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MULTIPLE POSITIVE SOLUTIONS FOR A DISCRETE FOURTH ORDER NONHOMOGENEOUS BOUNDARY VALUE PROBLEM

Johnny Henderson[1 §], Britney Hopkins[2]

[1]Department of Mathematics
Baylor University
Waco, Texas, 76798, USA
e-mail: Johnny_Henderson@baylor.edu

[2]Department of Mathematics and Statistics
University of Central Oklahoma
Edmond, Oklahoma, 73034, USA
e-mail: bhopkins3@uco.edu

Abstract: This paper examines the existence of positive solutions for the fourth order difference equation, $\Delta^4 u(t-2) = \lambda h(t, u(t), \Delta^2 u(t-1))$, for $t \in (0, N+2)_{\mathbb{Z}}$, satisfying the nonhomogeneous conjugate boundary conditions: $u(0) = \Delta^2 u(-1) = 0$, $u(N+2) = a$, and $\Delta^2 u(N+1) = -b$, where $h : [0, N+2]_{\mathbb{Z}} \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$ is continuous, $a, b, \lambda \geq 0$, and $a + b \neq 0$. By transforming the 4-th order equation into a system of second order boundary value problems and then applying the Guo-Krasnosel'skii Fixed Point Theorem, we guarantee the existence of at least three positive solutions. This result is then extended to include an even broader class of problems.

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1. Introduction

The study of multiple positive solutions for boundary value problems is a vastly researched field that has applications in modeling real world phenomena. Some examples of recent works concerning such results include [2], [3], [4], [6], [8], [9], [13], [14], [15], [16], [17], and the references therein. Many of these authors specifically focus on existence results for even order problems. For example, in [1], Agarwal gives an ex-

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§Correspondence author

istence and uniqueness result for the fourth order problem $x^{(4)} = f(t, x, x', x'', x^{(3)})$, arising from beam analysis. In [13], Marcos, Lorca, and Ubilla consider the fourth order differential equation $u^{(4)(t)} = \lambda h(t, u(t), u''(t))$ for $t \in (0, 1)$. Their work hinges on transforming the fourth order problem into a system of second order problems and then showing existence. That transformation technique serves as motivation for this work. Some other notable works on even order boundary value problems include, but are not limited to [2], [3], [5], [7], [9], [11], [12], [13], and [14]. In addition, several authors have studied positive solutions for various types of boundary value problems on both discrete domains and on time scales, as seen in [2], [6], and [9].

In this paper, we establish an existence result for the fourth order discrete boundary value problem,

$$\Delta^4 u(t-2) = \lambda h(t, u(t), \Delta^2 u(t-1)), \quad t \in (0, N+2)_{\mathbb{Z}}, \tag{1}$$

$$u(0) = 0, \quad \Delta^2 u(-1) = 0, \tag{2}$$

$$u(N+2) = a, \quad \Delta^2 u(N+1) = -b, \tag{3}$$

where $a, b, \lambda \geq 0$, $a+b > 0$, and $h : [0, N+2]_{\mathbb{Z}} \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$. We start by making a series of substitutions that ultimately transform the above difference equation into a system of second order difference equations satisfying homogeneous conjugate boundary conditions. We then show that the new system has multiple solutions by first constructing a sequence of four lemmas which culminate to allow the application of the Guo-Krasnosel'skii Fixed Point Theorem. This, in turn, yields our main result.

2. Preliminaries

Note that given any set $S \subset \mathbb{R}$, $S_{\mathbb{Z}}$ denotes the intersection of the set S with the integers; that is

$$S_{\mathbb{Z}} = S \cap \mathbb{Z}.$$

For $t \in [1, N+1]_{\mathbb{Z}}$, let $v(t) = -\Delta^2 u(t-1)$, $g(t, u(t), v(t)) = v(t)$, and $f(t, u, -v) = h(t, u, v)$. Thus, solutions of the fourth order difference equation (1)-(3) are in one-to-one correspondence with solutions of the system of discrete second order boundary value problems,

$$-\Delta^2 u(t-1) = g(t, u(t), v(t)), \quad t \in (0, N+2)_{\mathbb{Z}}, \tag{4}$$

$$-\Delta^2 v(t-1) = \lambda f(t, u(t), v(t)), \quad t \in (0, N+2)_{\mathbb{Z}}, \tag{5}$$

$$u(0) = v(0) = 0, \tag{6}$$

$$u(N+2) = a, \quad v(N+2) = b. \tag{7}$$

In order to achieve our main result, we place the following requirements on the function f:

