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EXISTENCE RESULTS FOR NONLOCAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract: In this paper, we study the existence of solutions for fractional differential inclusions of order $q \in (1, 2]$ with nonlocal multi-point boundary conditions involving convex and non-convex multivalued maps. Our results are based on the nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory.

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1. Introduction

Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory (see [11]) and take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors, for example, see [3, 4, 5, 15, 17, 19, 22, 28, 32, 33, 34, 36] and the references therein. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according

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to the censors located at intermediate points.

Differential equations and inclusions of fractional order have recently been addressed by several researchers for a variety of problems. The fractional calculus has found its applications in various disciplines of science and engineering such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc., see for instance [1, 2, 6, 7, 8, 13, 18, 20, 23, 25, 26, 27, 29, 30, 31].

In this paper, we consider the following fractional differential inclusions with nonlocal boundary conditions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) = 0, \quad x(1) = \sum\limits_{i=1}^{m-2} k_i x(\xi_i), \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $k_i > 0$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$.

Here we remark that the problem (1.1) has recently been considered in [35], where the existence of solutions for (1.1) was proved for compact and convex valued maps. In this paper, we discuss the case when the multivalued map in (1.1) is not necessarily convex valued. We also prove the existence of solutions for (1.1) when the right-hand side of (1.1) consists of non-convex valued maps.

2. Preliminaries

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1], \mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Now we recall some basic definitions on multi-valued maps (see [14, 21]).

For a normed space $(X, \|.\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is

$x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by Fix$G$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if:

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1([0,1], \mathbb{R}_+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_q(t) \quad \text{for all } \|x\|_\infty \leq q \text{ and for a.e. } t \in [0,1].$$

For each $y \in C([0,1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ is a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2. Let Y be a separable metric space and let

$$N : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$$

be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,T] \times \mathbb{R}) \to \mathcal{P}(L^1([0,T], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T]\},$$

which is called the Nymetzki operator associated with F.

Definition 2.3. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X, d) be a metric space induced from the normed space $(X; \|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [24]).

Definition 2.4. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.5. (see [9]) *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.6. (see [12]) *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\mathrm{Fix}N \neq \emptyset$.

Let us recall some definitions on fractional calculus, see [23, 30, 31].

Definition 2.7. For a function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1, \ q > 0,$$

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.8. The Riemann-Liouville fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 1 (2010)

Definition 2.9. The Riemann-Liouville fractional derivative of order q for a function g is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}}ds, \quad n=[q]+1,\ q>0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

In order to define the solution of (1.1), we consider the following lemma whose proof is given in [35].

Lemma 2.10. *Assume that* $\sum\limits_{i=1}^{m-2} k_i\xi_i \neq 1$. *For a given* $\rho \in C[0,1]$, *the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \rho(t), \quad 0<t<1, \quad 1<q\le 2, \\ x(0)=0, \quad x(1)=\sum\limits_{i=1}^{m-2} k_i x(\xi_i), \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\rho(s)\,ds - \frac{t}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\rho(s)\,ds \\ &\quad + \frac{t}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}\rho(s)\,ds. \end{aligned}$$

Definition 2.11. A function $x \in C^2([0,1])$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t)\in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)\,ds - \frac{t}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s)\,ds \\ &\quad + \frac{t}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}f(s)\,ds. \end{aligned}$$

3. Main Results

Consider first the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [9] for lower semi-continuous maps with decomposable values.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 1 (2010)

Theorem 3.1. *Assume that:*

(H_1) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty) \text{ for each } (t,x) \in [0,1]\times\mathbb{R};$$

(H_2) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\frac{1}{\Gamma(q)}\left(1 + \frac{1+\sum_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum_{i=1}^{m-2} k_i\xi_i\right|}\right)\psi(M)\|p\|_{L^1}} > 1;$$

