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LAWS OF LARGE NUMBERS OF SUBGRAPHS IN DIRECTED RANDOM GEOMETRIC NETWORKS

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Abstract: Given independent random points $\mathcal{X}_n = \{X_1, \cdots, X_n\}$ in $\mathbb{R}^2$, drawn according to some probability density function f on $\mathbb{R}^2$, and a cutoff $r_n > 0$ we construct a random geometric digraph $G(\mathcal{X}_n, \mathcal{Y}_n, r_n)$ with vertex set $\mathcal{X}_n$. Each vertex X_i is assigned uniformly at random a sector S_i, of central angle α with inclination Y_i, in a circle of radius r_n (with vertex X_i as the origin). An arc is present from X_i to X_j, if X_j falls in S_i. Another random geometric digraph $G(\mathcal{X}_n, \mathcal{R}_n)$ with random radius is also introduced. In this paper we investigate two kinds of small subgraphs – induced and isolated – in the above two directed networks, which contribute to understanding the local topology of many spatial networks, such as wireless communication networks. We give some strong laws of large numbers of subgraph counts thus extending those results of Penrose (see *Random Geometric Graphs*, Oxford University Press (2003)).

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1. Introduction

In the last decade there has been a resurgence of interest in the analysis of random geometric graphs (RGGs) particularly in the context of ad hoc wireless networks. An elegant written tutorial of random geometric graph theory is available in [13],

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and the paper [8] is a more recent survey emphasizing wireless networks. An RGG is usually constructed as follows. Let $||\cdot||$ be some norm on $\mathbb{R}^d$, and r_n be a real sequence. Let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$, $\{X_i\}$ are i.i.d. random d-vectors in $\mathbb{R}^d$ having a common probability density function f. We denote by $G(\mathcal{X}_n, r_n)$ the graph with vertex set $\mathcal{X}_n$ and with an edge X_iX_j if and only if $||X_i - X_j|| \leq r_n$ for $i \neq j$. Note that $G(\mathcal{X}_n, r_n)$ is isotropic and thus undirected, which is less appropriate in many practical applications such as wireless sensor networks. The issue of small subgraph counts are dealt for two kinds of directed models of RGG (for formal definitions see below). In the case of wireless networks, we are sometimes interested to know the number of a desirable local configuration involving a small number of transmitters and receivers. This is especially true if there are a small number of nodes with special capabilities, e.g. data collection centers in sensor network process the data collected by the beacon nodes that help in self-organization of the network. The small subgraph counts are also of independent interest in random graph theory in various guises. The concerned results on small subgraph for classical Erdös-Rényi random graphs are discussed in detail in [2], Chapter 4 and [9], Chapter 3, and for asymptotic results in random geometric graphs, see [13], Chapter 2 and [5], while for exact formulae treated in the circumstance of wireless network, see [16].

In this paper, we extend the method of Penrose [13] and establish some strong laws of large numbers of small subgraphs in directed geometric networks in some limiting regimes. We now define two random geometric digraphs to be used in this work. The archetype (with uniformly distributed points in $[0,1]^2$) of the first one has been proposed in [6], called "random scaled sector graph", to model the "Small Dust" sensor networks using optical communication. Some graph theoretic properties have been addressed for this model, see, e.g. [4, 6, 7, 10, 12, 14], mainly using combinatorial techniques.

Definition 1. Let $||\cdot||$ be Euclidean norm equipped on $\mathbb{R}^2$. Let $\alpha \in (0, 2\pi]$. Let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ be i.i.d. random vectors with a common density function $f : \mathbb{R}^2 \to \mathbb{R}$. Let $\mathcal{Y}_n = \{Y_1, Y_2, \cdots, Y_n\}$ be i.i.d. random variables, uniformly distributed on $[0, 2\pi)$. Associate every point $X_i \in \mathcal{X}_n$ a sector, which is centered at X_i, with radius r_n, amplitude α and elevation Y_i with respect to the horizontal direction anticlockwise. This sector is denoted as $S(X_i, Y_i, r_n)$. We denote by $G(\mathcal{X}_n, \mathcal{Y}_n, r_n)$ the digraph with vertex set $\mathcal{X}_n$, and with an arc (X_i, X_j), $i \neq j$, presents if and only if $X_j \in S(X_i, Y_i, r_n)$.