(H0) $f : [0, N+2]_{\mathbb{Z}} \times [0, \infty)^2 \to [0, \infty)$ is a continuous function which is nondecreasing in the last two variables.

(H1) Suppose there is an $\alpha_1, \beta_1 \in (0, N+2)_{\mathbb{Z}}$, where $\alpha_1 < \beta_1$, such that given $(u, v) \in [0, \infty)^2$, there is a $k > 0$ such that

$$f(t, u, v) > k$$

for $t \in [\alpha_1, \beta_1]_{\mathbb{Z}}$.

(H2) $\lim_{u+v \to 0^+} \frac{f(t,u,v)}{u+v} = 0$ uniformly for $t \in [0, N+2]_{\mathbb{Z}}$.

(H3) $\lim_{u+v \to \infty} \frac{f(t,u,v)}{u+v} = 0$ uniformly for $t \in [0, N+2]_{\mathbb{Z}}$.

It is easy to see that (4)-(7) can be transformed into the system of discrete second order boundary value problems:

$$-\Delta^2 u(t-1) = g\left(t, u(t) + \left(\frac{a}{N+2}\right)t, v(t) + \left(\frac{b}{N+2}\right)t\right), \; t \in (0, N+2)_{\mathbb{Z}}, \quad (8)$$

$$-\Delta^2 v(t-1) = \lambda f\left(t, u(t) + \left(\frac{a}{N+2}\right)t, v(t) + \left(\frac{b}{N+2}\right)t\right), \; t \in (0, N+2)_{\mathbb{Z}}, \quad (9)$$

$$u(0) = v(0) = 0, \quad (10)$$

$$u(N+2) = v(N+2) = 0. \quad (11)$$

Since (8)-(11) is merely a transformation of (4)-(7), solutions satisfying one system, automatically satisfy the other. Thus, it suffices to show that (8)-(11) has positive solutions.

For simplicity, set

$$A := \frac{a}{N+2} \text{ and } B := \frac{b}{N+2}.$$

As a result of the transformation process, we know that solutions to (8)-(11) are of the form

$$u(t) = \sum_{s=1}^{N+1} G(t,s) g(s, u(s) + As, v(s) + Bs),$$

$$v(t) = \lambda \sum_{s=1}^{N+1} G(t,s) f(s, u(s) + As, v(s) + Bs),$$

where $G(t, s)$ denotes the Green's function,

$$G(t,s) = \frac{1}{N+2} \begin{cases} t(N+2-s), & 0 \le t \le s \le N+1, \\ s(N+2-t), & 1 \le s \le t \le N+2. \end{cases}$$

Clearly, $G(t,s)$ is nonnegative and since both f and g are nonnegative by assumption, we see that u and v must also be positive. Another useful property of $G(t,s)$ is that

$$\max_{t\in[0,N+2]_{\mathbb{Z}}} \sum_{s=1}^{N+1} G(t,s) \leq \frac{(N+2)^2}{8}. \tag{12}$$

Let $(X, \|\cdot\|)$ be a real Banach space. A cone C in X is a nonempty, closed, convex subset of X satisfying both of the following properties:

