International Electronic Journal of Pure and Applied Mathematics

Volume 2 No. 2 2010, 21-46

EVENTUAL STABILITY FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH "SUPREMUM"

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Abstract: This paper investigates several types of eventual stability for nonlinear impulsive differential equations with "supremum". Both cases of regular norm as well as two different measures for the initial conditions and for the solution are studied. Sufficient conditions for eventual stability, for eventual practical stability and for eventual strong stability are obtained. Razumikhin method with piecewise continuous Lyapunov like functions has been applied. Comparison result for scalar impulsive ordinary differential equation has been employed.

AMS Subject Classification: 34D20, 34K20, 34K45

Key Words: eventual stability, Lyapunov function, impulsive differential equations with "supremum"

1. Introduction

One of the main qualitative questions in the theory of differential equation is stability. The problems of stability of solutions of differential equations via Lyapunov functions have been successfully investigated in the past. One type of stability, very useful in real world problems, deals with two different measures. Stability in terms of two measures for differential equations has been studied by means of various types of Lyapunov functions (see [1], [10], [7], [6] and references therein). In many real cases, it is obligatory the stability in Lyapunov sense is excluded. For example, such kind of situation could be seen in self-controlled systems of management. To be

Received: May 25, 2010

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solved such kind of a problem, a new type of stability, so called eventual stability, is introduced (see [4], [11], [12], [16]). At the same time for more generalized considerations some authors use two different measures to study stability properties of the solution (see [5], [8], [15], [17]). In this paper the concept of eventual stability is spreded to impulsive differential equations, in which the maximum value of unknown function over a past time interval is involved. These equations are adequate model of real world problem in which the present state depends significantly on its maximum value over a past time interval and at the same time the observed process has abrupt changes at certain moments. For example, in the theory of automatic control of various technical systems it often occurs that the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals. E.P. Popov (see [9]) in 1966 considered the system for regulating the voltage of a generator of constant current. The object of the experiment was a generator of constant current with parallel simulation and the regulated quantity was the voltage at the source electric current. Recently, the theory of differential equations, containing the maximum value of the unknown function is very fast developed (see [3], [2], [13], [14]).

In the paper various types of eventual stability in terms of two measures such as eventual stability, eventual practical stability as well as eventual strong practical stability are defined and studied. To generalize the studied object and to be obtained more general sufficient conditions, two different measures are applied. Razumikhin method and comparison results for scalar impulsive differential equations are employed. An appropriate example illustrates the application of the obtained sufficient conditions and the main advantages of the considered type of stability.

2. Preliminary Notes and Definitions

Let $\mathbb{R}^n$ be n-dimensional Euclidean space with a norm $||x||$, Ω be a bounded domain in $\mathbb{R}^n$ containing the origin and $\mathbb{R}_+ = [0, \infty)$.

Let $\{\tau_k\}_1^\infty$ be a sequence of fixed points in $\mathbb{R}_+$ such that $\tau_{k+1} > \tau_k$ and $\lim_{k\to\infty} \tau_k = \infty$. Let $r > 0$ be a fixed constant.

Consider the system of nonlinear impulsive differential equations with "supremum"

$$x' = f(t, x(t), \sup_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \geq t_0, \quad t \neq \tau_k, \tag{1}$$

$$x(\tau_k + 0) = I_k(x(\tau_k - 0)) \quad \text{for} \quad k = 1, 2, \ldots, \tag{2}$$

with initial condition

$$x(t) = \varphi(t) \qquad \text{for} \quad t \in [t_0 - r, t_0], \tag{3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, 3, \ldots$, $t_0 \in \mathbb{R}_+$, $\varphi : [t_0 - r, t_0] \to \mathbb{R}^n$.

Note that for $x: \ [t-r,t] \to \mathbb{R}^n, \ x = (x_1, x_2, \dots, x_n)$ we denote

$$\sup_{s\in[t-r,t]} x(s) = \Big(\sup_{s\in[t-r,t]} x_1(s), \ \sup_{s\in[t-r,t]} x_2(s), \dots, \ \sup_{s\in[t-r,t]} x_n(s) \Big).$$

Denote by $PC(X,Y) \ \ (X \subset \mathbb{R}, Y \subset \mathbb{R}^n)$ the set of all functions $u : X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $\tau_k \in X$ and which are continuous from the left at the points $\tau_k \in X$, $u(\tau_k) = u(\tau_k - 0)$.

In our further investigations we will assume that for any initial function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ the solution of the initial value problem for system of impulsive differential equations with "supremum" (1)-(3) exists on $[t_0, \infty)$.

Let $X \subset \mathbb{R}$. Denote by $Z(X)$ the set of all integers k such that $\tau_k \in X$.

We will define the set of measures:

$$\Gamma \ = \ \{h \in C([-r,\infty) \times \mathbb{R}^n, \mathbb{R}_+) : \min_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for each } t \in [-r,\infty)\}.$$

Let $h_0 \in \Gamma$, $t_0 \in \mathbb{R}_+$, $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n)$. We will use the following notation

$$H_0(t_0, \varphi) = \sup_{s\in[t_0-r,t_0]} h_0(s, \varphi(s)). \tag{4}$$

Let $\rho > 0$ be a fixed number and $h \in \Gamma$. Define:

$$S(h,\rho) = \{(t,x) \in [-r,\infty) \times \mathbb{R}^n : h(t,x) < \rho\},$$
$$\bar{S}(h,\rho) = \{(t,x) \in [-r,\infty) \times \mathbb{R}^n : h(t,x) \le \rho\},$$
$$S^C(h,\rho) = \{(t,x) \in [-r,\infty) \times \mathbb{R}^n : h(t,x) \ge \rho\}.$$

We will introduce the definition of various types of eventual stability for impulsive differential equations with "supremum", based on the ideas of stability in terms of two measures (see [5]).

Definition 1. Let $h, h_0 \in \Gamma$. System of impulsive differential equations with "supremum" (1), (2) is said to be:

(S1) *eventually stable in terms of two measures* if for every $\epsilon > 0$ there exists $\tau = \tau(\epsilon) > 0$ such that for any $t_0 \ge \tau$ there exists a positive $\delta = \delta(t_0, \epsilon) > 0$ such that for any $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ inequality $H_0(t_0, \phi) < \delta$ implies $h(t, (x(t; t_0, \phi)) < \epsilon$ for $t \ge t_0$, where $x(t; t_0, \phi)$ is any solution of the initial value problem for impulsive differential equations with "supremum"(1), (2), (3);

(S2) *uniformly eventually stable in terms of two measures* if δ in (S1) is independent on t_0.

(S3) *eventually practically stable in terms of two measures* if for any couple $(\lambda, A) : \ \lambda, A > 0$ there exists $\tau(\lambda, A) > 0$ such that for some $t_0 \ge \tau(\lambda, A)$ and for any $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$: $H_0(t_0, \phi) < \lambda$ the inequality $h(t, x(t; t_0, \phi)) < A$ holds for $t \geq t_0$;

(S4) *uniformly eventually practically stable in terms of two measures* if (S3) holds for all $t_0 \geq \tau(\lambda, A)$;

(S5) *eventually strongly practically stable in terms of two measures* if for any triple $(\lambda, A, T) :\ \lambda, A, T > 0$ and some $t_0 \in \mathbb{R}_+$ the inequality $H_0(t_0, \phi) < \lambda$, where $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0 + T$;

(S6) *uniformly eventually strongly practically stable in terms of two measures* if for any triple $(\lambda, A, T) :\ \lambda, A, T > 0$ and for any $t_0 \in \mathbb{R}_+$ and $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ the inequality $H_0(t_0, \phi) < \lambda$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0 + T$.

Remark 1. Note that if the system of impulsive differential equations with "supremum" (1), (2) is (uniformly) eventually practically stable in terms of two measures, then it is (uniformly) eventually strongly practically stable in terms of two measures.

In our further investigations we will use the comparison scalar impulsive differential equation

$$\begin{aligned} &u' = g(t, u), \quad t \geq t_0, \quad t \neq \tau_k, \\ &u(\tau_k + 0) = \xi_k(u(\tau_k)), \quad k = 1, 2, \dots, \end{aligned} \tag{5}$$

with an initial condition

$$u(t_0) = u_0, \tag{6}$$

where $u, u_0 \in \mathbb{R}$, $g :\ \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\xi_k :\ \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \dots$.

