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LIPSCHITZ STABILITY IN TERMS OF TWO MEASURES FOR DIFFERENTIAL EQUATIONS WITH "MAXIMA"

S. Hristova[1 §], S. Gluhcheva[2]
[1,2]Faculty of Mathematics and Informatics
Plovdiv University
Plovdiv, 4000, BULGARIA
[1]e-mail: snehri@uni-plovdiv.bg

Abstract: This paper investigates Lipschitz stability for nonlinear differential equations with "maxima". Two different measures for the initial condition and for the solution are employed. Several sufficient conditions for Lipschitz stability, uniformly Lipschitz stability as well as uniformly eventually Lipschitz stability are obtained. Lyapunov functions have been applied. A comparison scalar ordinary differential equation has been employed.

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1. Introduction

Differential equations with "maxima" are a special type of differential equations that contain the maximum of the unknown function over a previous interval(s). Such equations adequately model real world processes whose present state significantly depends on the maximum value of the state on a past time interval. For example, in the theory of automatic control in various technical systems often the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals. This requires the use of differential equations with "maxima" in the control theory. Recently, the interest in differential equations with "maxima" has increased. Note that several theoretical results for differential equations with

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§Correspondence author

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"maxima" are obtained by D.D. Bainov et al (see [1], [2], [3], [4], [5]).

One of the main problems in the qualitative theory of differential equations is stability of the solutions. There are various types of stability. One very useful stability is Lipschitz stability. This type of stability is introduced by F. Dannan and S. Elaydi (see [6]) and it becomes the one of the important stability notions through the technique of utilizing the linear variational system around an arbitrary solution. It lies somewhere between uniform stability on one side and the notions of asymptotic stability in variation and the uniform stability in variation on the other side. Lipschitz stability for different types of differential equations is studied by many authors (see, for example, [7], [11], [12], [14], [15], [16], [17]). At the same time, various types of stability for differential equations with "maxima" are studied in [8], [9], [10], [18], [19].

In the present paper the Lipschitz stability of differential equations with "maxima" is studied in context of Lyapunov functions and Razumikhin method. Several sufficient conditions for Lipschitz stability, uniform Lipschitz stability and eventually Lipschitz stability are obtained. Two different measures for the initial functions and for the solutions are applied. It helps the area of applications of the obtained results to real world problems to be wider. Note two different measures are applied by V. Lakshmikantham et al (see [13]) to study stability of differential equations. In this paper an example is given to illustrate the concept of the considered types of stability and the application of the obtained results.

2. Preliminary Notes and Definitions

Let $r > 0$ be a given number. Consider the nonlinear differential equations with "maxima"

$$x' = f(t, x(t), \max_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \geq t_0 \tag{1}$$

with initial condition

$$x(t) = \varphi(t) \quad \text{for} \quad t \in [t_0 - r, t_0], \tag{2}$$

where $x \in \mathbb{R}^n$, $f: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi: [t_0 - r, t_0] \to \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$.

Note that for $x: [t-r, t] \to \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$ we denote

$$\max_{s\in[t-r,t]} x(s) = \Big(\max_{s\in[t-r,t]} x_1(s), \max_{s\in[t-r,t]} x_2(s), \dots, \max_{s\in[t-r,t]} x_n(s)\Big).$$

We will introduce the class Λ of Lyapunov functions.

Definition 1. We will say that the function $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $\Delta \subset [-r, \infty)$, $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, belongs to class Λ if:

1. $V(t,x)$ is a continuous function in $\Delta \times \Omega$ and $V(t,0) \equiv 0$ for $t \in \Delta$;
2. Function $V(t,x)$ is Lipschitz with respect to its second argument.

Let $V(t,x) \in \Lambda$. For any $t \in \Delta$ and any function $\psi \in C([t-r,t],\mathbb{R}^n)$ we will define derivative of the function $V(t,x)$ along a trajectory of the system (1) as follows:

$$D_{(1)}V(t,\psi) = \limsup_{\epsilon \to 0} \frac{1}{\epsilon}\Big[V\Big(t+\epsilon, \psi(t) + \epsilon f(t,\psi(t), \max_{s\in[-r,0]} \psi(t+s))\Big) - V(t,\psi(t))\Big].$$

Note that the derivative $D_{(1)}V(t,\psi)$, defined above is a functional.