1. If $x \in C$, and $\lambda > 0$, then $\lambda x \in C$.
2. If $x \in C$ and $-x \in C$, then $x = 0$.

Next, set

$$Y = \{u(t) |\, u : [0, N+2]_{\mathbb{Z}} \to \mathbb{R}\}$$

and let $(X, \|\cdot\|)$ denote the Banach space $X = Y \times Y$, endowed with the norm

$$\|(u,v)\| = \|u\|_\infty + \|v\|_\infty,$$

where $\|u\|_\infty = \max\limits_{t\in[0,N+2]_{\mathbb{Z}}} |u(t)|$. Let Ω_r be the open set

$$\Omega_r = \{(u,v) \in X | \|(u,v)\| < r\}.$$

Define $C \subset X$ by

$$C = \{(u,v) \in X | (u,v)(0) = (u,v)(N+2) = (0,0) \text{ and } u, v \text{ are concave}\},$$

and note that C is a cone. Let $T : X \to X$ be the operator defined by $T(u,v) = (A_1(u,v), A_2(u,v))$, where

$$A_1 = \sum_{s=1}^{N+1} G(t,s) g(s, u(s) + As, v(s) + Bs),$$

$$A_2 = \lambda \sum_{s=1}^{N+1} G(t,s) f(s, u(s) + As, v(s) + Bs).$$

Thus, we have the following lemma concerning T.

Lemma 1. *T is a completely continuous cone preserving operator.*

The proof that T is cone preserving is straight forward. The fact that T is completely continuous follows from a standard Arzela–Ascoli argument.

Finally, we state the Guo-Krasnosel'skii Fixed Point Theorem, which is essential to acquiring our main result.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$ be a cone. Suppose Ω_1, Ω_2 are open subsets of X satisfying $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ is a completely continuous operator such that either*

i. $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_2$, *or*

ii. $\|Tu\| \geq \|u\|$ *for* $u \in C \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in C \cap \partial\Omega_2$,

then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Technical Results

Lemma 3. *Suppose* (H0) *and* (H1) *hold and let $\rho^* > 0$. Then there is a $\Lambda > 0$ such that, for every $\lambda \geq \Lambda$ and $(a, b) \in [0, \infty)^2$,*

$$\|T(u, v)\| \geq \|(u, v)\|,$$

for $(u, v) \in C \cap \partial\Omega_{\rho*}$.

Proof. Let $\rho^* > 0$ and $(u, v) \in C \cap \partial\Omega_{\rho^*}$. Pick r so that both $u(t) \geq r\|u\|_\infty$ and $v(t) \geq r\|u\|_\infty$ for $t \in [\alpha_1, \beta_1]$, and set

$$\Lambda = \left[rM \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) \right]^{-1},$$

where $[\![\cdot]\!]$ denotes the greatest integer function, and

$$M = \inf \left\{ \frac{f(t, ra_1, ra_2)}{r(a_1 + a_2)} : t \in [\alpha_1, \beta_1]_{\mathbb{Z}},\ a_1, a_2 > 0,\ \text{and } (a_1 + a_2) = \rho^* \right\}.$$

Thus, appealing to the nondecreasing property placed on f, we see that

$$\begin{aligned}
\|T(u, v)\| &\geq \|A_2(u, v)\|_\infty \\
&\geq \lambda \sum_{s=1}^{N+1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) f(s, u(s) + As, v(s) + Bs) \\
&\geq \lambda \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) f(s, r\|u\|_\infty, r\|v\|_\infty) \\
&= \lambda r \|(u, v)\| \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right) \frac{f(s, r\|u\|_\infty, r\|v\|_\infty)}{r\|(u, v)\|} \\
&\geq \lambda r M \|(u, v)\| \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2} \right]\!\right], s \right)
\end{aligned}$$

$$\geq \|(u,v)\|,$$

for each $\lambda \geq \Lambda$. This yields the desired result. □

Lemma 4. *Let* (H0) *and* (H1) *hold and fix* $\Lambda > 0$. *Then, for every* $\lambda \geq \Lambda$ *and* $(a,b) \in [0,\infty)^2$, *with* $a+b>0$, *there is a* $\rho_1 = \rho_1(\Lambda, a, b)$ *such that for every* $\rho \leq \rho_1$, *we have*

$$\|T(u,v)\| \geq \|(u,v)\|,$$

for $(u,v) \in C \cap \partial\Omega_\rho$.