For technical reasons we always assume $r_n \to 0$ as $n \to \infty$. We also assume that f is bounded and a.s. continuous throughout the paper. We mention that the above assumptions imposed on f are rather mild; in fact typical distributions such as normal distribution and $f = 1_{[0,1]^d}$ are clearly satisfied. Now we introduce another model $G(\mathcal{X}_n, \mathcal{R}_n)$ with random cutoffs motivated by Boolean model in continuum percolation [11]. For the sake of convenience, we still choose to employ the signs $\mathcal{X}_n$, f and $||\cdot||$ with a little ambiguous (see below), however, the right meaning will be clear in the context and no confusion will be incurred.

Definition 2. Let $||\cdot||$ be any norm equipped on $\mathbb{R}^d$, $d \geq 1$. Let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$ be i.i.d. random vectors with a common density function $f : \mathbb{R}^d \to \mathbb{R}$. Let R_n be a positive random variable with probability distribution function Q_n and density function q_n. For each point X_i, we associate a ball $B(X_i, R_{n,i})$ with radius $R_{n,i}$, centered at X_i, independent of other points. $\{R_{n,i}\}_{i=1}^n$ are independent copies of R_n and set $\mathcal{R}_n := \{R_{n,1}, R_{n,2}, \cdots, R_{n,n}\}$. We denote by $G(\mathcal{X}_n, \mathcal{R}_n)$ the digraph with vertex set $\mathcal{X}_n$, and with an arc (X_i, X_j) originating from X_i and terminating in X_j if and only if $X_j \in B(X_i, R_{n,i})$.

As usual, we shall impose a certain decaying condition on R_n. Here we assume $ER_n^d = \int_0^\infty r^d \mathrm{d}Q_n(r) \to 0$ as $n \to \infty$, throughout the paper. We will investigate two kinds of subgraphs in the above two models; one is induced subgraph and the other is isolated subgraph. Induced subgraph is defined in its usual meaning, see, e.g. [15]. Suppose G is a digraph, a subgraph H is isolated in G if H is an induced subgraph and there are no arcs leave H. If G is undirected, then a connected isolated subgraph of G is just a component.

Before going further we will need some other definitions. Given a finite set $\mathcal{X}$ in $\mathbb{R}^d$, let $\mathrm{card}(\mathcal{X})$ denote the number of points in $\mathcal{X}$ and let $|\cdot|$ be the d-dimensional Lebesgue measure, which is easy to discriminate from the sign for absolute value in the context. In the rest of the paper, let $f_{\max} := \sup\{t : |\{f(x) > t\}| > 0\}$ be the essential supremum of the probability density function f. As mentioned before, we assume $f_{\max} < \infty$. For a set $A \in \mathbb{R}^d$, let $\mathcal{X}(A)$ denote the number of points of $\mathcal{X}$ located in A. Denote $D(0,1)$ as the unit disk in $\mathbb{R}^2$, then $|D(0,1)| = \pi$ and also set $\theta := |B(0,1)|$ w.r.t some given norm. Let C, C', etc. be various positive constants, and the values may change from line to line.

The rest of this paper is organized as follows. Section 2 contains the statement of main results and proofs are provided in Section 3. We finally draw conclusions in Section 4.

2. Statement of Main Results

For $k \in \mathbb{N}$, let T be a fixed connected graph on k vertices. We say T is feasible if either $P(G(\mathcal{X}_k, \mathcal{Y}_k, r) \cong T) > 0$ for some $r > 0$ or $P(G(\mathcal{X}_k, \mathcal{R}_n) \cong T) > 0$ for some $\{r_{n,1}, r_{n,2}, \cdots, r_{n,k}\}$. Let $H_{n,T}$ and $\tilde{H}_{n,T}$ be the number of induced subgraphs and isolated subgraphs of $G(\mathcal{X}_n, \mathcal{Y}_n, r_n)$ isomorphic to T (T-subgraphs) respectively. Likewise, let $G_{n,T}$ and $\tilde{G}_{n,T}$ be the number of induced and isolated T-subgraphs of $G(\mathcal{X}_n, \mathcal{R}_n)$ respectively. For a finite set $\mathcal{X} \subseteq \mathbb{R}^2$ and a point $\mathcal{Y} \in [0, 2\pi)^{\mathrm{card}(\mathcal{X})}$, we define indicator random variables $h_T(\mathcal{X}, \mathcal{Y}) := 1_{[G(\mathcal{X},\mathcal{Y},1)\cong T]}$ and $h_{n,T}(\mathcal{X}, \mathcal{Y}) := 1_{[G(\mathcal{X},\mathcal{Y},r_n)\cong T]}$. For a finite set $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{R}_n = \{R_{n,1}, R_{n,2}, \cdots, R_{n,\mathrm{card}(\mathcal{X})}\}$, we define $g_{n,T}(\mathcal{X}, \mathcal{R}_n) := 1_{[G(\mathcal{X},\mathcal{R}_n)\cong T]}$.