We will define the set of measures:

$$\Gamma = \{h \in C([-r,\infty)\times\mathbb{R}^n, \mathbb{R}_+) : \min_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for each } t \in [-r,\infty)\}.$$

Let $h_0 \in \Gamma$, $t_0 \in \mathbb{R}_+$, $\psi \in C([t_0-r,t_0],\mathbb{R}^n)$. We will use the following notation

$$H_0(t_0,\psi) = \max_{s\in[t_0-r,t_0]} h_0(s,\psi(s)). \tag{3}$$

Let $\rho > 0$ be a fixed number and $h \in \Gamma$. Define the set:

$$S(h,\rho) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t,x) < \rho\}.$$

In our further investigations we will use the initial value problem for the comparison scalar ordinary differential equation

$$u' = g(t,u) \quad \text{for} \quad t \geq t_0, \tag{4}$$

$$u(t_0) = u_0, \tag{5}$$

where $u, u_0 \in \mathbb{R}$, $g \in C(\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R})$, $t_0 \in \mathbb{R}_+$.

We will define Lipschitz stability in terms of two measures.

Definition 2. Let both measures $h_0, h \in \Gamma$. The system of differential equations with "maxima" (1) is said to be:

1) *Uniformly eventually stable in terms of two measures*, if for $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $\tau(\epsilon) > 0$ such that for any initial point $t_0 \geq \tau(\epsilon)$ and any initial function $\varphi \in C([t_0-r,t_0],\mathbb{R}^n)$ such that $H_0(t_0,\varphi) \leq \delta$ the inequality

$$h(t, x(t;t_0,\varphi)) < \epsilon, \qquad t \geq t_0,$$

holds.

2) *Lipschitz stable in terms of two measures*, if for any initial point $t_0 \in R_+$ there exist $M > 0$ and $\delta = \delta(t_0) > 0$ such that for any initial function $\varphi \in C([t_0-r,t_0],\mathbb{R}^n)$ such that $H_0(t_0,\varphi) \leq \delta$ the inequality

$$h(t, x(t;t_0,\varphi)) < MH_0(t_0,\varphi), \qquad t \geq t_0,$$

holds.

3) *Uniformly Lipschitz stable in terms of two measures*, if there exist constants $M > 0$ and $\delta > 0$ such that for any initial point $t_0 \in R_+$ and for any initial function $\varphi \in C([t_0 - r, t_0], \mathbb{R}^n)$ such that $H_0(t_0, \varphi) \leq \delta$ the inequality

$$h(t, x(t; t_0, \varphi)) < MH_0(t_0, \varphi), \qquad t \geq t_0,$$

holds.

4) *Uniformly eventually Lipschitz stable in terms of two measures*, if for $\epsilon > 0$ there exist $M > 0$, $\delta = \delta(\epsilon) > 0$ and $\tau(\epsilon) > 0$ such that for any initial point $t_0 \geq \tau(\epsilon)$ and any initial function $\varphi \in C([t_0 - r, t_0], \mathbb{R}^n)$ such that $H_0(t_0, \varphi) \leq \delta$ the inequality

$$h(t, x(t; t_0, \varphi)) < MH_0(t_0, \varphi), \qquad t \geq t_0,$$

holds.

Lemma 1. *If system of differential equations with "maxima' (1) is uniformly eventually Lipschitz stable in terms of two measures, then (1) is uniformly eventually stable in terms of two measures.*

Proof. Let $\epsilon > 0$ there exist $M > 0$, $\delta_1 = \delta_1(\epsilon) > 0$ and $\tau(\epsilon) > 0$. Choose $\delta = \min\left(\delta_1, \frac{\epsilon}{M}\right)$. Then from the uniformly eventually Lipshitz stability we get for $H_0(t_0, \varphi) \leq \delta \leq \delta_1$ the validity of $h(t, x(t; t_0, \varphi)) < MH_0(t_0, \varphi) \leq \epsilon$. □

We note that the definition for Lipschitz stability of zero solution of ordinary differential equations is a partial case of Definition 2 for $h(t, x) = h_0(t, x) \equiv ||x||$ and it is introduced by F.M. Dannan, S. Elaydi (see [6]).