Proof. By (H1) and the nondecreasing property on f, there is a $k>0$ such that

$$f(t, u(t)+At, v(t)+Bt) \geq f(t, \alpha_1 A, \alpha_1 B) > k,$$

for $t \in [\alpha_1, \beta_1]_{\mathbb{Z}}$. Take $\rho_1 = \Lambda k \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right)$. Then, for $(u,v) \in C \cap \partial\Omega_\rho$, where $\rho \leq \rho_1$,

$$\begin{aligned}
\|T(u,v)\| &\geq \lambda \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right) f(s, \alpha_1 A, \alpha_1 B) \\
&\geq \lambda k \|(u,v)\| \sum_{s=\alpha_1}^{\beta_1} G\left(\left[\!\left[\frac{N+2}{2}\right]\!\right], s\right) (\|(u,v)\|)^{-1} \\
&\geq \|(u,v)\|.
\end{aligned}$$

□

Lemma 5. *Suppose* (H0) *and* (H2) *hold and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$, *there is a* $\rho_2 \in (0, \rho^*)$ *and a* $\delta > 0$ *such that for every* $(a,b) \in [0,\infty)^2$, *with* $0 < a+b < \delta$, *we have*

$$\|T(u,v)\| \leq \|(u,v)\|,$$

for $(u,v) \in C \cap \partial\Omega_{\rho_2}$.

Proof. Given $\lambda > 0$, pick $\epsilon > 0$ so that $\lambda\epsilon(N+2)^2 < 4$. Then, by (H2), there is a $\rho_2 \in (0, \rho^*)$ such that for $u+v = \rho_2$ and $a+b \leq \rho_2$, we have

$$f(t, u+a, v+b) \leq \epsilon[(u+a)+(v+b)], \;\; t \in [0, N+2]_{\mathbb{Z}}.$$

Take $(u,v) \in C \cap \partial\Omega_{\rho_2}$ and suppose $a+b < \rho_2$. Then,

$$\begin{aligned}
A_2(u,v)(t) &= \lambda \sum_{s=1}^{N+1} G(t,s) f(s, u(s)+As, v(s)+Bs) \\
&\leq \lambda \sum_{s=1}^{N+1} G(t,s) f(s, u(s)+a, v(s)+b) \\
&\leq \lambda \sum_{s=1}^{N+1} G(t,s) f(s, \|u\|_\infty + a, \|v\|_\infty + b)
\end{aligned}$$

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$$\begin{aligned}
&\leq \lambda\epsilon[\|(u,v)\| + (a+b)] \sum_{s=1}^{N+1} G(t,s) \\
&\leq 2\lambda\epsilon\|(u,v)\| \sum_{s=1}^{N+1} G(t,s) \\
&\leq \frac{\lambda\epsilon(N+2)^2}{4}\|(u,v)\|,
\end{aligned}$$

for $t \in [1, N+1]_{\mathbb{Z}}$, giving that

$$\|A_2(u,v)\|_\infty \leq \frac{\lambda\epsilon(N+2)^2}{4}\|(u,v)\|.$$

Next we consider A_1. First note that g is nondecreasing in the final two variables as a result of its projective nature. Moreover, there are both a

$$0 < \gamma < \frac{8}{(N+2)^2}$$

and a $q > 0$ such that, for every $(u,v) \in [0,\infty)^2$, with $u + v < q$, we have

$$g(t,u,v) \leq \gamma(u+v),$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Pick $\rho_2 < \frac{1}{2}q$. Then $[(u+a) + (v+b)] < 2\rho_2 < q$. Thus, the above gives that

$$g(t, u+a, v+b) \leq \gamma[(u+a) + (v+b)],$$

for $t \in [0, N+2]_{\mathbb{Z}}$. Let $\delta' < 1$ and set $\delta = \delta'\rho_2$. Take $a+b < \delta$ and $(u,v) \in C \cap \partial\Omega_{\rho_2}$. Then, for $t \in [1, N+1]_{\mathbb{Z}}$, we have

$$\begin{aligned}
A_1(u,v)(t) &= \sum_{s=1}^{N+1} G(t,s) g(s, u(s) + As, v(s) + Bs) \\
&\leq \sum_{s=1}^{N+1} G(t,s) g(s, \|u\|_\infty + a, \|v\|_\infty + b) \\
&\leq \gamma[\|(u,v)\| + (a+b)] \sum_{s=1}^{N+1} G(t,s) \\
&\leq \gamma(1+\delta')\|(u,v)\| \sum_{s=1}^{N+1} G(t,s) \\
&\leq \frac{\gamma(1+\delta')(N+2)^2}{8}\|(u,v)\|.
\end{aligned}$$

