International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 2 (2010)

**International Electronic Journal of Pure and Applied Mathematics**

**Volume 2 No. 2 2010, 13-20**

# OSCILLATIONS OF HIGHER ORDER DIFFERENTIAL INEQUALITIES WITH "MAXIMA"

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**Abstract:** The following differential inequalities with "maxima"

$$(-1)^n L_n x(t) \operatorname{sgn} x(t) \geq \sum_{j=1}^{m} p_j(t) f_j(M_t^j x)$$

and

$$(-1)^n L_n x(t) \operatorname{sgn} x(t) \geq p_0(t) \prod_{j=1}^{m} \varphi_j(M_t^j x)$$

are considered where $n \geq 1$, $m \geq 1$ are integers and

$$M_t^j x = \max_{\sigma_j(t) \leq s \leq \tau_j(t)} x(s), \quad j = 1, \dots, m.$$

Sufficient conditions are found under which all bounded solutions of these inequalities are oscillatory.

**AMS Subject Classification:** 34K15

**Key Words:** oscillation, almost oscillation, $n$-th order differential inequalities, deviating arguments depending on the unknown function

## 1. Introduction

We consider the $n$-th order differential inequalities with "maxima"

Received: May 23, 2010

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$$(-1)^n L_n x(t) \operatorname{sgn} x(t) \geq \sum_{j=1}^{m} p_j(t) f_j(M_t^j x) \tag{A}$$

and

$$(-1)^n L_n x(t) \operatorname{sgn} x(t) \geq p_0(t) \prod_{j=1}^{m} \varphi_j(M_t^j x), \tag{B}$$

where $n \geq 1$, $m \geq 1$ are integers, $t \in J = [\alpha, +\infty) \subseteq \mathbb{R}_+ = [0, +\infty)$, $M_t^j x = \max\limits_{\sigma_j(t) \leq s \leq \tau_j(t)} x(s)$, $j = 1, \ldots, m$, and

$$L_0 x(t) = x(t),\ \ L_k x(t) = r_k(t)(L_{k-1}x(t))',\ \ k = 1, \ldots, n.$$

The domain $D(L_n)$ of $L_n$ is defined to be the set of all functions $x : [t_x, +\infty) \to \mathbb{R}$ such that $L_k x(t), k = 1, \ldots, n$ exist and are continuous on an interval $[t_x, +\infty) \subseteq J$. By *proper* solution of inequality $(A)$ (or $(B)$) is meant a function $x \in D(L_n)$ that satisfies $(A)$ (or$(B)$) for all sufficiently large $t$ and $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq t_x$. We suppose that inequalities $(A)$ and $(B)$ do possess proper solutions. A proper solution of inequality $(A)$ (or $(B)$) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*.

The purpose of this article is to establish sufficient conditions under which all proper bounded solutions of inequalities $(A)$ and $(B)$ are oscillatory.

We note that the oscillatory and asymptotic behavior of the solutions of some neutral differential equations with "maxima" have been investigated by Bainov et al [1, 2].

Our main results generalize and improve the results given by Naito [8], where the case $L_n x(t) = x^{(n)}(t)$, $M_t^i x = x(g_j(t))$ is considered.

For related results we refer the reader to the papers of Gustafson [3], Koplatadze [5], Kusano and Onose [6], Markova and Simeonov [7], and Sfikas and Staikos [9, 10].

## 2. Preliminary Notes

Introduce the following conditions:

**H1.** $r_i \in C(J, (0, +\infty)), i = 1, \ldots, n$, $r_n = 1$ and

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty, \quad i = 1, \ldots, n-1.$$

**H2.** $p_j \in C(J, (0, +\infty)), j = 1, \ldots, m.$

**H3.** The functions $f_j(x)$ and $\varphi_j(x)$, $j = 1, \ldots, m$ are continuous and positive on $(-\infty, 0) \cup (0, +\infty)$ and $f_j(x) sngx$ and $\varphi_j(x) \operatorname{sgn} x$ are nondecreasing in $x$.

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**H4.** $\sigma_j, \tau_j \in C(J, \mathbb{R})$, $j = 1, \dots, m$ and

$$\lim_{t\to+\infty} \sigma_j(t) = +\infty, \ \ \sigma_j(t) \le \tau_j(t) \le t \ \text{ for } \ j = 1, \dots, m, \ \ t \in J.$$

**H5.** There exists a nondecreasing function $\tau : J \to \mathbb{R}$ having an inverse $\tau^{-1}(t)$ and such that $\tau_j(t) \le \tau(t) \le t$, $j = 1, \dots, m$, $t \in J$.