Initially we will prove a comparison result, which is the base of our main results.

Lemma 2. *Let the following conditions be fulfilled:*

1. *The functions $f \in C([t_0, T] \times \Omega \times \Omega, \mathbb{R}^n)$ and $g \in C([t_0, T] \times \mathbb{R}, \mathbb{R}_+)$, where $\Omega \subset \mathbb{R}^n$, and $t_0, T: \ 0 \leq t_0 < T < \infty$ are constants.*

2. *The initial value problem (1), (2) has a solution $x(t) = x(t; t_0, \varphi)$, such that $x(t) \in \Omega$ on $[t_0 - r, T]$, where $\varphi \in C([t_0 - r, t_0], \Omega)$.*

3. *The initial value problem (4), (5) has a maximal solution $u^*(t) = u^*(t; t_0, u_0)$, which is defined for $t \in [t_0, T]$ where $u_0 \in \mathbb{R}++$.*

4. *The function $V : [t_0 - r, T] \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$ is such that for any $t \in [t_0, T]$ and any $\psi \in C([t - r, t], \Omega)$ such that $V(t, \psi(t)) > V(t + s, \psi(t + s))$ for $s \in [-r, 0)$, the inequality*

$$D_{(1)}V(t, \psi) \leq g(t, V(t, \psi(t)))$$

holds.

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Then the inequality $\max_{s\in[t_0-r,t_0]} V(s,\varphi(s)) \le u_0$ *implies* $V(t,x(t)) \le u^*(t)$ *for* $t \in [t_0,T]$.

Proof. Let the inequality $\max_{s\in[t_0-r,t_0]} V(s,\varphi(s)) \le u_0$ hold.

For any natural number n we consider the initial value problem for the scalar ordinary differential equation

$$\begin{aligned} v' &= g(t,v) + \frac{1}{n}, \quad t \in [t_0,T], \\ v(t_0) &= u_0 + \frac{1}{n}. \end{aligned} \tag{6}$$

Denote by $v_n(t)$ the maximal solution of the initial value problem (6), which is defined for $t \in [t_0,T]$.

From $g(t,u)+\frac{1}{n} > 0$ on $[t_0,T]\times\mathbb{R}_+$ it follows that the function $v_n(t)$ is increasing in $[t_0,T]$.

Define a function $m(t) \in C([t_0-r,T],\mathbb{R}_+): \quad m(t) = V(t,x(t))$.

Because of the fact that $u^*(t) = \lim_{n\to\infty} v_n(t)$ it is enough to prove that for any natural number n the inequality

$$m(t) \le v_n(t) \quad \text{for } \ t \in [t_0,T] \tag{7}$$

holds.

Note that for any natural number n inequalities

$$m(t_0) \le \max_{s\in[t_0-r,t_0]} V(s,\varphi(s)) \le u_0 < v_n(t_0)$$

hold.

Assume inequality (7) is not true. Therefore there exist a natural number n and a point $t^* \in (t_0,T)$ such that

$$m(t^*) = v_n(t^*), \quad m(t) < v_n(t) \ \text{ for } t \in [t_0,t^*). \tag{8}$$

From inequality (8) it follows that

$$D_- m(t^*) \ge v_n'(t^*) = g(t^*,v_n(t^*)) + \frac{1}{n} = g(t^*,m(t^*)) + \frac{1}{n} > g(t^*,m(t^*)). \tag{9}$$

Case 1. Let $t^*-r \ge t_0$. From monotonicity of function $v_n(t)$ on $[t_0,T]$ it follows that $m(t^*) = v_n(t^*) > v_n(s) > m(s)$ for $s \in [t^*-r,t^*)$.

Case 2. Let $t^*-r < t_0$.

Let $\xi \in [t_0,t^*)$. As in Case 1 we obtain $m(t^*) > m(\xi)$.

Let $\xi \in [t^*-r,t_0)$. Then from the monotonicity of the function $v_n(t)$ we get $m(t^*) = v_n(t^*) \ge v_n(t_0+0) = u_0 + \frac{1}{n} > u_0 \ge \max_{s\in[t_0-r,t_0]} V(s,\phi(s)) \ge m(\xi)$. Therefore, $m(t^*) > m(\xi)$ for $\xi \in [t^*-r,t^*)$.