Therefore,

$$\|A_1(u,v)\|_\infty \leq \frac{\gamma(1+\delta')(N+2)^2}{8}\|(u,v)\|.$$

Letting $a+b<\delta$ and $(u,v)\in C\cap\partial\Omega_{\rho_2}$ yields

$$\|T(u,v)\| \leq (N+2)^2\left(\frac{\gamma(1+\delta')+2\lambda\epsilon}{8}\right)\|(u,v)\|.$$

Thus, if we select δ' and ϵ small enough so that $\gamma(1+\delta')+2\lambda\epsilon \leq 8(N+2)^{-2}$, we have our desired result. □

Lemma 6. *Let $\delta>0$ be given and let $0<a+b<\delta$. In addition, suppose assumptions* (H0) *and* (H3) *hold. Then, for every $\lambda>0$, there is a $\rho_3=\rho_3(\delta,\lambda)$ such that for $\rho\geq\rho_3$,*

$$\|T(u,v)\| \leq \|(u,v)\|,$$

where $(u,v)\in C\cap\partial\Omega_\rho$.

Proof. It follows, by construction, that g is nondecreasing in the last two variables, and there are both an

$$0<\eta<\frac{8}{(N+2)^2}$$

and a $p>0$ such that for $(u,v)\in[0,\infty)^2$, with $u+v>p$, we have

$$g(t,u,v)\leq\eta(u+v),$$

for $t\in[0,N+2]_{\mathbb{Z}}$. Thus, given any $q_1\geq p$, where p is defined above, we have

$$g(t,u+a,v+b)\leq\eta[(u+a)+(v+b)],$$

where $u+v\geq q_1$. Let $\epsilon>0$ and pick q_1 large enough so that $\epsilon>\frac{\eta\delta}{q_1}$. Then

$$\begin{aligned} g(t,u+a,v+b) &\leq \eta(u+v)+\eta(a+b)\\ &\leq \eta(u+v)+\epsilon(u+v)\\ &= (\eta+\epsilon)(u+v), \end{aligned}$$

for $t\in[0,N+2]_{\mathbb{Z}}$. Therefore, for any $(u,v)\in C\cap\partial\Omega_{q_1}$, we have

$$\begin{aligned} A_1(u,v)(t) &= \sum_{s=1}^{N+1}G(t,s)g(s,u(s)+As,v(s)+Bs)\\ &\leq \sum_{s=1}^{N+1}G(t,s)g(s,u(s)+a,v(s)+b) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{s=1}^{N+1} G(t,s) g(s, \|u\|_\infty + a, \|v\|_\infty + b) \\
&\leq (\eta + \epsilon)\|(u,v)\| \sum_{s=1}^{N+1} G(t,s) \\
&\leq \frac{(\eta+\epsilon)(N+2)^2}{8} \|(u,v)\|,
\end{aligned}$$

for $t \in [1, N+1]_{\mathbb{Z}}$, yielding

$$\|A_1(u,v)\|_\infty \leq \frac{(\eta+\epsilon)(N+2)^2}{8}\|(u,v)\|.$$

Next we consider $A_2(u,v)$. Let $\delta' > 0$. By (H0) and (H3), there is a $q_2 > 0$ such that, for every $(u,v) \in [0,\infty)^2$ with $u + v \geq q_2$,

$$f(t, u + At, v + Bt) \leq f(t, u + a, v + b) \leq \delta'[(u + a) + (v + b)].$$