In the paper we use the following notations:

$$f(x) = \min_{1\le j\le m} f_j(x), \ \ \varphi(x) = \prod_{j=1}^{m} \varphi_j(x), \ \ I_0 \equiv 1,$$

$$I_j(t, s; a_j, \dots, a_1) = \int_s^t \frac{1}{a_j(u)} I_{j-1}(u, s; a_{j-1}, \dots, a_1) du, \ \ j = 1, 2, \dots$$

where $a_j \in C(J, (0, +\infty))$, $j = 1, 2, \dots$.

We need the following lemmas:

**Lemma 1.** *Suppose condition H1 holds and the functions $L_n x$ and $x \in D(L_n)$ are of constant sign and not identically zero for $t \ge t_* \ge \alpha$. Then there exist a $t_k \ge t_*$ and an integer $k$, $0 \le k \le n$ with $n + k$ even for $x(t)L_n x(t)$ nonnegative or $n + k$ odd for $x(t)L_n x(t)$ nonpositive and such that for every $t \ge t_k$*

$$x(t)L_i x(t) > 0, \qquad i = 0, 1, \dots, k,$$

$$(-1)^{i-k} x(t)L_i x(t) > 0, \qquad i = k, k + 1, \dots, n.$$

**Lemma 2.** *If $x \in D(L_n)$, then for $s, t \in J$ and $0 \le i < \nu \le n$*

$$L_i x(s) = \sum_{j=i}^{\nu-1} (-1)^{j-i} I_{j-i}(t, s; r_j, \dots, r_{i+1}) L_j x(t) + (-1)^{\nu-i} \int_s^t I_{\nu-i-1}(u, s; r_{\nu-1}, \dots, r_{i+1}) \frac{L_\nu x(u)}{r_\nu(u)} du.$$

Lemma 1 generalizes the well-known lemma of Kiguradze [4] and can be proved simiralry.

Lemma 2 is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

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## 3. Main Results

**Theorem 1.** *Assume that conditions H1-H5 hold and*

$$\limsup_{t\to+\infty} \int\limits_{\tau(t)}^{t} I_{n-1}(u,\tau(t);r_{n-1},\dots,r_1)\sum_{j=1}^{m} p_j(u)du > \limsup_{x\to 0}\frac{|x|}{f(x)}. \tag{1}$$

*Then, if $n \geq 2$ all proper bounded solutions of inequality $(A)$ are oscillatory, while if $n = 1$ all proper solutions of $(A)$ are oscillatory.*

*Proof.* Suppose there exists a bounded nonoscillatory solution $x(t)$ of $(A)$. Without loss of generality we suppose that $x(t)$ is eventually positive: $x(t) > 0$, $t \geq t_0 \geq \alpha$. It follows from $(A)$ and conditions H2-H4 that $(-1)^n L_n x(t) \geq 0$, $t \geq t_1 \geq t_0$. By the boundedness of $x(t)$ and Lemma 1 it follows that there exists a $t_k \geq t_1$ such that

$$(-1)^i L_i x(t) > 0,\ \ t \geq t_k,\ \ i = 1,\dots,n. \tag{2}$$

From Lemma 2 with $\nu = n$ and $i = 0$ we have

$$x(s) = \sum_{j=0}^{n-1}(-1)^j I_j(t,s;r_j,\dots,r_1)L_j x(t) + (-1)^n \int\limits_{s}^{t} I_{n-1}(u,s;r_{n-1},\dots,r_1)L_n x(u)du,\ \ t \geq s \geq t_k.$$

Using (2) and $(A)$ we get

$$x(s) \geq x(t) + \int\limits_{s}^{t} I_{n-1}(u,s;r_{n-1},\dots,r_1)\sum_{j=1}^{m} p_j(u) f_j(M_u^j x)du \tag{3}$$

for $t \geq s \geq t_k$.

By (2) $x(t)$ is decreasing for $t \geq t_k$ so that $M_t^j x \geq x(\tau_j(t)) \geq x(\tau(t))$ for $t \geq t_2 \geq t_k$, $j = 1,\dots,m$.