From condition 4 it follows $D_- m(t^*) \le g(t^*,m(t^*))$ which contradicts (9). Therefore the inequality (7) is true.

From the inequality (7) it follows that $m(t) \le u(t)$ for $t \in [t_0,T]$. □

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3. Main Results

We will obtain some sufficient conditions for Lipschitz stability applying two measures from the class Γ, Lyapunov functions from Λ and Razumikhin method.

Theorem 1. *Let the following conditions be fulfilled:*

1. *Function* $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.
2. *Functions* $h_0, h \in \Gamma$.
3. *There exists a number* $\rho > 0$ *and a function* $V : [-r, \infty) \times S(h, \rho) \to \mathbb{R}_+$, $V \in \Lambda$ *such that:*

   (i) for any number $t \in \mathbb{R}_+$ *and any function* $\psi \in C([t-r, t], S(h, \rho))$ *such that* $V(t, \psi(t)) > V(t+s, \psi(t+s))$ *for* $s \in [-r, 0)$ *the inequality*

   $$D_{(1)}V(t, \psi) \le g(t, V(t, \psi(t)))$$

   holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $g(t, 0) \equiv 0$;

   (ii) there exist constants $C_1, C_2 > 0$ *such that* $C_1 h(t, x) \le V(t, x) \le C_2 h_0(t, x)$ *for* $x \in S(h, \rho)$, $t \in [-r, \infty)$.
4. *For any* $t_0 \in \mathbb{R}_+$ *and* $\varphi \in C([t_0 - r, t_0], \mathbb{R}^n)$ *the initial value problem (1), (2) has a solution* $x(t) = x(t; t_0, \varphi)$ *defined for* $t \ge t_0$.
5. *For any initial point* $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ *the initial value problem (4), (5) has a maximal solution, defined for* $t \ge t_0$.

Then:

(A) If zero solution of scalar ordinary differential equation (4) is Lipschitz stable, then system of differential equations with "maxima" (1) is Lipschitz stable in terms of two measures.

(B) If zero solution of scalar ordinary differential equation (4) is uniformly Lipschitz stable, then system of differential equations with "maxima" (1) is uniformly Lipschitz stable in terms of two measures.

(C) If zero solution of scalar ordinary differential equation (4) is uniformly eventually stable, then system of differential equations with "maxima" (1) is uniformly eventually stable in terms of two measures.

(D) If zero solution of scalar ordinary differential equation (4) is uniformly eventually Lipschitz stable, then system of differential equations with "maxima" (1) is uniformly eventually Lipschitz stable in terms of two measures.

Proof. According to Lemma 1 the claim (C) follows from (D). The proof of the other three claims (A), (B), (D) are similar and we will give only the proof of (B).

Let zero solution of scalar ordinary differential equation (4) be uniformly Lipschitz stable. Then there exist $M > 0$ and $\delta_1 > 0$ such that for any $u_0 \in \mathbb{R}$: $|u_0| < \delta_1$ the following inequality

$$|u(t; t_0, u_0)| < M|u_0|, \quad t \geq t_0, \tag{10}$$

holds where $u(t; t_0, u_0)$ is a solution of the initial value problem (4), (5).

Choose $M_1 > 1$ and $\delta_2 > 0$:

$$M_1 > \frac{C_2}{C_1}, \qquad M_1 > \frac{MC_2}{C_1}, \qquad M_1\delta_2 < \rho. \tag{11}$$

Let $\delta = \min\{\frac{\delta_1}{C_2}, \delta_1, \delta_2\}$.

Let point $t_0 \in \mathbb{R}_+$ be fixed and choose the initial function $\varphi \in C([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0, \varphi) < \delta. \tag{12}$$

From conditions $3(ii)$ and inequalities (11), (12) we get the following inequalities

$$h(t, \varphi(t)) \leq \frac{C_2}{C_1} h_0(t, \varphi(t)) \leq \frac{C_2}{C_1} H_0(t_0, \varphi) < \frac{C_2}{C_1}\delta_2 < \rho, \quad t \in [t_0 - r, t_0], \tag{13}$$

and

$$h(t, \varphi(t)) \leq \frac{C_2}{C_1} H_0(t_0, \varphi) < M_1 H_0(t_0, \varphi), \quad t \in [t_0 - r, t_0]. \tag{14}$$

According to (13) the inclusion $\varphi(t) \in S(h, \rho)$ is valid for $t \in [t_0 - r, t_0]$.