(H_3) $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that:*

(a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable,*

(b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$*;*

(H_4) *for each* $\sigma > 0$*, there exists* $\varphi_\sigma \in L^1([0,1],\mathbb{R}_+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \varphi_\sigma(t) \text{ for all } \|x\|_\infty \le \sigma \text{ and for a.e.} t \in [0,1].$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. It follows from (H_3) and (H_4) that F is of l.s.c. type. Then from Lemma 2.5, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad t \in [0,1], \quad 1 < q \le 2, \\ x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} k_i x(\xi_i). \end{cases} \tag{3.2}$$

Observe that if $x \in C^2([0,1])$ is a solution of (3.2), then x is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator Ω as

$$\Omega(x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))\,ds - \frac{t}{1-\sum_{i=1}^{m-2} k_i\xi_i}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(x(s))\,ds$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 1 (2010)

$$+\frac{t}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{q-1}}{\Gamma(q)} f(x(s))\, ds.$$

(i) Ω *is continuous.* Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
|\Omega(y_n)(t) - \Omega(y)(t)| &= \Big| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[f(y_n)(s) - f(y)(s)]\, ds \\
&- \frac{t}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}[f(y_n)(s) - f(y)(s)]\, ds \\
&+ \frac{t}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{q-1}}{\Gamma(q)}[f(y_n)(s) - f(y)(s)]\, ds\Big| \\
\leq \quad & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(y_n)(s) - f(y)(s)\|_\infty\, ds \\
&- \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f(y_n)(s) - f(y)(s)\|_\infty\, ds \\
&+ \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{q-1}}{\Gamma(q)}\|f(y_n)(s) - f(y)(s)\|_\infty\, ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\Omega(y_n) - \Omega(y)\|_\infty &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(y_n)(s) - f(y)(s)\|_\infty\, ds \\
&- \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f(y_n)(s) - f(y)(s)\|_\infty\, ds \\
&+ \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i - s)^{q-1}}{\Gamma(q)}\|f(y_n)(s) - f(y)(s)\|_\infty\, ds \\
&\to 0 \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Thus Ω is continuous.

(ii) Ω *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $x \in B_r = \{x \in$

$C([0,1],\mathbb{R}) : \|x\|_\infty \leq r\}$, we have $\|\Omega(x)\|_\infty \leq \ell$. From (H_4) we have:

$$\begin{aligned}|\Omega(x)(t)| &\leq \int_0^1 \frac{(t-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds + \frac{1}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds \\ &\quad + \frac{1}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds \\ &\leq \frac{1}{\Gamma(q)}\left(1+\frac{1+\sum\limits_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\right)\int_0^1 \phi_r(s)\,ds.\end{aligned}$$

Hence

$$\|\Omega(x)\|_\infty \leq \frac{1}{\Gamma(q)}\left(1+\frac{1+\sum\limits_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\right)\int_0^1 \phi_r(s)\,ds := \ell.$$

(iii) Ω *maps bounded sets into equicontinuous sets in* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1], t_1 < t_2$ and B_r be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}|\Omega(x)(t_2)-\Omega(x)(t_1)| &\leq \left|\int_0^{t_1} \frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds\right| \\ &+ \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds\right| \\ &+ \frac{t_2-t_1}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds \\ &+ \frac{t_2-t_1}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)}\phi_r(s)\,ds.\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero.

(iv) Finally, we discuss *a priori bounds on solutions*. Let x be a solution of (1.1). In view of (H_1), for each $t \in [0,1]$, we obtain

$$|x(t)| \leq \frac{1}{\Gamma(q)}\left(1+\frac{1+\sum\limits_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum\limits_{i=1}^{m-2} k_i\xi_i\right|}\right)\psi(\|x\|_\infty)\|p\|_{L^1}.$$

Consequently, we have

$$\frac{\|x\|_\infty}{\frac{1}{\Gamma(q)}\left(1+\frac{1+\sum_{i=1}^{m-2}k_i\xi_i}{\left|1-\sum_{i=1}^{m-2}k_i\xi_i\right|}\right)\psi(\|x\|_\infty)\|p\|_{L^1}}\leq 1.$$

In view of (H_2), there exists M such that $\|x\|_\infty \neq M$. Let us set

$$U=\{x\in C([0,T],\mathbb{R}):\|x\|_\infty<M+1\}.$$

Note that the operator $\Omega:\overline{U}\to C([0,T],\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\mu\Omega(x)$ for some $\mu\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (see [16]), we deduce that Ω has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [12].