In our further investigations we will assume that for any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar impulsive equation (5) exists on $[t_0, \infty)$, $t_0 \geq 0$. For some existence results of (5) see the book of D. Bainov et al [1].

In the case $r = 0$ and $h_0(t, x) = h(t, x) = ||x||$ the above given definitions (S1)-(S6) reduce to definitions for the corresponding types of *eventual stability of the zero solution* of impulsive differential equations which will be used in our further investigations.

In the case $r = 0$, $I_k(x) \equiv x$, $k = 1, 2, \dots$, and $h_0(t, x) = h(t, x) = ||x||$ the above given definitions (S1)-(S6) reduce to definitions for the corresponding types of eventual stability of the zero solution of ordinary differential equations, given in [4].

Since we will use different types of eventual stability of the zero solution of impulsive differential equation (5) in our main results, we will give them:

Definition 2. The zero solution of impulsive differential equation (5) is said to be:

(S7) *eventually stable* if for every $\epsilon > 0$ there exists $\tau = \tau(\epsilon) > 0$ such that for any $t_0 \geq \tau$ there exists a positive $\delta = \delta(t_0, \epsilon) > 0$ such that for any $u_0 :\quad |u_0| < \delta$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

the iequality $u(t;t_0,u_0) < \epsilon$ holds for $t \geq t_0$, where $u(t;t_0,u_0)$ is a solution of (5), (6);

(S8) *uniformly eventually stable* if δ in (S7) is independent on t_0.

(S9) *eventually practically stable* if for any couple (λ, A) : $0 < \lambda < A$ there exists $\tau(\lambda, A) > 0$ such that for some $t_0 \geq \tau(\lambda, A)$ and for any u_0 : $|u_0| < \lambda$ the inequality $u(t;t_0,u_0) < A$ holds for $t \geq t_0$;

(S10) *uniformly eventually practically stable* if (S9) holds for all $t_0 \geq \tau(\lambda, A)$;

(S11) *eventually strongly practically stable* if (S9) holds and for any triple (λ, A, T) : $0 < \lambda < A, 0 < T$ and some $t_0 \in \mathbb{R}_+$ the inequality $|u_0| < \lambda$ implies $u(t;t_0,u_0) < A$ for $t \geq t_0 + T$;

(S12) *uniformly eventually strongly practically stable* if (S10) holds and for any triple (λ, A, T) : $0 < \lambda < A, 0 < T$ and for any $t_0 \in \mathbb{R}_+$ and u_0 $|u_0| < \lambda$ the inequality $u(t;t_0,u_0) < A$ holds for $t \geq t_0 + T$.

Remark 2. Note that since in Definition 2 the inequality $\lambda < A$ is required in (S9)-(S12), this inequality is missing in Definition 1 (S3)-(S6). But in the sufficient conditions for the stability we have to be sure that the constants λ, A and the measures h_0, h are such that $H_0(t_0, \phi) < \lambda$ implies $h(t, \phi(t)) < A$ on $[t_0 - r, t_0]$.

We will study the connection between eventual stability of the scalar impulsive differential equation (5) and the corresponding eventual stability in terms of two measures for the system of impulsive differential equations with "supremum" (1), (2).

Introduce the following notations

$$G_k = \{t \in [-r, \infty) : \; t \in (\tau_k, \tau_{k+1})\}, \quad k = 1, 2, \ldots, \qquad \mathcal{G} = \bigcup_{k=1}^{\infty} G_k.$$

We will introduce the class Λ of piecewise continuous Lyapunov functions which will be used to investigate defined above types of eventual stability of impulsive differential equations with "supremum".

Definition 3. We will say that the function $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $\Delta \subset [-r, \infty)$, $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, belongs to class Λ if:

1. $V(t,x)$ is a continuous function in $(\Delta \cap \mathcal{G}) \times \Omega$ and $V(t,0) \equiv 0$ for $t \in \Delta$;
2. For every $k \in Z(\Delta)$ and $x \in \Omega$ there exist the finite limits

$$V(\tau_k, x) = V(\tau_k - 0, x) = \lim_{t \uparrow \tau_k} V(t,x), \qquad V(\tau_k + 0, x) = \lim_{t \downarrow \tau_k} V(t,x).$$

3. Function $V(t,x)$ is Lipschitz with respect to its second argument in the set $\Delta \times \Omega$.

Let $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$. For any $t \in \Delta \cap \mathcal{G}$ and any function $\psi \in PC([t-r,t], \Omega)$ we will define a derivative of the function V along a trajectory of the solution of (1), (2) as follows:

$$D_{(1),(2)}V(t,\psi) = \limsup_{\epsilon\to 0} \frac{1}{\epsilon}\Big[V\Big(t+\epsilon, \psi(t)+\epsilon f(t,\psi(t), \max_{s\in[-r,0]}\psi(t+s))\Big) - V(t,\psi(t))\Big]. \tag{7}$$

Definition 4. Let $h_0 \in \Gamma$. The function $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$ is strongly-h_0-decrescent if there exist a function $a \in K$ and a constant $\rho > 0$ such that $h_0(t,x) < \rho$ implies $V(t,x) \le a(h_0(t,x))$, where $(t,x) \in \Delta \times \Omega$.

Let $\rho > 0$ be a given number. Consider the following sets:

$$K = \{a \in C(\mathbb{R}_+,\mathbb{R}_+) : \ a(r) \text{ is strictly increasing and } a(0) = 0\};$$
$$\mathcal{K} = \{a \in C(\mathbb{R}_+,\mathbb{R}_+) : \ a(r) \text{ is strictly increasing and } a(s) \ge s,\ a(0) = 0\}.$$

Definition 5. Let $h, h_0 \in \Gamma$. The function h_0 is *uniformly finer* than h if there exist a function $\psi \in K$ and a constant $\rho > 0$ such that $h_0(t,x) < \rho$ for some $(t,x) \in [-r,\infty) \times \mathbb{R}^n$ implies $h(t,x) < \psi(h_0(t,x))$.

In the further investigations, we will use the following comparison result:

Lemma 1. (see S. Hristova [3]) *Let the following conditions be fulfilled:*

1. *The functions $f \in PC([t_0,T]\times\Omega\times\Omega,\mathbb{R}^n)$ and $I_k \in C(\Omega,\Omega)$ for $k \in Z([t_0,T))$, where $\Omega \subset \mathbb{R}^n$, and $t_0, T :\ 0 \le t_0 < T < \infty$ are constants.*
2. *The function $\varphi \in PC([t_0-r,t_0],\Omega)$.*
3. *The initial value problem (1), (2), (3) has a solution $x(t) = x(t;t_0,\varphi)$, such that $x(t) \in \Omega$ on $[t_0-r,T]$.*
4. *The functions $g \in PC([t_0,T]\times\mathbb{R}_+,\mathbb{R}_+)$, $g(t,0) \equiv 0$ for $t \in [t_0,T]$ and $\xi_k \in \mathcal{K}$, $k \in Z((t_0,T))$.*
5. *For any initial point $u_0 \in \mathbb{R}_+$ the initial value problem for the scalar impulsive differential equation (5) has a maximal solution $u^*(t) = u^*(t;t_0,u_0)$, which is defined for $t \in [t_0,T]$.*
6. *The function $V : [t_0-r,T]\times\Omega \to \mathbb{R}_+$, $V \in \Lambda$ is such that:*

   (i) for any number $t \in [t_0,T] :\ t \ne \tau_k,\ k \in Z((t_0,T))$ and any function $\psi \in PC([t-r,t],\Omega)$ such that $V(t,\psi(t)) \ge V(t+s,\psi(t+s))$ for $s \in [-r,0)$ the inequality

   $$D_{(1),(2)}V(t,\psi(t)) \le g(t,V(t,\psi(t)))$$

   holds.

   (ii) $V(\tau_k+0,I_k(x)) \le \xi_k(V(\tau_k,x))$, $\quad k \in Z((t_0,T)), x \in \Omega$.

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Then the inequality $\sup_{s\in[-r,0]} V(t_0+s,\varphi(t_0+s)) \le u_0$ *implies the inequality* $V(t,x(t)) \le u^*(t)$ *for* $t \in [t_0,T]$.

Remark 3. Note that the claim of Lemma 1 remains valid if $T=\infty$. Then only the considered intervals are opened from the right.

3. Main Results

3.1. Eventual Stability in Terms of Two Measures

We will obtain sufficient conditions for various types of eventual stability in terms of two measures for impulsive differential equations with "supremum". We will use Lyapunov functions from class Λ. The proof is based on Razumikhin method and a comparison method employing scalar impulsive differential equations.