The basic tool we shall need in the proofs is the following Azuma's inequality; we refer the reader to [1] for a proof and a wealth of materials regarding that topic.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

Lemma 1. *Suppose $M_1, M_2, \cdots, M_n$ is a martingale with corresponding martingale difference sequence $D_1, D_2, \cdots, D_n$, where $D_i := M_i - M_{i-1}$ $(2 \le i \le n)$ and $D_1 := M_1 - EM_1$. Then for any $a > 0$, we have*

$$P\Big(\Big|\sum_{i=1}^{n} D_i\Big| > a\Big) \le 2\exp\Big(-\frac{a^2}{2\sum_{i=1}^{n}||D_i||_\infty^2}\Big),$$

where $||D_i||_\infty^2 := \inf\{b : P(|D_i| \le b) = 1\}$.

In the sequel we sometimes use a generalized version with "tolerance" of Azuma's inequality [3]:

Lemma 2. (see [3]) *Suppose $M_1, M_2, \cdots, M_n$ is a martingale with corresponding martingale difference sequence $D_1, D_2, \cdots, D_n$, where $D_i := M_i - M_{i-1}$ $(2 \le i \le n)$ and $D_1 := M_1 - EM_1$. Then for any $a, b > 0$,*

$$P\Big(\Big|\sum_{i=1}^{n} D_i\Big| > a\Big) \le 2\exp\Big(-\frac{a^2}{32nb^2}\Big) + \Big(1 + \frac{2\sup_i ||D_i||_\infty}{a}\Big)\sum_{i=1}^{n} P\big(|D_i| > b\big).$$

By the notations defined in the beginning of this section, we have the following strong laws of large numbers under various regimes:

Theorem 1. *Suppose T is a connected feasible graph of order k, $k \in \mathbb{N}$. Let $nr_n^2 \to \lambda \in (0, \infty)$. Then*

$$\lim_{n\to\infty} n^{-1}\tilde{H}_{n,T} = k^{-1}\int_{\mathbb{R}^2} \varphi_T(\lambda f(x)) f(x)\mathrm{d}x \quad a.s.,$$

where

$$\varphi_T(t) := \begin{cases} \frac{t^{k-1}}{(k-1)!(2\pi)^k} \underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{k} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{k-1} h_T(0, x_2, \cdots, x_k, y_1, \cdots, y_k) & \\ \quad \cdot e^{-t|S(0,x_2,\cdots,x_k,y_1,\cdots,y_k)|}\mathrm{d}x_2\cdots\mathrm{d}x_k\mathrm{d}y_1\cdots\mathrm{d}y_k & k \ge 2; \\ e^{-\frac{t\alpha}{2}} & k = 1, \end{cases}$$

and $S(x_1, \cdots, x_k, y_1, \cdots, y_k) := \cup_{i=1}^{k} S(x_i, y_i, 1)$.

Theorem 2. *Let T be a single vertex. Suppose $nR_n^d \to \lambda \in [0, \infty)$ in probability. Then*

$$\lim_{n\to\infty} n^{-1}\tilde{G}_{n,T} = \int_{\mathbb{R}^d} e^{-\theta\lambda f(x)} f(x)\mathrm{d}x \quad a.s.$$

Theorem 3. *For $k \ge 2$, let T be a connected feasible graph of order k. Suppose $nr_n^2 \to 0$ and $\ln n = o\big(n^{2k-1}r_n^{4(k-1)}\big)$, as $n \to \infty$. Then*

$$\lim_{n\to\infty} n^{-k}r_n^{-2(k-1)}\tilde{H}_{n,T} = \mu_T \quad a.s.,$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

where

$$\mu_T := \frac{1}{k!(2\pi)^k}\underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{k}\underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{k} h_T(0,z_2,\cdots,z_k,y_1,\cdots,y_k) \cdot f^k(x)\mathrm{d}x\mathrm{d}z_2\cdots\mathrm{d}z_k\mathrm{d}y_1\cdots\mathrm{d}y_k.$$