Theorem 3.2. *Assume that the following conditions hold:*

(H_5) $F:[0,1]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ *is such that* $F(.,x):[0,1]\to P_{cp}(\mathbb{R})$ *is measurable for each* $x\in\mathbb{R}$.

(H_6) $H_d(F(t,x),F(t,\bar{x}))\leq m(t)|x-\bar{x}|$ *for almost all* $t\in[0,1]$ *and* $x,\bar{x}\in\mathbb{R}$ *with* $m\in L^1([0,1],\mathbb{R})$ *and* $d(0,F(t,0))\leq m(t)$ *for almost all* $t\in[0,1]$.

Then the boundary value problem (1.1) has at least one solution on $[0,1]$ *if*

$$\frac{\|m\|_{L^1}}{\Gamma(q)}\left(1+\frac{1+\sum_{i=1}^{m-2}k_i\xi_i}{\left|1-\sum_{i=1}^{m-2}k_i\xi_i\right|}\right)<1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x\in C([0,1],\mathbb{R})$ by the assumption (H_5), so F has a measurable selection (see Theorem III.6 [10]). Now we show that the operator Ω satisfies the assumptions of Lemma 2.6. To show that $\Omega(x)\in P_{cl}((C[0,1],\mathbb{R}))$ for each $x\in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\geq 0}\in\Omega(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u\in C([0,1],\mathbb{R})$ and there exists $v_n\in S_{F,x}$ such that, for each $t\in[0,1]$,

$$u_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)\,ds-\frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_n(s)\,ds$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 1 (2010)

$$+\frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}v_n(s)\,ds.$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) \to u(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v(s)\,ds - \frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)\,ds \\ &\quad +\frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}v(s)\,ds. \end{aligned}$$

Hence $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x),\Omega(\bar{x})) \le \gamma\|x-\bar{x}\|_\infty \quad \text{for each} \quad x,\bar{x} \in C([0,1],\mathbb{R}).$$

Let $x,\bar{x} \in C([0,1],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)\,ds - \frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_1(s)\,ds \\ &\quad +\frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}v_1(s)\,ds. \end{aligned}$$

By (H_6), we have

$$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $V(t)\cap F(t,\bar{x}(t))$ is measurable (see Proposition III.4 in [10]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t,\bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t)-v_2(t)| \le m(t)|x(t)-\bar{x}(t)|$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 2, No. 1 (2010)

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)\,ds - \frac{t}{1-\sum_{i=1}^{m-2} k_i\xi_i} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v_2(s)\,ds \\ &+ \frac{t}{1-\sum_{i=1}^{m-2} k_i\xi_i} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)} v_2(s)\,ds. \end{aligned}$$

Thus

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |v_1(s)-v_2(s)|ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} k_i\xi_i} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |v_1(s)-v_2(s)|ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} k_i\xi_i} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)} |v_1(s)-v_2(s)|ds \\ &\le \frac{1}{\Gamma(q)} \left(1 + \frac{1}{\left|1-\sum_{i=1}^{m-2} k_i\xi_i\right|} + \frac{\sum_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum_{i=1}^{m-2} k_i\xi_i\right|} \right) \int_0^1 m(s)\|x-\overline{x}\|ds. \end{aligned}$$

Hence

$$\|h_1(t)-h_2(t)\|_\infty \le \frac{\|m\|_{L^1}}{\Gamma(q)} \left(1 + \frac{1+\sum_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum_{i=1}^{m-2} k_i\xi_i\right|} \right) \|x-\overline{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x),\Omega(\bar{x})) &\le \gamma\|x-\bar{x}\|_\infty \\ &\le \frac{\|m\|_{L^1}}{\Gamma(q)} \left(1 + \frac{1+\sum_{i=1}^{m-2} k_i\xi_i}{\left|1-\sum_{i=1}^{m-2} k_i\xi_i\right|} \right) \|x-\overline{x}\|_\infty. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 2.10 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

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