In the case when the Lyapunov function satisfies globally the desired conditions we obtain the following result:

Theorem 1. *Let the following conditions be fulfilled:*

1. *The function* $f \in PC(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ *and* $f(t,0,0) \equiv 0$.
2. *The functions* $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$ *and* $I_k(0)=0$ *for* $k \in Z(\mathbb{R}_+)$.
3. *The functions* $h_0, h \in \Gamma$.
4. *There exists a function* $V(t,x) : [-r,\infty)\times\mathbb{R}^n \to \mathbb{R}_+$ *with* $V\in\Lambda$ *such that:*
   - *(i)* $b(h(t,x)) \le V(t,x) \le a(h_0(t,x))$ *for* $(t,x) \in [-r,\infty)\times\mathbb{R}^n$ *where* $a,b \in K$;
   - *(ii) for any number* $t\in\mathbb{R}_+ : \ t \ne \tau_k, \ k \in Z(\mathbb{R}^+)$ *and any function* $\psi \in PC([t-r,t],\mathbb{R}^n)$ *such that* $V(t,\psi(t)) > V(t+s,\psi(t+s))$ *for* $s\in[-r,0)$ *the inequality*
     $$D_{(1),(2)}V(t,\psi(t)) \le g(t,V(t,\psi(t)))$$
     holds, where $g \in PC(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$ *and* $g(t,0)\equiv 0$;
   - *(iii)* $V(\tau_k+0, I_k(x)) \le \xi_k(V(\tau_k,x))$, *for* $x\in\mathbb{R}^n$, $k \in Z(\mathbb{R}^+)$, *where* $\xi_k \in \mathcal{K}$.

Then:

(A) the (uniform) eventual stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);

(B) the (uniform) eventual practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);

(C) the (uniform) eventual strong practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual strong practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2).

Proof. (A) Let $\epsilon > 0$ be an arbitrary number and the zero solution of scalar impulsive differential equation (5) be eventually stable. Therefore there exist $\tau = \tau(\epsilon) > 0$ such that for any $t_0 \geq \tau$ there exists a number $\delta = \delta(t_0, \epsilon) > 0$ such that inequality $|u_0| < \delta$ implies

$$|u(t; t_0, u_0)| < b(\epsilon) \qquad \text{for} \quad t \geq t_0, \tag{8}$$

where $u(t; t_0, u_0)$ is a solution of (5), (6).

Let $t_0 \geq \tau$ and $\delta_1 > 0$ be such that $a(\delta_1) < \delta$. Note that $\delta_1 = \delta_1(t_0, \epsilon)$. Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0, \phi) < \delta_1. \tag{9}$$

Let $x(t; t_0, \phi)$ be a solution of (1), (2) with initial condition (3).

Let $u_0 = \sup_{s \in [-r,0]} V(t_0 + s, \phi(t_0 + s))$. From Lemma 1 for the function $V(t, x)$ defined on $[-r, \infty) \times \mathbb{R}^n$ it follows the validity of the inequality

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) \quad \text{for} \quad t \geq t_0. \tag{10}$$

From condition 4(i) of Theorem 1 we obtain

$$V(t_0 + s, \phi(t_0 + s)) \leq a(h_0(t_0 + s, \phi(t_0 + s))) \leq a(H_0(t_0, \phi)) < a(\delta_1) < \delta, \quad s \in [-r, 0],$$

or

$$u_0 < \delta. \tag{11}$$

From inequalities (8), (10), and (11) it follows that

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) < b(\epsilon) \quad \text{for} \quad t \geq t_0. \tag{12}$$

From inequality (12) and condition 4(i) we get for $t \geq t_0$

$$b(h(t, x(t; t_0, \phi))) \leq V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) < b(\epsilon), \tag{13}$$

or

$$h(t, x(t; t_0, \phi)) < \epsilon. \tag{14}$$

Note that if $\delta = \delta(\epsilon)$, then $\delta_1 = \delta_1(\epsilon)$ which proves the unform eventual stability of (1), (2).

(B) Let couple $(\lambda, A): \quad \lambda, A > 0, \quad a(\lambda) < b(A)$ be given and let scalar impulsive differential equation (5) be eventually practically stable. Therefore there exist $\tau(\lambda, A) > 0$ and $t_0 > \tau(\lambda, A)$ such that for any $u_0 \in \mathbb{R}_+ : \ |u_0| < a(\lambda)$ the inequality

$$|u(t; t_0, u_0)| < b(A) \qquad \text{for} \ \ t \geq t_0 \tag{15}$$

holds where $u(t; t_0, u_0)$ is a solution of (5).

Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0, \phi) < \lambda. \tag{16}$$

The rest of the proof is similar to the one of case (A).

The proof of claim (C) is similar to the one of (B) and we omit it. □

Corollary 1. *Let the following conditions be fulfilled:*

1. *Conditions 1, 2, 3 of Theorem 1 are satisfied.*
2. *There exists a function $V(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V \in \Lambda$ such that:*
   - *(i) $b(h(t, x)) \leq V(t, x) \leq a(h_0(t, x))$ for $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ where $a, b \in K$;*
   - *(ii) for any number $t \in \mathbb{R}_+ : \ t \neq \tau_k, \ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t - r, t], \mathbb{R}^n)$ such that $V(t, \psi(t)) > V(t + s, \psi(t + s))$ for $s \in [-r, 0)$ the inequality*
     $$D_{(1),(2)} V(t, \psi(t)) \leq 0$$
     holds;
   - *(iii) $V(\tau_k + 0, I_k(x)) \leq V(\tau_k, x)$ for $x \in \mathbb{R}^n, \ \ k \in Z(\mathbb{R}_+)$.*

Then the system of impulsive differential equations with "supremum" (1), (2) is uniformly eventually practically stable in terms of two measures.

Proof. The proof of Corollary 1 follows from the one of Theorem 1 for $g(t, x) \equiv 0$ and $\xi(x) \equiv x$. In this case it easy to prove that zero solution of scalar differential equation $u' = 0$ is uniformly eventually practically stable and the claim of Corollary 1 follows from (B) of Theorem 1. □

In the case when Lyapunov function does not satisfy globally the condition 4 of Theorem 1, we obtain the following sufficient conditions:

Theorem 2. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 1 are satisfied.*

2. *The functions $h_0, h \in \Gamma$ and there exists a positive constant ρ such that for any $\beta \in (0, \rho]$ inequality $h(\tau_k, x) < \beta$ implies $h(\tau_k, I_k(x)) \neq \beta$ for $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.*

3. *There exists a function $V(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V \in \Lambda$ such that*

   (i) $b(h(t, x)) \leq V(t, x) \leq a(h_0(t, x))$ for $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ where $a, b \in K$;

   (ii) for any number $t \in \mathbb{R}_+ : \; t \neq \tau_k, \; k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t - r, t], \mathbb{R}^n)$ such that $(t, \psi(t)) \in S(h, \rho)$ and $V(t, \psi(t)) > V(t + s, \psi(t + s))$ for $s \in [-r, 0)$ the inequality

$$D_{(1),(2)} V(t, \psi(t)) \leq g(t, V(t, \psi(t)))$$

   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g(t, 0) \equiv 0$;

   (iii) $V(\tau_k + 0, I_k(x)) \leq \xi_k(V(\tau_k, x))$ for $(\tau_k, x) \in S(h, \rho)$, $k \in Z(\mathbb{R}_+)$, where $\xi_k \in \mathcal{K}$.

Then:

(A) the (uniform) eventual stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);

(B) the (uniform) eventual practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);

(C) the (uniform) eventual strong practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual strong practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2).

Proof. (A) Let $\epsilon > 0$ be given such that $\epsilon < \rho$. Let the zero solution of scalar impulsive differential equation (5) be eventually stable. Then there exist $\tau = \tau(\epsilon) > 0$ such that for any $t_0 \geq \tau$ there exists a number $\delta = \delta(t_0, \epsilon) > 0$ such that inequality $|u_0| < \delta$ implies (8).

From the properties of the function $a(u)$ it follows that there exists a number $\delta_1 = \delta_1(t_0, \epsilon) >$ such that $a(\delta_1) \leq \delta$, $a(\delta_1) < b(\epsilon)$, $\delta_1 < \rho$.

Let $t_0 \geq \tau$ and a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ be such that

$$H_0(t_0, \phi) < \delta_1. \tag{17}$$

Let $x(t; t_0, \phi)$ be a solution of (1), (2), (3) with initial function ϕ.