Theorem 4. *For $k \geq 2$, let T be a connected feasible graph of order k. Suppose f has bounded support set (denoted by $\operatorname{supp} f$). Suppose $r_n \to 0$ and there is a constant $\delta > 0$ such that $\liminf n^{2k-1-\delta} r_n^{4(k-1)} > 0$. Then*

$$\lim_{n\to\infty} n^{-k} r_n^{-2(k-1)} H_{n,T} = \mu_T \quad a.s.$$

3. Proofs

To prove our main theorems, we first derive several asymptotic results (Propositions 1-4) for the means of subgraph counts $H_{n,T}$, $\tilde{H}_{n,T}$ and $\tilde{G}_{n,T}$.

Proposition 1. *For $k \geq 2$, let T be a connected feasible graph of order k. Suppose $r_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} n^{-k} r_n^{-2(k-1)} EH_{n,T} = \mu_T$$

where μ_T is defined in Section 2.

Proof. From the linearity of expectation, $EH_{n,T} = \binom{n}{k} Eh_{n,T}(\mathcal{X}_k, \mathcal{Y}_k)$. Thereby

$$\begin{aligned} EH_{n,T} = {} & \frac{1}{(2\pi)^k}\binom{n}{k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_{n,T}(x_1,\cdots,x_k,y_1,\cdots,y_k) \\ & \cdot f^k(x_1)\mathrm{d}x_1\cdots\mathrm{d}x_k\mathrm{d}y_1\cdots\mathrm{d}y_k \\ & + \frac{1}{(2\pi)^k}\binom{n}{k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_{n,T}(x_1,\cdots,x_k,y_1,\cdots,y_k) \\ & \cdot\left(\prod_{i=1}^k f(x_i) - f^k(x_1)\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\mathrm{d}y_1\cdots\mathrm{d}y_k. \qquad (1) \end{aligned}$$

Let $x_1 = x$ and $x_i = x_1 + r_n z_i$ for $2 \leq i \leq k$, then the first term on the right hand side of (1) equals

$$\frac{1}{(2\pi)^k}\binom{n}{k} r_n^{2(k-1)}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_{n,T}(x, x+r_n z_2,\cdots,x+r_n z_k, y_1,\cdots,y_k)$$

$$\cdot f^k(x)\mathrm{d}x\mathrm{d}z_2\cdots\mathrm{d}z_k\mathrm{d}y_1\cdots\mathrm{d}y_k.$$

We have $h_{n,T}(x, x+r_nz_2,\cdots,x+r_nz_k,y_1,\cdots,y_k) = h_T(0,z_2,\cdots,z_k,y_1,\cdots,y_k)$. Since f is bounded and T is a connected graph, $f^k(x)$ and $h_T(0,z_2,\cdots,z_k,y_1,\cdots,y_k)$ are integrable on $\mathbb{R}^2$ and $(\mathbb{R}^2)^{k-1}\times[0,2\pi)^k$ respectively. Hence, the first term on the right hand side of (1) tends to $n^k r_n^{2(k-1)}\mu_T$ by the dominated convergence theorem.

Set $\eta_n(x_1) := \int_{D(x_1,kr_n)}\cdots\int_{D(x_1,kr_n)} r_n^{-2(k-1)}\left|\prod_{i=2}^k f(x_i) - f^{k-1}(x_1)\right|\mathrm{d}x_2\cdots\mathrm{d}x_k$. It is easy to see that the absolute value of the second term on the right hand side of (1) multiplied by $n^{-k}r_n^{-2(k-1)}$ is bounded by $\int_{\mathbb{R}^2}\eta_n(x_1)f(x_1)\mathrm{d}x_1$. If x_1 is a continuous point of f, then $\eta_n(x_1)\to 0$ as $n\to\infty$ by the mean value theorem for integrals. Therefore, by the dominated convergence theorem and the assumption of almost everywhere continuity of f, we have $\int_{\mathbb{R}^2}\eta_n(x_1)f(x_1)\mathrm{d}x_1\to 0$. This concludes the proof. □

Proposition 2. *For $k\ge 2$, let T be a connected feasible graph of order k. Suppose $nr_n^2\to 0$ as $n\to\infty$. Then*

$$\lim_{n\to\infty} n^{-k}r_n^{-2(k-1)}E\tilde{H}_{n,T} = \mu_T,$$

where μ_T is defined in Section 2.