We will prove that if inequality (12) is satisfied then

$$h(t, x(t; t_0, \varphi)) < M_1 H_0(t_0, \varphi), \quad t \geq t_0 - r, \tag{15}$$

where $x(t) = x(t; t_0, \varphi)$ is the solution of the initial value problem (1), (2) with the chosen above initial function φ.

Note that validity of inequality (15) for $t \in [t_0 - r, t_0]$ follows from (14).

Assume that (15) is not true for $t > t_0$. Therefore there exists a point t^* such that $t^* > t_0$, $t^* < \infty$ and

$$h(t^*, x(t^*)) = M_1 H_0(t_0, \varphi), \quad h(t, x(t)) < M_1 H_0(t_0, \varphi), \quad t \in [t_0 - r, t^*). \tag{16}$$

From inequalities (11), (12), (16), and according to the choice of δ it follows

$$h(t, x(t)) \leq M_1 H_0(t_0, \varphi) < M_1\delta \leq M_1\delta_2 < \rho, \quad t \in [t_0 - r, t^*], \tag{17}$$

i.e. $x(t) \in S(h, \rho)$ on $[t_0 - r, t^*]$.

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From condition 3(i) of Theorem 1 it follows the validity of the condition 4 of Lemma 2 for $\Omega = S(h,\rho)$.

Let $u^*(t) = u^*(t; t_0, u_0)$ be the maximal solution of (4), (5), where

$$u_0 = \max_{s\in[t_0-r,t_0]} V(s,\varphi(s)).$$

Apply Lemma 2 for $\Omega = S(h,\rho)$, $T = t^*$ and obtain

$$V(t,x(t)) \le u^*(t), \quad t \in [t_0, t^*]. \tag{18}$$

From the choice of function φ and the constant δ follows that $V(s,\varphi(s)) \le C_2 h_0(s,\varphi(s)) \le C_2 H_0(t_0,\varphi) < C_2\delta \le \delta_1$ for every $s \in [t_0 - r, t_0]$, i.e. $u_0 < \delta_1$.

From condition 3(ii) of Theorem 1, inequalities (10), (18) we get

$$\begin{aligned} M_1\ H_0(t_0,\varphi) &= h(t^*,x(t^*)) \le \frac{1}{C_1}V(t^*,x(t^*)) \le \frac{1}{C_1}u^*(t^*) < \frac{M}{C_1}u_0 \\ &= \frac{M}{C_1}\max_{s\in[t_0-r,t_0]} V(s,\varphi(s)) = \frac{M}{C_1}V(\xi,\varphi(\xi)) \le \frac{MC_2}{C_1}h_0(\xi,\varphi(\xi)) \\ &\le \frac{MC_2}{C_1}\ H_0(t_0,\varphi) < M_1\ H_0(t_0,\varphi), \end{aligned}$$

where $\xi \in [t_0 - r, t_0]$.

The obtained contradiction proves the validity of inequality (15) and the claim of Theorem 1. □

Corollary 1. *Let conditions of Theorem 1 be fulfilled for $g(t,x) \equiv 0$. Then system of differential equations with "maxima" (1) is uniformly Lipschitz stable in terms of two measures.*

Proof. For $g(t,x) \equiv 0$ the solution of the initial value problem (4), (5) is $u(t) = u_0$ and zero solution is uniformly Lipschitz stable. According to the claim (B) of Theorem the conclusion of Corollary 1 is true. □

As a partial case of Theorem 1 we obtain the following sufficient conditions:

Theorem 2. *Let the following conditions be fulfilled:*

1. *The conditions 1, 4, and 5 of Theorem 1 are satisfied.*

2. *Functions $h_0, h \in \Gamma$ and there exist constants $C_1, C_2, \rho_1 > 0$ such that $C_1 h(t,x) \le ||x|| \le C_2 h_0(t,x)$ for $x \in S(h,\rho_1)$, $t \in \mathbb{R}_+$, where $||x||$ is a norm in $\mathbb{R}^n$.*