We will prove that

$$h(t, x(t; t_0, \phi)) < \epsilon \tag{18}$$

holds for $t \geq t_0$.

From condition 3(i) of Theorem 2 we obtain that

$$b\big(h(s, \phi(s))\big) \leq a\big(h_0(s, \phi(s))\big) \leq a(H_0(t_0, \phi)) < a(\delta_1) < b(\epsilon)$$

for $s \in [t_0 - r, t_0]$, i.e. inequality (18) holds on $[t_0 - r, t_0]$.

Assume (18) does not hold for $t > t_0$. Consider the following three case:

Case 1A. Let there exist a point $t^* > t_0$, $t^* \neq \tau_k$, $k \in Z((t_0, \infty))$ such that

$$h(t^*, x(t^*; t_0, \phi)) = \epsilon \quad \text{and} \quad h(t, x(t; t_0, \phi)) < \epsilon \ \text{ for } t \in [t_0 - r, t^*). \tag{19}$$

Let $u_0 = \sup_{s \in [-r,0]} V(t_0 + s, \phi(t_0 + s))$. From Lemma 1 for the function $V(t, x)$ defined on the set $\{(t, x) \in [t_0, t^*] \times \mathbb{R}^n : \ h(t, x) \leq \epsilon\}$ it follows

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) \quad \text{for} \ \ t \in [t_0, t^*]. \tag{20}$$

From condition 3(i) of Theorem 2 we obtain

$$V(t_0 + s, \phi(t_0 + s)) \leq a(h_0(t_0 + s, \phi(t_0 + s))) \leq a(H_0(t_0, \phi)) < a(\delta_1) \leq \delta, \quad s \in [-r, 0],$$

or

$$u_0 < \delta. \tag{21}$$

From inequalities (8), (20), and (21) it follows that

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) < b(\epsilon) \quad \text{for} \ \ t \in [t_0, t^*]. \tag{22}$$

From inequality (22) and condition 3(i) of Theorem 2 we get

$$b(\epsilon) = b(h(t^*, x(t^*; t_0, \phi))) \leq V(t^*, x(t^*; t_0, \phi)) \leq u^*(t^*; t_0, u_0) < b(\epsilon). \tag{23}$$

The obtained contradiction proves the validity of (18) for $t > t_0$.

Case 2A. Let there exist a number $k \in Z((t_0, \infty))$ such that $h(t, x(t; t_0, \phi)) < \epsilon$ for $t \in [t_0 - r, \tau_k)$ and $h(\tau_k, x(\tau_k; t_0, \phi)) = \epsilon$. Then as in Case 1A for $t^* = \tau_k$ we obtain a contradiction.

Case 3A. Let there exist a natural number $k \in Z((t_0, \infty))$ such that

$$h(t, x(t; t_0, \phi)) < \epsilon, \ t \in [t_0 - r, \tau_k] \quad h\Big(\tau_k, I_k\big(x(\tau_k; t_0, \phi)\big)\Big) \geq \epsilon.$$

Since $\epsilon < \rho$ from condition 2 of Theorem 2 it follows that $h\Big(\tau_k, I_k\big(x(\tau_k; t_0, \phi)\big)\Big) > \epsilon$.

Then as in Case 1A we prove the validity of inequality (22) for $t \in [t_0, \tau_k]$. Applying conditions 3(i) and 3(iii) of Theorem 2 we get

$$\begin{aligned} b(\epsilon) < b\Big(h\Big(\tau_k, I_k\big(x(\tau_k;t_0,\phi)\big)\Big)\Big) &= b\Big(h\Big(\tau_k+0, I_k\big(x(\tau_k;t_0,\phi)\big)\Big)\Big) \\ \le V\Big(\tau_k+0, I_k\big(x(\tau_k;t_0,\phi)\big)\Big) &\le \xi_k\big(V(\tau_k, x(\tau_k;t_0,\phi))\big) \\ \le \xi_k\big(u^*(\tau_k;t_0,u_0)\big) &= u^*(\tau_k+0;t_0,u_0) \le b(\epsilon). \end{aligned} \tag{24}$$

The obtained contradictions prove the validity of (18) for $t > t_0$.

(B) Let couple λ, A be fixed such that $\lambda, A > 0,\ a(\lambda) < b(A)$.

Let the zero solution of scalar impulsive differential equation (5) be eventually practical stable. Therefore for the couple $(a(\lambda), b(A))$ there exists $\tau(\lambda, A) > 0$ such that for some $t_0 \ge \tau(\lambda, A)$ inequality $|u_0| < a(\lambda)$ implies

$$|u(t;t_0,u_0)| < b(A) \qquad \text{for } \ t \ge t_0, \tag{25}$$

where $u(t;t_0,u_0)$ is a solution of (5), (6).

Choose a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0,\phi) < \lambda \tag{26}$$

and let $x(t;t_0,\phi)$ be a solution of (1), (2) with initial condition (3). Then similarly to the proof of claim (A) we obtain the inequality $h(t, x(t;t_0,\phi)) < A$.

The proof of (C) is similar to the proof of (B) and we omit it. □

In the case when the measure h_0 is uniformly finer than h we obtain the following sufficient conditions:

Theorem 3. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 1 are satisfied.*
2. *The functions $h_0, h \in \Gamma$, h_0 is uniformly finer than h and there exist a positive constant ρ such that for any $\beta \in (0, \rho]$ inequality $h(\tau_k, x) < \beta$ implies $h(\tau_k, I_k(x)) < \beta$ for $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.*
3. *There exists a function $V(t,x) : \bar{S}(h,\rho) \to \mathbb{R}_+$ with $V \in \Lambda$ such that*

   (i) *$b(h(t,x)) \le V(t,x) \le a(h_0(t,x))$ for $(t,x) \in \bar{S}(h,\rho)$ where $a, b \in K$;*

   (ii) *for any number $t \in \mathbb{R}_+ :\ t \ne \tau_k,\ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t-r,t], \mathbb{R}^n)$ such that $(t, \psi(t)) \in S(h,\rho)$ and $V(t,\psi(t)) > V(t+s, \psi(t+s))$ for $s \in [-r, 0)$ the inequality*

   $$D_{(1),(2)}V(t,\psi(t)) \le g(t, V(t,\psi(t)))$$

   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g(t,0) \equiv 0$;

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(iii) $V(\tau_k+0, I_k(x)) \leq \xi_k(V(\tau_k, x))$ *for* $(\tau_k, x) \in S(h,\rho)$, $k \in Z(\mathbb{R}_+)$, *where* $\xi_k \in \mathcal{K}$.

Then:

(A) *the (uniform) eventual stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);*

(B) *the (uniform) eventual practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);*

(C) *the (uniform) eventual strong practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual strong practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2).*

Proof. (A) Since h_0 is uniformly finer than h, there exists a constant ρ_0 and a function $\psi \in K$ such that inequality $h_0(t,x) < \rho_0$ implies

$$h(t,x) < \psi(h_0(t,x)). \tag{27}$$

Without loss of generality we could assume that $\rho_0 < \rho$, $\psi(\rho_0) \leq \rho$.

Let $\epsilon > 0$ be given such that $\epsilon < \rho_0$.

Let zero solution of scalar impulsive differential equation (5) be eventually stable. Therefore there exists $\tau = \tau(\epsilon) > 0$ and a point $t_0 \in \mathbb{R}_+$ for which there exists $\delta = \delta(t_0, \epsilon) > 0$ such that the inequality $|u_0| < \delta$ implies (8) for $t \geq t_0$.

From the properties of the function $a(u)$ it follows there exists a number $\delta_1 = \delta_1(t_0,\epsilon) >$ such that $a(\delta_1) \leq \delta$, $a(\delta_1) < b(\epsilon)$, $\delta_1 < \rho_0$.

Let $t_0 \geq \tau$ and a function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ be such that inequality (17) holds.

We will prove that inequality

$$h(t, x(t; t_0, \phi)) < \epsilon \tag{28}$$

holds for $t \geq t_0$.