Proof. Let $\tilde{h}_{n,T}(\mathcal{X},\mathcal{Y}) := 1_{[G(\mathcal{X},\mathcal{Y},r_n)\text{ is isolated and }\cong T]}$. Therefore,

$$\begin{aligned} E\tilde{H}_{n,T} &= \binom{n}{k}E\tilde{h}_{n,T}(\mathcal{X}_k,\mathcal{Y}_k) \\ &= \binom{n}{k}Eh_{n,T}(\mathcal{X}_k,\mathcal{Y}_k)\cdot P(G(\mathcal{X}_k,\mathcal{Y}_k,r_n)\text{ is isolated}|G(\mathcal{X}_k,\mathcal{Y}_k,r_n)\cong T) \\ &:= EH_{n,T}\cdot P_1. \end{aligned}$$

Since T is a connected graph and by the assumed asymptotic behavior of r_n, we get as $n\to\infty$,

$$1\ge P_1\ge (1-P(X_1\in D(0,kr_n)))^{n-k}\ge(1-f_{\max}\pi(kr_n)^2)^{n-k}\to 1.$$

Consequently, $E\tilde{H}_{n,T} = (1+o(1))EH_{n,T}$. By using Proposition 1, we complete the proof. □

Proposition 3. *For $k\in\mathbb{N}$, let T be a connected feasible graph of order k. Suppose $nr_n^2\to\lambda\in(0,\infty)$ as $n\to\infty$. Then*

$$\lim_{n\to\infty} n^{-1}E\tilde{H}_{n,T} = k^{-1}\int_{\mathbb{R}^2}\varphi_T(\lambda f(x))f(x)\mathrm{d}x,$$

where $\varphi_T(\cdot)$ is defined in Section 2.

Proof. By the definition of $h_{n,T}$ and similar with the beginning of the proof of Proposition 1, we have

$$n^{-1}E\tilde{H}_{n,T} = \frac{1}{n(2\pi)^k}\binom{n}{k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_{n,T}(x_1,\cdots,x_k,y_1,\cdots,y_k)$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

$$\cdot\Big(1-\int_{\cup_{i=1}^k S(x_i,y_i,r_n)} f(x)\mathrm{d}x\Big)^{n-k} f^k(x_1)\mathrm{d}x_1\cdots\mathrm{d}x_k\mathrm{d}y_1\cdots\mathrm{d}y_k$$
$$+\frac{1}{n(2\pi)^k}\binom{n}{k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_{n,T}(x_1,\cdots,x_k,y_1,\cdots,y_k)$$
$$\cdot\Big(1-\int_{\cup_{i=1}^k S(x_i,y_i,r_n)} f(x)\mathrm{d}x\Big)^{n-k}$$
$$\cdot\Big(\prod_{i=1}^k f(x_i)-f^k(x_1)\Big)\mathrm{d}x_1\cdots\mathrm{d}x_k\mathrm{d}y_1\cdots\mathrm{d}y_k. \tag{2}$$

Let $x_i = x_1 + r_n z_i$ for $2 \le i \le k$, then the first term on the right hand side of (2) tends to

$$\frac{1}{n(2\pi)^k}\binom{n}{k} r_n^{2(k-1)}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_{n,T}(x_1,x_1+r_nz_2,\cdots,x_1+r_nz_k,y_1,\cdots,y_k)$$
$$\cdot\, e^{(n-k)\ln\left(1-\int_{S(x_1,y_1,r_n)\cup\cup_{i=2}^k S(x_1+r_nz_i,y_i,r_n)} f(x)\mathrm{d}x\right)} f^k(x_1)\mathrm{d}x_1\mathrm{d}z_2\cdots\mathrm{d}z_k\mathrm{d}y_1\cdots\mathrm{d}y_k$$

which is further asymptotic to

$$\frac{\lambda^{k-1}}{k!(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2} h_T(0,z_2,\cdots,z_k,y_1,\cdots,y_k)$$
$$\cdot\, e^{-\lambda f(x_1)|S(0,z_2,\cdots,z_k,y_1,\cdots,y_k)|} f^k(x_1)\mathrm{d}x_1\mathrm{d}z_2\cdots\mathrm{d}z_k\mathrm{d}y_1\cdots\mathrm{d}y_k$$

by using the mean value theorem for integrals and the dominated convergence theorem. Thereby, the first term on the right hand side of (2) tends to $k^{-1}\int_{\mathbb{R}^2}\varphi_T(\lambda f(x_1))$ $f(x_1)\mathrm{d}x_1$.