3. *There exist a function $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $g(t,0) \equiv 0$ and a constant $\rho_2 > 0$ such that for any function $\psi \in C([-r,\infty), \mathbb{R}^n)$: if there exists $t \in \mathbb{R}_+$ such that $\psi(t) \in S(h,\rho_2)$ and $||\psi(t+s)|| < ||\psi(t)||$, $s \in [-r,0)$ then the inequality*

$$||\psi(t) + \epsilon f(t,\psi(t), \max_{s\in[-r,0]} \psi(t+s)|| \le ||\psi(t)|| + \epsilon g(t, ||\psi(t)||) + h(\epsilon)$$

holds, where $\lim_{\epsilon\to 0} \frac{h(\epsilon)}{\epsilon} = 0$.

Then:

(A) *If zero solution of scalar ordinary differential equation (4) is uniformly Lipschitz stable, then system of differential equations with "maxima" (1) is uniformly Lipschitz stable in terms of two measures.*

(B) *If zero solution of scalar ordinary differential equation (4) is uniformly eventually stable, then system of differential equations with "maxima" (1) is uniformly eventually stable in terms of two measures.*

(C) *If zero solution of scalar ordinary differential equation (4) is uniformly eventually Lipschitz stable, then system of differential equations with "maxima" (1) is uniformly eventually Lipschitz stable in terms of two measures.*

Proof. Let $\rho = \min\{\rho_1, \rho_2\}$. Consider the function $V(t,x) = ||x||$, $x \in \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$. Note that $V \in \Lambda$.

From condition 2 of Theorem 2 it follows the validity of condition $3(ii)$ of Theorem 1 for the chosen above function V.

Let $t \in \mathbb{R}_+$ and function $\psi \in C([t-r,t], S_\rho)$ be such that $||\psi(t)|| > ||\psi(t+s)||$ for $s \in [-r, 0)$. Then according to condition 2 of Theorem 2 we get

$$\begin{aligned} D_{(1)}V(t,\psi) =& \limsup_{\epsilon \to 0} \frac{1}{\epsilon}\Big[V\Big(t+\epsilon, \psi(t) + \epsilon f\big(t, \psi(t), \max_{s\in[-r,0]} \psi(t+s)\big)\Big) - V(t,\psi(t))\Big] \\ =& \limsup_{\epsilon \to 0} \frac{1}{\epsilon}\Big[||\psi(t) + \epsilon f(t, \psi(t), \max_{s\in[-r,0]} \psi(t+s))|| - ||\psi(t)||\Big] \\ \leq& g(t, ||\psi(t)||) + \limsup_{\epsilon \to 0} \frac{1}{\epsilon} h(\epsilon) \\ =& g(t, V(t, \psi(t))). \end{aligned}$$

Therefore the condition $3(i)$ of Theorem 1 is satisfied.

We apply Theorem 1 and prove the claims of Theorem 2. □

Remark 1. Let, for example, $h(t,x) = e^{-t}||x||$ and $h_0(t,x) = e^t||x||$. Then both measures h_0, h satisfies condition 2 of Theorem 2.

Now we will give an example to illustrate some of the obtained sufficient conditions on a differential equation with "maxima".

Example. Consider the scalar differential equation with "maxima"

$$2xx'(t) = \Big(\max_{s\in[t-r,t]} x(s)\Big)^2 - e^t \quad \text{for} \quad t \geq t_0 \tag{19}$$

with an initial condition

$$x(t) = \varphi(t) \quad \text{for} \quad t \in [t_0 - r, t_0], \tag{20}$$

where $t_0 \in \mathbb{R}_+$ is an arbitrary point, $r > 0$ is fixed.

The equation (19) is equivalent to

$$x'(t) = \frac{\Big(\max\limits_{s\in[t-r,t]} x(s)\Big)^2 - e^t}{2x}, \quad t \geq t_0.$$

There is no explicit solution of the considered equation, but we will apply the above results to investigate the Lipschitz stability of the equation.

Let $V(t,x) = e^{-t}x^2 \in \Lambda$.