From inequality (17) it follows that $h_0(t,\phi(t)) < \delta_1 \leq \rho_0$ for $t \in [t_0 - r, t_0]$ and according to (27) we get $h(t,\phi(t)) < \psi(h_0(t,\phi(t))) < \psi(\rho_0) \leq \rho$, i.e.

$$(t, \phi(t)) \in \bar{S}(h,\rho) \quad \text{for} \quad t \in [t_0 - r, t_0].$$

Then from condition 3(i) of Theorem 3 we obtain $b\big(h(s,\phi(s))\big) \leq a\big(h_0(s,\phi(s))\big) \leq a(H_0(t_0,\phi)) < a(\delta_1) < b(\epsilon)$ for $s \in [t_0 - r, t_0]$, i.e. inequality (28) holds on $[t_0 - r, t_0]$.

Then the rest of the proof is similar to the one of Theorem 2, where, according to condition 2 of Theorem 3, Case 3A is not possible.

The proofs of (B) and (C) are similar to the proof of (A) and we omit them. □

Remark 4. Note that if in condition 2 for any $\beta \in (0, \rho]$ inequality $h(\tau_k, x) < \beta$ implies $h(\tau_k, I_k(x)) \leq \beta$ for $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$, than the claims of Theorem 3 remain if functions $\xi \in \mathcal{K}$, $\xi(x) > x$, $k \in Z(\mathbb{R}_+)$.

We will study eventual stability in the case when the Lyapunov function does not satisfy condition 3(i) of Theorem 2 and 3(i) of Theorem 3. Then we will use a perturbing Lyapunov function.

In this case we will use scalar comparison impulsive differential equation (5) and the following scalar impulsive differential equation

$$\begin{aligned} &v' = p(t, v), \quad t \geq t_0, \quad t \neq \tau_k, \\ &v(\tau_k + 0) = \eta_k(v(\tau_k)), \quad k = 1, 2, \dots, \end{aligned} \tag{29}$$

with initial condition

$$v(t_0) = v_0, \tag{30}$$

where $v, v_0 \in \mathbb{R}$, $p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\eta_k \in \mathbb{R} \to \mathbb{R}$.

In our further investigations we will assume that for any initial point $(t_0, v_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of (29), (30) exists on $[t_0, \infty)$, $t_0 \geq 0$.

Theorem 4. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 2 of Theorem 1 and condition 2 of Theorem 2 are satisfied.*
2. *There exists a function $V_1(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V_1 \in \Lambda$ which is strongly h_0-decrescent, and*

   (i) *for any number $t \in \mathbb{R}_+ :\ t \neq \tau_k,\ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t - r, t], \mathbb{R}^n)$ such that $V_1(t, \psi(t)) \geq V_1(t + s, \psi(t + s))$ for $s \in [-r, 0)$ and $(t, \psi(t)) \in S(h, \rho)$ the inequality*

   $$D_{(1),(2)} V_1(t, \psi(t)) \leq g(t, V_1(t, \psi(t)))$$

   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g(t, 0) \equiv 0$.

   (ii) *$V_1(\tau_k + 0, I_k(x)) \leq \xi_k(V_1(\tau_k, x)), \quad (\tau_k, x) \in S(h, \rho),\ k \in Z(\mathbb{R}_+)$ where functions $\xi_k \in \mathcal{K}$.*
3. *For any number $\mu > 0$ there exists a function $V_2^{(\mu)}(t, x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V_2^{(\mu)} \in \Lambda$ such that:*

   (iii) *$b(h(t, x)) \leq V_2^{(\mu)}(t, x) \leq a(h_0(t, x))$ for $(t, x) \in [-r, \infty) \times \mathbb{R}^n$, where $a, b \in K$;*

(iv) *for any* $t \in \mathbb{R}_+ : \ t \neq \tau_k, \ k \in Z(\mathbb{R}_+)$ *and any function* $\psi \in PC([t-r,t],\mathbb{R}^n)$ *such that* $(t,\psi(t))) \in S(h,\rho) \bigcap S^C(h_0,\mu)$ *and* $V(t,\psi(t)) > V(t+s,\psi(t+s))$ *for* $s \in [-r,0)$ *the inequality*

$$D_{(1),(2)}V_1(t,\psi(t)) + D_{(1),(2)}V_2^{(\mu)}(t,\psi(t)) \\ \leq p\Big(t, V_1(t,\psi(t)) + V_2^{(\eta)}(t,\psi(t)))\Big)$$

holds, where $p \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *and* $p(t,0) \equiv 0$;

(v) $V_1(\tau_k+0, I_k(x)) + V_2^{(\mu)}(\tau_k+0, I_k(x)) \leq \eta_k\Big(V_1(\tau_k,x) + V_2^{(\mu)}(\tau_k,x)\Big)$ *for* $(\tau_k,x) \in S(h,\rho)\bigcap S^C(h_0,\mu), \ k \in Z(\mathbb{R}_+)$ *where functions* $\eta_k \in \mathcal{K}$.

Then:

(A) *the uniform eventual stability of the zero solution of scalar impulsive differential equations (5) and (29) implies uniform eventual stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);*

(B) *the uniform eventual practical stability of the zero solution of scalar impulsive differential equations (5) and (29) implies uniform eventual practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2);*

(C) *the uniform eventual strong practical stability of the zero solution of scalar impulsive differential equations (5) and (29) implies uniform eventual strong practical stability in terms of two measures of system of impulsive differential equations with "supremum" (1), (2).*

Proof. (A) Since function $V_1(t,x)$ is strongly-h_0-decrescent, there exist a constant $\rho_1 > 0$ and a function $\psi_1 \in K$ such that $h_0(t,x) < \rho_1$ implies

$$V_1(t,x) \leq \psi_1(h_0(t,x)), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{31}$$

Since h_0 is uniformly finer than h, there exist a constant ρ_0 and a function $\psi_2 \in K$ such that inequality $h_0(t,x) < \rho_0$ implies inequality

$$h(t,x) \leq \psi_2(h_0(t,x)) \quad \text{for} \ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{32}$$

We will assume that $\rho_0 < \rho_1 < \rho$ and $\psi_2(\rho_0) < \rho_1$.

Let ϵ be fixed such that $0 < \epsilon < \rho_0$. Choose a positive number δ_1 such that $\psi_2(\delta_1) < \epsilon$.

Since zero solution of scalar impulsive differential equation (29) is uniformly eventually stable, there exist $T_1 = T_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for all $t_0 \geq T_1$ inequality $|v_0| < \delta_2$ implies

$$|v(t; t_0, v_0)| < b(\epsilon), \quad t \geq t_0, \tag{33}$$

where $v(t; t_0, v_0)$ is the maximal solution of (29), (30).

Since zero solution of scalar impulsive differential equation (5) is uniformly eventually stable, there exist $T_2 = T_2(\epsilon) > 0$ and $\delta_3 = \delta_3(\epsilon) > 0$ such that for any $t_0 \geq T_2$ inequality $u_0 : \quad |u_0| < \delta_3$ implies

$$u(t; t_0, u_0) < \frac{\delta_2}{2} \quad \text{for } t \geq t_0, \tag{34}$$

where $u(t; t_0, u_0)$ is the maximal solution of (5), (6).

Let $\delta_4 > 0$ be such that $\psi_1(\delta_4) < \delta_3$, $\psi_1(\delta_4) < \frac{\delta_2}{2}$, and $a(\delta_4) < \frac{\delta_2}{2}$.

Now let $T(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$ and $\delta = \min_{1 \leq i \leq 4} \delta_i$. For any $t_0 > T(\epsilon)$ we consider the function $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n$ such that

$$H_0(t_0, \varphi) < \delta. \tag{35}$$

We will prove that if inequality (35) is satisfied then

$$h(t, x(t; t_0, \phi)) < \epsilon, \qquad t \geq t_0 - r, \tag{36}$$

where $x(t; t_0, \phi)$ is a solution of initial value problem (1)-(3).

From inequalities (32), (35) it follows that $h(t_0 + s, \phi(t_0 + s)) \leq \psi_2(h_0(t_0 + s, \phi(t_0 + s)) < \psi_2(\delta) \leq \psi_2(\delta_1) < \epsilon$ for $s \in [-r, 0]$, i.e. inequality (36) holds on $[t_0 - r, t_0]$.

Assume inequality (36) is not true for $t > t_0$.

Case 1A. Let exists a point $t^* \neq \tau_k, \quad k \in Z((t_0, \infty))$ such that

$$h(t^*, x(t^*; t_0, \phi)) = \epsilon, \qquad h(t, x(t; t_0, \phi)) < \epsilon, \quad t \in [t_0 - r, t^*). \tag{37}$$

Define $x(s) = x(s; t_0, \phi)$. Note $(t, x(t)) \in S(h, \epsilon)$ on $[t_0, t^*)$, or $(t, x(t)) \in S(h, \rho)$ on $[t_0, t^*]$.