On the other hand, note that $n^{-1}\binom{n}{k} \le Cr_n^{-2(k-1)}$ for some positive constant C, then the absolute value of the second term on the right hand side of (2) is bounded by $C\int_{\mathbb{R}^2} f(x_1)\eta_n(x_1)\mathrm{d}x_1$, where $\eta_n(x_1)$ is defined by

$$\eta_n(x_1)=\int_{D(x_1,kr_n)}\cdots\int_{D(x_1,kr_n)} r_n^{-2(k-1)}\cdot\Big|\prod_{i=2}^k f(x_i)-f^{k-1}(x_1)\Big|\mathrm{d}x_2\cdots\mathrm{d}x_k.$$

By the mean value theorem for integrals, $\eta_n(x_1)$ tends to 0 if x_1 is a continuous point of f, then by the dominated convergence theorem, as in the proof of Proposition 1, $\int_{\mathbb{R}^2} f(x_1)\eta_n(x_1)\mathrm{d}x_1 \to 0$ as $n\to\infty$. Thus the proof is completed. □

Proposition 4. *For $k = 1$, let T be a graph of order k, that is, T is a single point. Suppose $nR_n^d \to \lambda \in [0,\infty)$ in probability, as $n\to\infty$. Then*

$$\lim_{n\to\infty} n^{-1}E\tilde{G}_{n,T} = \int_{\mathbb{R}^d} e^{-f(x)\theta\lambda} f(x)\mathrm{d}x$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

Proof. By the definition of $g_{n,T}$,

$$n^{-1}E\tilde{G}_{n,T} = \int_0^\infty \int_{\mathbb{R}^d} g_{n,T}(x_1, r_{n,1})\Big(1-\int_{B(x_1,r_{n,1})} f(x)\mathrm{d}x\Big)^{n-1} f(x_1)q_n(r_{n,1})\mathrm{d}x_1\mathrm{d}r_{n,1}$$
$$= \int_0^\infty \int_{\mathbb{R}^d} \Big(1-\int_{B(x_1,r_{n,1})} f(x)\mathrm{d}x\Big)^{n-1} f(x_1)q_n(r_{n,1})\mathrm{d}x_1\mathrm{d}r_{n,1}$$
$$\sim \int_0^\infty \int_{\mathbb{R}^d} e^{-nf(x_1)\theta r_{n,1}^d} f(x_1)q_n(r_{n,1})\mathrm{d}x_1\mathrm{d}r_{n,1}.$$

Hence using the dominated convergence theorem and the assumption of R_n, the above expression tends to $\int_{\mathbb{R}^d} e^{-f(x_1)\theta\lambda} f(x_1)\mathrm{d}x_1$ as n tends to infinity, which concludes the proof. □

Now we are in position to prove our strong laws of large numbers.

Proof of Theorem 1. In order to use Lemma 1, we shall first define a filtration. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ be the trivial σ-field, and $\mathcal{F}_i = \sigma\{(X_1, Y_1), \cdots, (X_i, Y_i)\}$ for $i \geq 1$. Define a martingale difference sequence as $D_{n,i} := E(\tilde{H}_{n,T}|\mathcal{F}_i) - E(\tilde{H}_{n,T}|\mathcal{F}_{i-1})$, therefore we can write $\tilde{H}_{n,T} - E\tilde{H}_{n,T} = \sum_{i=1}^n D_{n,i}$. Let $\tilde{H}^i_{n,T}$ denote the number of isolated T-subgraphs in $G(\mathcal{X}_{n+1}\backslash\{X_i\}, \mathcal{Y}_{n+1}\backslash\{Y_i\}, r_n)$ and $\tilde{H}'_{n+1,T}$ denote the number of isolated T-subgraphs in $G(\mathcal{X}_{n+1}, \mathcal{Y}_{n+1}, r_n)$. Thus we have $D_{n,i} = E(\tilde{H}_{n,T} - \tilde{H}^i_{n,T}|\mathcal{F}_i)$. Adding a point to a finite set in $\mathbb{R}^d$ can cause the number of isolated T-subgraphs to increase by at most 1, and can cause it to decrease by less than a constant M (here $d = 2$, so we may take $M = 6$), thereby we get