From (3) and the monotonicity of $f_j$, $j = 1,\dots,m$ it follows that

$$x(s) \geq x(t) + f(x(\tau(t)))\int\limits_{s}^{t} I_{n-1}(u,s;r_{n-1},\dots,r_1)\sum_{j=1}^{m} p_j(u)du$$

for $t \geq s \geq t_2$. Therefore

$$x(\tau(t)) \geq x(t) + f(x(\tau(t)))\int\limits_{\tau(t)}^{t} I_{n-1}(u,\tau(t);r_{n-1},\dots,r_1)\sum_{j=1}^{m} p_j(u)du, \tag{4}$$

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for $t \geq t_3 \geq t_2$, where $t_3$ is so large that $\tau(t) \geq t_2$ for $t \geq t_3$.

Since $x(t)$ is decreasing and positive, then $\lim_{t\to+\infty} x(t) = c \geq 0$. It follows from (4) and (1) that $c = 0$. Then

$$\frac{x(\tau(t))}{f(x(\tau(t)))} \geq \int_{\tau(t)}^{t} I_{n-1}(u, \tau(t); r_{n-1}, \ldots, r_1) \sum_{j=1}^{m} p_j(u)du, \quad t \geq t_3, \tag{5}$$

and taking the limit superior as $t \to +\infty$ of both sides of (5) we obtain a contradiction to (1). The proof in the case $n \geq 2$ is complete.

In the case $n = 1$ all solutions of $(A)$ are oscillatory because every nonoscillatory solution of $(A)$ is bounded. □

The following theorem can be proved similarly.

**Theorem 2.** *Assume that conditions H1-H5 hold and*

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} I_{n-1}(u, \tau(t); r_{n-1}, \ldots, r_1) p_0(u)du > \limsup_{x\to0} \frac{|x|}{\varphi(x)}. \tag{6}$$

*Then if $n \geq 2$ all bounded solutions of inequality $(B)$ are oscillatory, while if $n = 1$ all solutions of $(B)$ are oscillatory.*

**Remark 1.** In the particular case when $r_j \equiv 1$, $\tau_j(t) = \sigma_j(t)$, $M_t^j x = x(\tau_j(t))$, $j = 1, \ldots, m$ inequalities $(A)$, $(B)$ take the form

$$(-1)^n x^{(n)}(t) \operatorname{sgn} x(t) \geq \sum_{j=1}^{m} p_j(t) f_j(x(\tau_j(t))), \tag{$A_0$}$$

$$(-1)^n x^{(n)}(t) \operatorname{sgn} x(t) \geq p_0(t) \prod_{j=1}^{m} \varphi_j((x(\tau_j(t))). \tag{$B_0$}$$

These inequalities have been considered in Naito [8] under conditions H2-H4. For $(A_0)$ and $(B_0)$ conditions (1) and (6) reduce to the conditions

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} [u - \tau(t)]^{n-1} \sum_{j=1}^{m} p_j(u)du > (n-1)! \limsup_{x\to0} \frac{|x|}{f(x)}, \tag{$1^*$}$$

and

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} [u - \tau(t)]^{n-1} p_0(u)du > (n-1)! \limsup_{x\to0} \frac{|x|}{\varphi(x)}, \tag{$6^*$}$$

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respectively.

If we choose $\tau(t) = \max_{1\leq j\leq m} \tau_j(t)$, conditions $(1^*)$ and $(6^*)$ coincide with conditions (1) and (7) of Theorems 1 and 2 from [8].

**Theorem 3.** *Assume that conditions H1-H5 hold,*

$$\int\limits_{+0}^{+a} \frac{dx}{f(x)} < +\infty, \quad \int\limits_{-a}^{-0} \frac{dx}{f(x)} < +\infty \ \textit{for some} \ a > 0, \tag{7}$$

*and*

$$\int^{\infty} \frac{1}{r_1(s)} \Big( \int\limits_{s}^{\tau^{-1}(t)} I_{n-2}(u, s; r_{n-1}, \ldots, r_2) \sum_{j=1}^{m} p_j(u) du \Big) ds = +\infty. \tag{8}$$