If we assume that $h_0(t^*, x(t^*)) \leq \delta$ then $h(t^*, x(t^*)) \leq \psi_2(h_0(t^*, x(t^*))) \leq \psi_2(\delta) \leq \psi_2(\delta_1) < \epsilon$ that contradicts (37). Therefore

$$h_0(t^*, x(t^*)) > \delta, \quad h_0(t_0, \phi(t_0))) < \delta. \tag{38}$$

There exists a point $t_0^* \in (t_0, t^*)$ such that

$$(t, x(t)) \in S(h, \rho) \bigcap S^c(h_0, \delta), \quad t \in [t_0^*, t^*]. \tag{39}$$

Let $r_1(t;t_0,u_0)$ be the maximal solution of (5), (6), where $u_0 = \sup_{s\in[-r,0]} V_1(t_0 + s, \varphi(t_0+s))$. Then $u_0 < \psi_1(\delta) \le \psi_1(\delta_4) < \delta_3$. From inequalities (34) it follows that

$$|r_1(t;t_0,u_0)| < \frac{\delta_2}{2} \quad \text{for } t \ge t_0. \tag{40}$$

From Lemma 1 for the function $V_1(t,x)$ defined on the set $\{(t,x) \in [t_0,t^*] \times \mathbb{R}^n : h(t,x) \le \epsilon\}$ we obtain

$$V_1(s,x(s)) \le r_1(s;t_0,u_0), \quad s \in [t_0,t^*]. \tag{41}$$

We will prove that

$$V_1(t_0^* + s, x(t_0^* + s)) < \frac{\delta_2}{2} \quad \text{for} \quad s \in [-r,0]. \tag{42}$$

Indeed, if $t_0^* - r < t_0$ then from (31) it follows that $V_1(s,x(s)) \le \psi_1(h_0(s,\phi(s)) \le \psi_1(\delta) \le \psi_1(\delta_4) < \frac{\delta_2}{2}$ for $s \in [t_0^* - r, t_0]$ and from (40) and (41) we obtain that $V_1(s,x(s)) < \frac{\delta_2}{2}$ for $s \in [t_0, t_0^*]$, i.e. inequality (42) holds. If $t_0^* - r \ge t_0$ then from (40) and (41) it follows (42).

Consider the function $V_2^{(\delta)}(t,x)$, defined in condition 3 of Theorem 4, and let

$$m(t,x) = V_1(t,x) + V_2^{(\delta)}(t,x), \qquad t \ge t_0 - r. \tag{43}$$

From inequality (35) and condition 3(iii) of Theorem 4 it follows that for $s \in [-r,0]$

$$V_2^{(\delta)}(t_0^* + s, x(t_0^* + s)) \le a(h_0(t_0^* + s, x(t_0^* + s))) \le a(\delta_4) < \frac{\delta_2}{2}. \tag{44}$$

From inequalities (42) and (44) we obtain

$$m(t_0^* + s, x(t_0^* + s)) < \delta_2 \quad \text{for } s \in [-r,0]. \tag{45}$$

From Lemma 1 for the function $m(t,x)$ defined on the set $\{(t,x) \in [t_0^*,t^*] \times \mathbb{R}^n : h(t,x) \le \epsilon,\ h_0(t,x) \ge \delta\}$ we get

$$m(t, x(t;t_0,\phi)) \le r^*(t;t_0^*,v_0^*), \quad t \in [t_0^*, t^*], \tag{46}$$

where $r^*(t;t_0^*,v_0^*)$ is the maximal solution of (29), (30) with $v_0^* = \sup_{s\in[-r,0]} m(t_0^* + s, x(t_0^* + s; t_0, \phi)$.

From inequality (45) it follows that $|v_0^*| < \delta_2$ and therefore according to inequality (33)

$$r^*(t;t_0^*,v_0^*) < b(\epsilon), \quad t \ge t_0^*. \tag{47}$$

From inequalities (46), (47), the choice of the point t^*, and condition 3(iii) of Theorem 4 we obtain

$$b(\epsilon) \quad > \quad r^*(t^*;t_0^*,w_0^*) \ge m(t^*, x(t^*;t_0,\phi))$$

$$\geq \quad V_2^{(\delta)}(t^*, x(t^*;t_0,\phi)) \geq b(h(t^*, x(t^*;t_0,\phi))) = b(\epsilon).$$

The obtained contradiction proves the validity of inequality (36) for $t \geq t_0$.

Case 2A. Let there exists a number $k \in Z((t_0,\infty))$ such that $h(t, x(t;t_0,\phi)) < \epsilon$ for $t \in [t_0 - r, \tau_k)$ and $h(\tau_k, x(\tau_k;t_0,\phi)) = \epsilon$. Then as in Case 1A for $t^* = \tau_k$ we obtain a contradiction.

Case 3A. Let there exists a natural number $k \in Z((t_0,\infty))$ such that

$$h(t, x(t;t_0,\phi)) < \epsilon,\ t \in [t_0 - r, \tau_k] \quad h\Big(\tau_k, I_k\big(x(\tau_k;t_0,\phi)\big)\Big) \geq \epsilon.$$

Since $\epsilon < \rho$ from condition 2 of Theorem 2 it follows that $h\Big(\tau_k, I_k\big(x(\tau_k;t_0,\phi)\big)\Big) > \epsilon$. The rest of the proof is similar to the proof of Case 3A of Theorem 2.

The obtained contradictions prove the validity of the inequality (36).

(B) As in the case (A) there exist constants $\rho_0, \rho_1 > 0$, and functions $\psi_1, \psi_2 \in K$ such that (31) and (32) hold.

Choose a function $\psi_3 \in K$ such that $\psi_3(s) > \max\big(\psi_1(s),\ a(s)\big)$.

Let the couple (λ, A) be fixed such that $\lambda, A \in (0, \rho_0)$, $\psi_2(\lambda) < A$, and $2\psi_3(\lambda) < b(A)$.

Since zero solution of scalar impulsive differential equation (5) is uniformly eventually practically stable, for the couple $(\psi_1(\lambda), \psi_3(\lambda))$ there exists $T_1(\lambda) > 0$ such that for $t_0 \geq T_1(\lambda)$ the inequality $|u_0| < \psi_1(\lambda)$ implies

$$u(t;t_0,u_0) < \psi_3(\lambda) \qquad \text{for } t \geq t_0, \tag{48}$$

where $u(t;t_0,u_0)$ is the maximal solution of (5), (6).

Since zero solution of scalar impulsive differential equation (29) is uniformly eventually practically stable, for the couple $(2\psi_3(\lambda), b(A))$ there exists $T_2(\lambda, A) > 0$ such that for all $t_0 \geq T_2(\lambda, A)$ inequality $|v_0| < 2\psi_3(\lambda)$ implies

$$|v(t;t_0,v_0)| < b(A), \quad t \geq t_0, \tag{49}$$

where $v(t;t_0,v_0)$ is the maximal solution of (29), (30).

Now let $T(\lambda, A) = \max(T_1(\lambda), T_2(\lambda, A))$. For any $t_0 > T(\lambda, A)$ we consider the function $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n$ such that

$$H_0(t_0, \varphi) < \lambda. \tag{50}$$

We will prove that if inequality (50) is satisfied then

$$h(t, x(t;t_0,\phi)) < A, \qquad t \geq t_0 - r, \tag{51}$$

where $x(t;t_0,\phi)$ is a solution of initial value problem (1)-(3).

From inequalities (32), (50) it follows that $h(t_0 + s, \phi(t_0 + s)) \leq \psi_2(h_0(t_0 + s, \phi(t_0 + s)) < \psi_2(\lambda) < A$ for $s \in [-r, 0]$, i.e. inequality (51) holds on $[t_0 - r, t_0]$.

Assume inequality (51) is not true for $t > t_0$.

Case 1B. Let there exists a point $t^* \neq \tau_k, \quad k \in Z((t_0, \infty))$ such that

$$h(t^*, x(t^*; t_0, \phi)) = A, \qquad h(t, x(t; t_0, \phi)) < A, \quad t \in [t_0 - r, t^*). \tag{52}$$

Define $x(s) = x(s; t_0, \phi)$. Note $(t, x(t)) \in S(h, A) \subset S(h, \rho)$ on $[t_0, t^*)$.