$$\begin{aligned}|\tilde{H}_{n,T} - \tilde{H}^i_{n,T}| &\leq |\tilde{H}_{n,T} - \tilde{H}'_{n+1,T}| + |\tilde{H}'_{n+1,T} - \tilde{H}^i_{n,T}| \\ &\leq (M+1) + (M+1) = 2(M+1)\end{aligned}$$

and then $D_{n,i} \leq 2(M+1)$. Now for any $\varepsilon > 0$, by Lemma 1, we have $P(|\tilde{H}_{n,T} - E\tilde{H}_{n,T}| > \varepsilon n) \leq 2\exp\big(-\frac{\varepsilon^2 n}{8(M+1)^2}\big)$, which is summable in n. The result follows by Borel-Cantelli Lemma and Proposition 3. □

Proof of Theorem 2. The proof parallels to that of Theorem 1. Define a filtration: $\mathcal{F}_{n,0} = \{\emptyset, \Omega\}$ and $\mathcal{F}_{n,i} = \sigma\{(X_1, R_{n,1}), \cdots, (X_i, R_{n,i})\}$ for $i \geq 1$. A martingale difference sequence is defined by $D_{n,i} = E(\tilde{G}_{n,T}|\mathcal{F}_{n,i}) - E(\tilde{G}_{n,T}|\mathcal{F}_{n,i-1})$, and then we have $\tilde{G}_{n,T} - E\tilde{G}_{n,T} = \sum_{i=1}^n D_{n,i}$. Let $\tilde{G}^i_{n,T}$ denote the number of isolated vertices in $G(\mathcal{X}_{n+1}\backslash\{X_i\}, \mathcal{R}_n \cup R_{n,n+1}\backslash\{R_{n,i}\})$. It is easy to see that $D_{n,i} = E(\tilde{G}_{n,T} - \tilde{G}^i_{n,T}|\mathcal{F}_{n,i})$. Reason similarly as in the proof of Theorem 1, there exists a constant $M > 0$ depends only on d such that $|\tilde{G}_{n,T} - \tilde{G}^i_{n,T}| \leq M$, hence $|D_{n,i}| \leq M$ a.s. For $\varepsilon > 0$, by Lemma 1, $P(|\tilde{G}_{n,T} - E\tilde{G}_{n,T}| > \varepsilon n) \leq 2\exp\big(-\frac{\varepsilon^2 n}{2M^2}\big)$, which is summable in n. The result then follows by Borel-Cantelli Lemma and Proposition 4. □

Proof of Theorem 3. Proceeding along the same line as the proof of Theorem 1,

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

by Lemma 1, we get

$$P\big(|\tilde{H}_{n,T} - E\tilde{H}_{n,T}| > \varepsilon n^k r_n^{2(k-1)}\big) \leq 2\exp\left(-\frac{\varepsilon^2 n^{2k-1} r_n^{4(k-1)}}{8(M+1)^2}\right),$$

which is summable in n by the assumption. The result follows by the Borel-Cantelli Lemma and Proposition 2. □

In the next proof, we substitute Lemma 2 for Lemma 1 since this time the basic Azuma' inequality is no longer valid.

Proof of Theorem 4. Set $\eta := \frac{1\wedge\delta}{3(k-1)}$ and partition $\mathbb{R}^2$ into squares $(A_{n,i},\ i \in \mathbb{N})$, each of side r_n. Let

$$E := \{\mathcal{X} \subset \mathbb{R}^2 | \\ \mathrm{card}(\mathcal{X}) = n, \mathcal{X}(A_{n,i}) \leq n^\eta(nr_n^2 \vee 1) \text{ for every } A_{n,i}\ s.t.\ A_{n,i} \cap \mathrm{supp}f \neq \emptyset\}.$$