Case 1. Let $h_0(t,x) = x^2$, $h(t,x) = e^{-t}x^2$. Then $e^{-t} \leq e^r$ for $t \geq -r$ and $C_1 h(t,x) \leq V(t,x) \leq C_2 h_0(t,x)$ for $C_1 = \frac{1}{2},\ C_2 = e^r$.

Therefore condition $3(ii)$ of Theorem 1 is fulfilled.

Let $t \in \mathbb{R}_+$, and the function $\psi \in C([t-r,t],\mathbb{R})$ be such that

$$e^{-(t+s)}\psi^2(t+s) < e^{-t}\psi^2(t) \quad \text{for} \ \ s \in [-r,0)$$

or

$$V(t+s,\psi(t+s)) < V(t,\psi(t)) \quad \text{for} \ \ s \in [-r,0).$$

Therefore

$$e^{-t}\max_{s\in[t-r,t]}\Big(\psi^2(s)\Big) \leq \max_{s\in[t-r,t]}\Big(e^{-s}\psi^2(s)\Big) = e^{-t}\psi^2(t),$$

or

$$\max_{s\in[t-r,t]}\Big(\psi^2(s)\Big) \leq \psi^2(t).$$

Then the inequality

$$\begin{aligned} D_{(19)}V(t,\psi) &= -e^{-t}\Big(\psi(t)\Big)^2 + 2e^{-t}\psi(t)\frac{\Big(\max\limits_{s\in[t-r,t]}\psi(s)\Big)^2 - e^t}{2\psi(t)} \\ &= e^{-t}\Big(\Big(\max_{s\in[t-r,t]}\psi(s)\Big)^2 - \psi^2(t)\Big) - 1 \leq -1 < 0, \quad (21)\end{aligned}$$

holds.

According to Corollary 1 differential equation with "maxima" (19) is uniformly Lipschitz stable in both measures h_0 and h, i.e. there exist constants M and $\delta > 0$ such that for $\max_{s\in[t_0-r,t_0]}\Big(\varphi(s)\Big)^2 < \delta$ the inequality $e^{-t}\Big(x(t;t_0,\varphi)\Big)^2 < M\max_{s\in[t_0-r,t_0]}\Big(\varphi(s)\Big)^2$ holds, or

$$\Big(x(t;t_0,\varphi)\Big)^2 < Me^t\max_{s\in[t_0-r,t_0]}\Big(\varphi(s)\Big)^2.$$

Case 2. Let $h_0(t,x) = e^t x^2$, $h(t,x) = e^{-t}x^2$. Then for $t \geq -r$ it follows $e^t \geq e^{-r}$, $e^r \geq e^{-t}$ and $e^{2r}e^t x^2 \geq e^r x^2 \geq e^{-t}x^2$ or $C_1 h(t,x) \leq V(t,x) \leq C_2 h_0(t,x)$ for $C_1 = \frac{1}{2},\ C_2 = e^{2r}$.

Then the conditions of Theorem 1 are satisfied and according to Corollary 1 there exist constants M and $\delta > 0$ such that for $\max_{s\in[t_0-r,t_0]} e^s\Big(\varphi(s)\Big)^2 < \delta$ the inequality $e^{-t}\Big(x(t;t_0,\varphi)\Big)^2 < M \max_{s\in[t_0-r,t_0]} e^s\Big(\varphi(s)\Big)^2$ holds, or

$$\Big(x(t;t_0,\varphi)\Big)^2 < Me^t \max_{s\in[t_0-r,t_0]} e^s\Big(\varphi(s)\Big)^2.$$

To compare both different selections of measures, let us consider $\varphi(t) \equiv K$ where K is a constant.

Then in the first case, if $K^2 < \delta$ then $\Big(x(t;t_0,\varphi)\Big)^2 < Me^t K^2$.

In the second case if $\max_{s\in[t_0-r,t_0]} e^s K^2 = e^{t_0}K^2 < \delta$ then $\Big(x(t;t_0,\varphi)\Big)^2 < Me^t \max_{s\in[t_0-r,t_0]} e^s K^2 = Me^{t+t_0}K^2$.

It is obviously that in the first case the initial point t_0 is not involved in the bounds, since it appears in the bound of the solution in the second case.

This example shows that we could choose both measures in an appropriate way such that to be given higher weight of the initial conditions as well as to be avoided some restrictions on the initial conditions.

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