If we assume that $h_0(t^*, x(t^*)) \leq \lambda$ then $h(t^*, x(t^*)) \leq \psi_2(h_0(t^*, x(t^*))) \leq \psi_2(\lambda) < A$ that contradicts (52). Therefore

$$h_0(t^*, x(t^*)) > \lambda, \quad h_0(t_0, \phi(t_0))) < \lambda. \tag{53}$$

There exists a point $t_0^* \in (t_0, t^*)$ such that

$$(t, x(t)) \in S(h, \rho) \bigcap S^c(h_0, \lambda), \quad t \in [t_0^*, t^*]. \tag{54}$$

Let $r_1(t; t_0, u_0)$ be the maximal solution of (5), (6), where $u_0 = \sup_{s \in [-r,0]} V_1(t_0 + s, \varphi(t_0 + s))$. Then $u_0 < \psi_1(\lambda)$. From inequalities (48) follows that

$$|r_1(t; t_0, u_0)| < \psi_3(\lambda) \text{ for } t \geq t_0. \tag{55}$$

From Lemma 1 for the function $V_1(t, x)$ defined on the set $\{(t, x) \in [t_0, t^*] \times \mathbb{R}^n : h(t, x) \leq A\}$ we obtain

$$V_1(s, x(s)) \leq r_1(s; t_0, u_0), \quad s \in [t_0, t^*]. \tag{56}$$

We will prove that

$$V_1(t_0^* + s, x(t_0^* + s)) < \psi_3(\lambda) \quad \text{for} \quad s \in [-r, 0]. \tag{57}$$

Indeed, if $t_0^* - r < t_0$ then from (31) it follows that $V_1(s, x(s)) \leq \psi_1(h_0(s, \phi(s)) \leq \psi_1(\lambda) < \psi_3(\lambda)$ for $s \in [t_0^* - r, t_0]$ and from (55) and (56) we obtain that $V_1(s, x(s)) < \psi_3(\lambda)$ for $s \in [t_0, t_0^*]$, i.e. inequality (57) holds. If $t_0^* - r \geq t_0$ then from (55) and (56) it follows (57).

Consider the function $V_2^{(\lambda)}(t, x)$, defined in condition 3 of Theorem 4, and let

$$m(t, x) = V_1(t, x) + V_2^{(\lambda)}(t, x), \qquad t \geq t_0 - r. \tag{58}$$

From inequality (50) and condition 3(*iii*) of Theorem 4 follows that for $s \in [-r, 0]$

$$V_2^{(\lambda)}(t_0^* + s, x(t_0^* + s)) \leq a(h_0(t_0^* + s, x(t_0^* + s))) \leq a(\lambda) < \psi_3(\lambda). \tag{59}$$

From inequalities (57) and (59) we obtain

$$m(t_0^* + s, x(t_0^* + s)) < 2\psi_3(\lambda) \text{ for } s \in [-r, 0]. \tag{60}$$

From Lemma 1 for the function $m(t,x)$ defined on the set $\{(t,x) \in [t_0^*, t^*] \times \mathbb{R}^n : h(t,x) \leq A,\ h_0(t,x) \geq \lambda\}$ we get

$$m(t, x(t; t_0, \phi)) \leq r^*(t; t_0^*, v_0^*), \quad t \in [t_0^*, t^*], \tag{61}$$

where $r^*(t; t_0^*, v_0^*)$ is the maximal solution of (29), (30) for $v_0^* = \sup_{s\in[-r,0]} m(t_0^* + s, x(t_0^* + s; t_0, \phi)$.

From inequality (60) follows that $|v_0^*| < 2\psi_3(\lambda)$ and therefore according to inequality (49)

$$r^*(t; t_0^*, v_0^*) < b(A), \quad t \geq t_0^*. \tag{62}$$

From inequalities (61), (62), the choice of the point t^*, and condition $3(iii)$ of Theorem 4 we obtain

$$\begin{aligned} b(A) &> r^*(t^*; t_0^*, w_0^*) \geq m(t^*, x(t^*; t_0, \phi)) \\ &\geq V_2^{(\lambda)}(t^*, x(t^*; t_0, \phi)) \geq b(h(t^*, x(t^*; t_0, \phi))) = b(A). \end{aligned}$$

The obtained contradiction proves the validity of inequality (51) for $t \geq t_0$.

Case 2B. Let there exists a number $k \in Z((t_0, \infty))$ such that $h(t, x(t; t_0, \phi)) < A$ for $t \in [t_0 - r, \tau_k)$ and $h(\tau_k, x(\tau_k; t_0, \phi)) = A$. Then as in Case 1B for $t^* = \tau_k$ we obtain a contradiction.

Case 3. Let there exists a natural number $k \in Z((t_0, \infty))$ such that

$$h(t, x(t; t_0, \phi)) < A,\ t \in [t_0 - r, \tau_k] \quad h\Big(\tau_k, I_k\big(x(\tau_k; t_0, \phi)\big)\Big) \geq A.$$

Since $A < \rho$ from condition 2 of Theorem 2 it follows that $h\Big(\tau_k, I_k\big(x(\tau_k; t_0, \phi)\big)\Big) > A$. The rest of the proof is similar to the proof of Case 3A of Theorem 2.

The obtained contradictions prove the validity of the inequality (51).

The proof of (C) is similar to the proof of (B) and we omit it. □

3.2. Eventual Stability with Regular Norm

In the special case when both measures coincide any norm $||.||$ in $\mathbb{R}^n$, i.e. $h_0(t,x) = h(t,x) \equiv ||x||$, we obtain sufficient conditions for the studied types of eventual stability of impulsive differential equations with "supremum" (1), (2). We will state only the results since they are partial cases of the proved above results:

Theorem 5. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 1 are satisfied.*
2. *There exists a function $V(t,x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V \in \Lambda$ such that conditions 4(ii) and 4(iii) of Theorem 1 are satisfied and*

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(i) $b(||x||) \le V(t,x) \le a(||x||)$ *for* $(t,x) \in [-r,\infty) \times \mathbb{R}^n$ *where* $a, b \in K$;

Then:

(A) *the (uniform) eventual stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual stability of system of impulsive differential equations with "supremum" (1), (2);*

(B) *the (uniform) eventual practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual practical stability of system of impulsive differential equations with "supremum" (1), (2);*

(C) *the (uniform) eventual strong practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual strong practical stability of system of impulsive differential equations with "supremum" (1), (2).*

Theorem 6. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 1 are satisfied and there exists a positive constant* ρ *such that for any* $\beta \in (0,\rho)$ *inequality* $||x|| < \beta$ *implies* $||I_k(x)|| < \beta$ *for* $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.
2. *There exists a function* $V(t,x) : [-r,\infty) \times \{x \in \mathbb{R}^n : \ ||x|| < \rho\} \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that:*

   (i) $b(||x||)) \le V(t,x) \le a(||x||)$ *for* $t \in \mathbb{R}_+,\ ||x|| < \rho$ *where* $a, b \in K$;

   (ii) for any number $t \in \mathbb{R}_+ :\ t \ne \tau_k,\ k \in Z(\mathbb{R}_+)$ *and any function* $\psi \in PC([t-r,t],\mathbb{R}^n)$ *such that* $||\psi(t)|| < \rho$ *and* $V(t,\psi(t)) > V(t+s,\psi(t+s))$ *for* $s \in [-r,0)$ *the inequality*

   $$D_{(1),(2)}V(t,\psi(t)) \le g(t,V(t,\psi(t)))$$

   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R},\mathbb{R}_+)$ *and* $g(t,0) \equiv 0$;

   (iii) $V(\tau_k+0,I_k(x)) \le \xi_k(V(\tau_k,x))$ *for* $||x|| < \rho,\ \ k \in Z(\mathbb{R}_+)$, *where* $\xi_k \in \mathcal{K}$.

Then:

(A) *the (uniform) eventual stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual stability of system of impulsive differential equations with "supremum" (1), (2);*

(B) *the (uniform) eventual practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual practical stability of system of impulsive differential equations with "supremum" (1), (2);*

(C) the (uniform) eventual strong practical stability of the zero solution of scalar impulsive differential equation (5) implies (uniform) eventual strong practical stability of system of impulsive differential equations with "supremum" (1), (2).