Let $q_3 = \max\{\delta, q_2\}$ and recall $a + b < \delta$. Then for every $(u,v) \in [0,\infty)^2$, with $u + v \geq q_3$, we have

$$f(t, u + ta, v + tb) \leq \delta'(u + v) + \delta'\delta \leq 2\delta'(u + v),$$

for $t \in [0, N+2]_{\mathbb{Z}}$. It then follows that

$$\|A_2(u,v)\|_\infty \leq \frac{\delta'\lambda(N+2)^2}{4}\|(u,v)\|$$

for $(u,v) \in C \cap \partial\Omega_{q_3}$. Therefore,

$$\|T(u,v)\| \leq (N+2)^2 \left(\frac{\eta + \epsilon + 2\delta'\lambda}{8}\right) \|(u,v)\|.$$

Picking ϵ and δ' so that $\epsilon + 2\delta'\lambda \leq 8(N+2)^{-2} - \eta$, gives the desired result. □

4. Main Result

Theorem 7. *Let f satisfy* (H0)-(H3). *Then there exists a $\Lambda > 0$ such that, for any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that, for every $a, b \geq 0$ with $0 < a + b < \delta$, the system* (8)-(11) *has at least three positive solutions.*

Proof. Suppose f satisfies hypotheses (H0)-(H3) and fix $\rho^* > 0$. By Lemma 3, there is a $\Lambda > 0$ such that for every $\lambda \geq \Lambda$ and $a, b \geq 0$,

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho^*}.$$

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Now, fix $\lambda \geq \Lambda$. Lemmas 4 – 6 give that there is a $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$, with $\rho_1 < \rho_2 < \rho^* < \rho_3$, such that for $(a, b) \in [0, \infty)^2$, satisfying $0 < a + b < \delta$, we have

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_1},$$
$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_2},$$
$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for } (u,v) \in C \cap \partial\Omega_{\rho_3}.$$

Therefore, by appealing to the Guo-Krasnosel'skii Fixed Point Theorem, there exist three positive solutions, $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in C$ of (8)-(11) such that,

$$\rho_1 < \|(u_1, v_1)\| < \rho_2 < \|(u_2, v_2)\| < \rho^* < \|(u_3, v_3)\| < \rho_3. \qquad \square$$

This result can be generalized to any system of the form

$$-\Delta^2 u(t-1) = g\left(t, u(t) + \left(\frac{a}{N+2}\right)t, v(t) + \left(\frac{b}{N+2}\right)t\right), \; t \in (0, N+2)_{\mathbb{Z}}, \quad (13)$$

$$-\Delta^2 v(t-1) = \lambda f\left(t, u(t) + \left(\frac{a}{N+2}\right)t, v(t) + \left(\frac{b}{N+2}\right)t\right), \quad t \in (0, N+2)_{\mathbb{Z}}, \quad (14)$$

$$u(0) = v(0) = u(N+2) = v(N+2) = 0, \quad (15)$$

by amending (H0) to

(H0′) $f, g : [0, N+2]_{\mathbb{Z}} \times [0, \infty)^2 \to [0, \infty)$ are continuous and nondecreasing in the last two variables,

and adding the assumptions:

(H4) There are both a $0 < \gamma < \frac{8}{(N+2)^2}$ and a $q > 0$ such that, for every $(u, v) \in [0, \infty)^2$, with $u + v < q$, we have

$$g(t, u, v) \leq \gamma(u + v),$$

for $t \in [0, N+2]_{\mathbb{Z}}$.

(H5) There are both an $0 < \eta < \frac{8}{(N+2)^2}$ and a $p > 0$ such that, for $(u, v) \in [0, \infty)^2$, with $u + v > p$, we have

$$g(t, u, v) \leq \eta(u + v),$$

for $t \in [0, N+2]_{\mathbb{Z}}$.

Thus, we have the following corollary.

Corollary 8. *Let f, and g satisfy* (H0′)-(H5). *Then there exists a $\Lambda > 0$ such that, given any $\lambda \geq \Lambda$, there is a $\delta > 0$ such that, for every $a, b, \geq 0$ satisfying $0 < a + b < \delta$, the system* (13)-(15) *has at least three positive solutions.*

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