y_n) \nrightarrow F(y_n)\ as\ n \to \infty$. Therefore $f_n * g_n(t) \nrightarrow f * g(t)\ \ as\ \ n \to \infty$ which is contradiction. Hence we must have $F_n \to F$ uniformly on $[0,t]$.

ii) Suppose $F_n \to F$ pointwise on $[0,t]$, F_n is uniformly bounded and $V(g_n - g) \to 0$. Consider $f_n * g_n(t) - f * g(t) = I_1 + I_2$, where $I_1 = \int_0^t f_n(u)(g_n(t-u) - g(t-u))du$, $I_2 = \int_0^t (f_n(u) - f(u))g(t-u)du$.

By integrating parts we get,

$$I_1 = (g_n(t-u) - g(t-u))F_n - \int_0^t F_n d(g_n - g).$$

Since $\{F_n\}$ is uniformly bounded, we have $|F_n| \le M\,, \forall\, n$, for some constant M.

Therefore $I_1 \le |g_n(t-u) - g(t-u)|M - M\ \ V(g_n - g)$. By the assumption we can write $I_1 \to 0$ as $n \to \infty$,

$$I_2 = g(t-u)\{F_n(t) - F(t)\} - \int_0^t (F_n - F)dg.$$

By the assumption and by the dominated convergence theorem for Riemann-Stieltjes integral we write $I_2 \to 0$ as $n \to \infty$. Thus $f_n * g_n(t) \to f * g(t)$ as $n \to \infty$.

Now for the converse part suppose there is $c \in (a, b)$ such that $F_n(c) - F(c) \nrightarrow 0$ as $n \to \infty$. Let $g_n(t - x) = g(t - x) = H(x - c)$.

Then

$$\int_0^t f_n(u) g_n(t-u) du = F_n(t) - F_n(c),$$

$$\int_0^t f(u) g(t-u) du = F(t) - F(c).$$

Since $F_n(t) \to F(t)$ and $F_n(c) \nrightarrow F(c)$, we have $f_n * g_n(t) \nrightarrow f * g(t)$ as $n \to \infty$ which is a contradiction. Hence we must have $F_n \to F$ pointwise on $[0, t]$.

Now if $\{F_n\}$ is not uniformly bounded, then there is a sequence $\{x_n\}_{n\in\Lambda}$ in $[0, t]$ on which $|F_n| \to \infty$ and there is a subsequence $\{y_n\}_{n\in I}$ of $\{x_n\}_{n\in\Lambda}$, $I \subset \Lambda$, such that $F_n(y_n) \geq 1$, $\forall\, n$ and $F_n(y_n) \to \infty$ and $y_n \to y$.

Without loss of generality let us take $0 \leq y_n \leq y \leq t$. Define $g_n(t-u) = \frac{H(u-y_n)}{\sqrt{F_n(y_n)}}$. Then $V(g_n) \leq 1$, $g = 0$ and $V(g_n - g) = 0$ and

$$\int_0^t f_n(u) g_n(t-u) du = \int_0^t f_n(u) \frac{H(u-y_n)}{\sqrt{F_n(y_n)}} du.$$

Therefore

$$\int_0^t f_n(u) g_n(t-u) du \to -\infty \quad \text{as} \quad n \to \infty$$

and

$$\int_0^t f(u) g(t-u) du = 0.$$

Hence $f_n * g_n \nrightarrow f * g$ as $n \to \infty$ which is contradiction.

Thus we have that $\{F_n\}$ is uniformly bounded. □

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