Since $\mathcal{X}_n(A_{n,i}) \sim \mathrm{Bin}(n, \int_{A_{n,i}} f(x)\mathrm{d}x)$ is a binomial random variable, by a Chernoff bound (see, e.g. [13], p. 16), we have $P(\mathcal{X}_n(A_{n,i}) > n^\eta(nr_n^2 \vee 1)) \leq e^{-n^\eta}$ when n is large enough. Since supp$f$ is bounded, we have

$$P(\mathcal{X}_n \notin E) \leq Cn^{\frac{2k-1-\delta}{2k-2}} e^{-n^\eta}, \tag{3}$$

where C is some positive constant. Now let us define a filtration. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ be the trivial σ-field, and $\mathcal{F}_i = \sigma\{(X_1, Y_1), \cdots, (X_i, Y_i)\}$ for $i \geq 1$. We have $H_{n,T} - EH_{n,T} = \sum_{i=1}^n D_{n,i}$ with the martingale differences $D_{n,i} := E(H_{n,T}|\mathcal{F}_i) - E(H_{n,T}|\mathcal{F}_{i-1})$. Let $H_{n,T}^i$ denote the number of induced T-subgraphs in $G(\mathcal{X}_{n+1}\backslash\{X_i\}, \mathcal{Y}_{n+1}\backslash\{Y_i\}, r_n)$, then $D_{n,i} = E(H_{n,T} - H_{n,T}^i|\mathcal{F}_i)$. Set an event $E_{n,i} = \{\mathcal{X}_n \in E, \mathcal{X}_{n+1}\backslash\{X_i\} \in E\}$, hence we may derive $|H_{n,T} - H_{n,T}^i| \cdot 1_{E_{n,i}} \leq C \cdot [n^\eta(nr_n^2 \vee 1)]^{k-1}$ for some constant C, and $|H_{n,T} - H_{n,T}^i| \leq n^k$, since changing the position of one point in a configuration can only at most affect the subgraphs constructed by points in surrounding nine squares. Consequently,

$$\begin{aligned} |D_{n,i}| &\leq E(|H_{n,T} - H_{n,T}^i| \cdot 1_{E_{n,i}}|\mathcal{F}_i) + E(|H_{n,T} - H_{n,T}^i| \cdot 1_{E_{n,i}^c}|\mathcal{F}_i) \\ &\leq C \cdot [n^\eta(nr_n^2 \vee 1)]^{k-1} + n^k P(E_{n,i}^c|\mathcal{F}_i). \end{aligned} \tag{4}$$

Define an event $F_{n,i} := \{P(E_{n,i}^c|\mathcal{F}_i) \leq n^{-k}\}$, thus

$$P(F_{n,i}^c) \leq n^k E(P(E_{n,i}^c|\mathcal{F}_i)) = n^k P(E_{n,i}^c) \leq C' n^{k+\frac{2k-1-\delta}{2k-2}} e^{-n^\eta},$$

by Markov's inequality and (3). Note by (4) when $F_{n,i}$ occurs, $|D_{n,i}| \leq C \cdot [n^\eta(nr_n^2 \vee 1)]^{k-1}$. Therefore, by using Lemma 2, we obtain for any $\varepsilon > 0$

$$P(|H_{n,T} - EH_{n,T}| > \varepsilon n^k r_n^{2(k-1)}) \leq 2\exp\left(- \frac{\varepsilon^2 n^{2k} r_n^{4(k-1)}}{Cn[n^\eta (nr_n^2 \vee 1)]^{2(k-1)}} \right)$$

$$+ \left(1 + \frac{2n^k}{\varepsilon n^k r_n^{2(k-1)}}\right) \cdot C' n^{k+1+\frac{2k-1-\delta}{2k-2}} e^{-n^\eta}$$

$$\leq 2e^{-\frac{\varepsilon^2}{C}\left(n^{1-2\eta(k-1)} \wedge n^{2k-1-2\eta(k-1)} r_n^{4(k-1)}\right)} + C'' n^{3k-\delta} e^{-n^\eta},$$

which is summable in n. The result then follows by the Borel-Cantelli Lemma and Proposition 1. □

4. Conclusion and Future Work

We have derived various strong laws of large numbers of subgraph counts in two types of directed random geometric networks. These models of directed graphs are natural generalizations of standard geometric graphs applicable to analysis of a variety of spatial networks exemplified as wireless communication networks and sensor networks. Our results obtained for the second model are limited and a very natural question would be to ask what happens for other sorts of small subgraphs.

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