*Then all bounded solutions of inequality $(A)$ are oscillatory.*

*Proof.* Suppose there exists a bounded nonoscillatory solution $x(t)$ of inequality $(A)$. Without loss of generality we suppose that $x(t)$ is eventually positive: $x(t) > 0$, $t \geq t_0 \geq \alpha$. It follows from $(A)$ and conditions H2-H4 that $(-1)^n L_n x(t) \geq 0$, $t \geq t_1 \geq t_0$. By the boundedness of $x(t)$ and Lemma 1 we conclude that (2) holds for some $t_k \geq t_1$ sufficiently large. From Lemma 2 with $\nu = n$ and $i = 1$ we have

$$L_1 x(s) = \sum_{j=1}^{n-1} (-1)^{j-1} I_{j-1}(t, s; r_j, \ldots, r_2) L_j x(t) + (-1)^{n-1} \int\limits_{s}^{t} I_{n-2}(u, s; r_{n-1}, \ldots, r_2) L_n x(u) du, \ \ t \geq s \geq t_k.$$

Then

$$-r_1(s) x'(s) = \sum_{j=1}^{n-1} (-1)^{j} I_{j-1}(t, s; r_j, \ldots, r_2) L_j x(t) + (-1)^{n} \int\limits_{s}^{t} I_{n-2}(u, s; r_{n-1}, \ldots, r_2) L_n x(u) du, \ \ t \geq s \geq t_k. \tag{9}$$

Since $x(t)$ is decreasing for $t \geq t_k$, then $M_t^j x \geq x(\tau_j(t)) \geq x(\tau(t))$ for $t \geq t_2 \geq t_k$, $j = 1, \ldots, m$. Putting $t = \tau^{-1}(s)$ in (9) and taking the monotonicity of $f_j$, $j = 1, \ldots, m$ and (2) into account we obtain

$$-x'(s) \geq \frac{f(x(s))}{r_1(s)} \int\limits_{s}^{\tau^{-1}(s)} I_{n-2}(u, s; r_{n-1}, \ldots, r_2) \sum_{j=1}^{m} p_j(u) du \tag{10}$$

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for $t \geq t_2$.

Dividing the both sides of (10) by $f(x(s))$ and then integrating from $t_2$ to $t$ we obtain

$$\int_{x(t)}^{x(t_2)} \frac{dx}{f(x)} \geq \int_{t_2}^{t} \frac{1}{r_1(s)} \Big( \int_{s}^{\tau^{-1}(s)} I_{n-2}(u, s; r_{n-1}, \ldots, r_2) \sum_{j=1}^{m} p_j(u) du \Big) ds. \tag{11}$$

It follows from (2) that $\lim_{t \to +\infty} x(t) = c \geq 0$. Therefore by (7) the left hand side of (11) remains bounded as $t \to +\infty$, which contradicts condition (8). □

The following theorem can be proved similarly.

**Theorem 4.** *Assume that conditions H1-H5 hold,*

$$\int_{+0}^{+a} \frac{dx}{\varphi(x)} < +\infty, \quad \int_{-a}^{-0} \frac{dx}{\varphi(x)} < +\infty \text{ for some } a > 0, \tag{12}$$

*and*

$$\int^{\infty} \frac{1}{r_1(s)} \Big( \int_{s}^{\tau^{-1}(s)} I_{n-2}(u, s; r_{n-1}, \ldots, r_2) p_0(u) du \Big) ds = +\infty. \tag{13}$$

*Then all bounded solutions of inequality* $(B)$ *are oscillatory.*

**Remark 2.** For inequalities $(A_0)$, $(B_0)$ conditions (8) and (13) take the form

$$\int^{\infty} \Big( \int_{s}^{\tau^{-1}(s)} (u - s)^{n-2} \sum_{j=1}^{m} p_j(u) du \Big) ds = +\infty \tag{8*}$$

and

$$\int^{\infty} \Big( \int_{s}^{\tau^{-1}(s)} (u - s)^{n-2} p_0(u) du \Big) ds = +\infty, \tag{13*}$$

respectively.

In Theorems 3 and 4 of [8] it is supposed that $\tau(t) = \max_{1 \leq j \leq m} \tau_j(t)$, and additionally, $\tau(t)$ is differentiable and $\tau'(t) \geq 0$ for $t \in J$. Under this supposition conditions (8∗) and (13∗) reduce to the conditions

$$\int^{\infty} \tau'(t) \Big( \int_{\tau(t)}^{t} (u - \tau(t))^{n-2} \sum_{j=1}^{m} p_j(u) du \Big) dt = +\infty$$

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and

$$\int^{\infty} \tau'(t)\Big(\int_{\tau(t)}^{t} (u-\tau(t))^{n-2} p_0(u)du\Big)dt = +\infty,$$

which coincide with the corresponding conditions of Theorems 3 and 4, in [8].

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