Theorem 7. *Let the following conditions be fulfilled:*

1. *Conditions 1 and 2 of Theorem 1 and there exists a positive constant ρ such that for any $\beta \in (0, \rho)$ inequality $||x|| < \beta$ implies $||I_k(x)|| \neq \beta$ for $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.*

2. *There exists a function $V_1(t,x) : [-r,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V_1 \in \Lambda$ and:*

   (i) $V(t,x) \leq c(||x||)$ for $t \in \mathbb{R}_+,\ ||x|| < \rho$, where $c \in K$;

   (ii) for any number $t \in \mathbb{R}_+ :\ t \neq \tau_k,\ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ such that $V_1(t,\psi(t)) \geq V_1(t+s,\psi(t+s))$ for $s \in [-r,0)$ and $||\psi(t)|| < \rho$ the inequality

   $$D_{(1),(2)}V_1(t,\psi(t)) \leq g(t, V_1(t,\psi(t)))$$

   holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $g(t,0) \equiv 0$.

   (ii) $V_1(\tau_k+0, I_k(x)) \leq \xi_k(V_1(\tau_k,x)),\quad ||x|| < \rho,\ k \in Z(\mathbb{R}_+)$, where functions $\xi_k \in \mathcal{K}$.

3. *For any number $\mu \in (0,\rho)$ there exists a function $V_2^{(\mu)}(t,x) : [-r,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ with $V_2^{(\mu)} \in \Lambda$ such that:*

   (iii) $b(||x||) \leq V_2^{(\mu)}(t,x) \leq a(||x||)$ for $(t,x) \in [-r,\infty) \times \mathbb{R}^n$, where $a, b \in K$;

   (iv) for any number $t \in \mathbb{R}_+ :\ t \neq \tau_k,\ k \in Z(\mathbb{R}_+)$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ such that $\mu \leq ||\psi(t))|| < \rho$ and $V(t,\psi(t)) > V(t+s,\psi(t+s))$ for $s \in [-r,0)$ the inequality

   $$D_{(1),(2)}V_1(t,\psi(t)) + D_{(1),(2)}V_2^{(\mu)}(t,\psi(t)) \leq p\Big(t, V_1(t,\psi(t)) + V_2^{(\eta)}(t,\psi(t)))\Big)$$

   holds, where $p \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $p(t,0) \equiv 0$;

   (v) $V_1(\tau_k+0, I_k(x)) + V_2^{(\mu)}(\tau_k+0, I_k(x)) \leq \eta_k\Big(V_1(\tau_k,x) + V_2^{(\mu)}(\tau_k,x)\Big)$ for $\mu \leq ||x|| < \rho,\ k \in Z(\mathbb{R}_+)$ where functions $\eta_k \in \mathcal{K}$.

Then:

(A) *the uniform eventual stability of the zero solution of scalar impulsive differential equations (5) and (29) implies uniform eventual stability of system of impulsive differential equations with "supremum" (1), (2);*

(B) *the uniform eventual practical stability of the zero solution of scalar impulsive differential equations (5) and (29) implies uniform eventual practical stability of system of impulsive differential equations with "supremum" (1), (2);*

(C) *the uniform eventual strong practical stability of the zero solution of scalar impulsive differential equations (5) and (29) implies uniform eventual strong practical stability of system of impulsive differential equations with "supremum" (1), (2).*

4. Applications

Consider the following system of impulsive differential equations with "supremum"

$$\begin{aligned}
&x'(t) = y(t)\Big(x^2(t) + y^2(t)\Big)\sin^2 t + e^{-t} \sup_{s\in[t-r,t]} x(s),\\
&y'(t) = -\frac{1}{2}x(t)\Big(x^2(t) + y^2(t)\Big)\sin^2 t + e^{-t} \sup_{s\in[t-r,t]} y(s), \qquad t \geq t_0, t \neq k,\\
&x(k+0) = ax(k), \qquad y(k+0) = by(k),
\end{aligned} \tag{63}$$

with initial conditions

$$x(t) = \phi_1(t), \quad y(t) = \phi_2(t) \quad \text{for } t \in [t_0 - r, t_0], \tag{64}$$

where $x, y \in \mathbb{R}$, $r > 0$ is enough small constant, $t_0 \geq 0$, and $a, b \in (1, 2)$, k is a natural number.

Let $h_0(t, x, y) = |x| + \sqrt{2}|y|$, $h(t, x, y) = x^2 + 2y^2$. Consider $V : \mathbb{R}^2 \to \mathbb{R}_+$, $V(x, y) = \frac{3}{2}(x^2 + 2y^2)$. It is easy to check Condition (i) of Theorem 1 for functions $a(s) = \frac{3}{2}s^2 \in K$ and $b(s) = s \in K$.

Let $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k = 1, 2, \dots$ and $\psi \in PC([t - r, t], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that

$$\psi_1^2(t) + 2\psi_2^2(t) > \psi_1^2(t + s) + 2\psi_2^2(t + s) \quad \text{for} \quad s \in [-r, 0), \tag{65}$$

or $V(\psi_1(t), \psi_2(t)) > V(\psi_1(t + s), \psi_2(t + s))$.

Let $i = 1, 2$. If there exists a point $\eta \in [t-r, t]$ such that $\sup_{s\in[t-r,t]} \psi_i(s) = \psi_i(\eta)$, then $\big(\sup_{s\in[t-r,t]} \psi_i(s)\big)^2 = (\psi_i(\eta))^2 \leq \sup_{s\in[t-r,t]} \big(\psi_1^2(s)+2\psi_2^2(s)\big) = \psi_1^2(t)+2\psi_2^2(t)$.

The above inequality could analogously be proved if $\sup_{s\in[t-r,t]} \psi_i(s) > \psi_i(\eta)$ for all $\eta \in [-r, t]$, i.e. there exists a natural number $k \in (t - r, t)$ such that $\sup_{s\in[t-r,t]} \psi_i(s) = \psi_i(k + 0)$.

Then for $i = 1, 2$ we obtain

$$\psi_i(t) \sup_{s\in[t-r,t]} \psi_i(s) \leq |\psi_i(t)| \, |\sup_{s\in[t-r,t]} \psi_i(s)| = \sqrt{(\psi_i(t))^2} \sqrt{(\sup_{s\in[t-r,t]} \psi_i(s))^2}$$
$$\leq \frac{3}{2}\Big(\psi_1^2(t) + 2\psi_2^2(t)\Big) = V(\psi_1(t), \psi_2(t)).$$

Therefore, if inequality (65) is fulfilled, we have

$$D_{(63)}V(\psi_1(t), \psi_2(t)) = 3e^{-t}\Big(\psi_1(t) \max_{s\in[t-r,t]} \psi_1(s) + 2\psi_2(t) \max_{s\in[t-r,t]} \psi_2(s)\Big)$$
$$\leq 6e^{-t}V(\psi_1(t), \psi_2(t)).$$

For any k we obtain

$$V(ax, by) = \frac{3}{2}(a^2x^2 + 2b^2y^2) \leq c^2\frac{3}{2}(x^2 + 2y^2) = c^2V(x, y),$$

where $c = \max(a, b) > 1$.

Now, consider the initial value problem for the scalar comparison impulsive differential equation

$$u' = 6e^{-t}u \quad \text{for } t \neq k, \qquad u(k + 0) = c^2u(k), \qquad u(t_0) = u_0,$$

whose solution is $u(t) = \Big(\prod_{k:\ t_0\leq t\leq k+1}(c^2-1)\Big)u_0e^{6\left(e^{-t_0}-e^{-t}\right)}$ and $|u(t)| \leq |u_0|e^{6e^{-t_0}}$ for $t \geq t_0$. For any numbers $0 < \lambda < A$, we choose a number $\tau > \max\{0, \ln 6 - \ln(\ln(\frac{A}{\lambda}))\} > 0$. Note $\tau = \tau(\lambda, A) > 0$. It is easy to check that for $t_0 > \tau$ and $|u_0| < \lambda$ the inequality $|u(t)| < A$ holds, i.e. the zero solution of scalar comparison equation is uniformly eventually practically stable. Therefore, according to Theorem 1 system of impulsive differential equations with "supremum" (63) is uniformly eventually practically stable in terms of two measures, i.e. for any positive numbers $\lambda < A$, there exists a number $\tau = \tau(\lambda, A) > 0$ such that, if $t_0 > \tau$ then the inequality $\sup_{s\in[t_0-r,t_0]}(|\phi_1(s)| + 2|\phi_2(s)|) < \lambda$ implies $x^2(t; t_0, \phi) + 2y^2(t; t_0, \phi) < A$ for $t \geq t_0$.

Acknowledgments

This research was partially supported by RS09FMI018.

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