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OSCILLATION CRITERIA FOR SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH MIXED NONLINEARITIES

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Abstract: The authors study the oscillatory behavior of solutions of the equation

$$(r(t)(z'(t))^{\alpha})' + q(t)x^{\alpha}(\tau(t)) + \sum_{j=1}^{n} q_j(t)x^{\alpha_j}(\tau_j(t)) = 0$$

where $z(t) = x(t) + p(t)x(\sigma(t))$ and $\alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha > \alpha_{m+1} > \ldots > \alpha_n > 0$. The results obtained extend and improve some existing results in the literature for second order neutral and delay differential equations. Examples illustrating the main results are given.

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1. Introduction

In this paper, we study the oscillatory behavior of solutions of the second order neutral delay differential equation

$$(r(t)(z'(t))^{\alpha})' + q(t)x^{\alpha}(\tau(t)) + \sum_{j=1}^{n} q_j(t)x^{\alpha_j}(\tau_j(t)) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $z(t) = x(t) + p(t)x(\sigma(t))$ and α, α_1, α_2, $\ldots$, α_n are all ratios of odd positive

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integers. In what follows we assume that

(c_1) $\alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha > \alpha_{m+1} > \ldots > \alpha_n > 0$;

(c_2) $r(t)$ is positive and differentiable for all $t \geq t_0$;

(c_3) $p(t)$ is continuous with $-1 < p \leq p(t) \leq p_1 < 1$ for all $t \geq t_0$;

(c_4) $q(t)$ and $q_j(t)$ $(j = 1, 2, 3, \ldots, n)$ are continuous and non-negative for all $t \geq t_0$;

(c_5) there exists $\delta(t) \in C^1[t_0, \infty)$ such that $\delta(t) = \min\limits_{1 \leq j \leq n} \{\tau(t),\ \tau_j(t)\}$, $\delta(t) \leq t$, $\lim\limits_{t \to \infty} \delta(t) = \infty$ and $\delta'(t) > 0$ for $t \geq t_0$;

(c_6) $\sigma(t) \in C[t_0, \infty)$, $\sigma(t) \leq t$ for $t \geq t_0$ and $\lim\limits_{t \to \infty} \sigma(t) = \infty$.

By a *solution* of (1.1), we mean a function $x \in C^1([t_x, \infty), \mathbb{R})$, $t_x \geq t_0$, with $r(t)\ (z'(t))^\alpha \in C^1([t_x, \infty), \mathbb{R})$ and which satisfies equation (1.1) for all $t \geq t_x$. We restrict our attention to those solutions $x(t)$ of equation (1.1) that exist on some half-line $[t_x, \infty)$ with $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. As usual, a solution of equation (1.1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

If $\alpha = 1$, $p(t) = 0$, and $\tau(t) = \tau_j(t) = t$, $j = 1, 2, \ldots, n$, then equation (1.1) becomes

$$(r(t)x'(t))' + p(t)x(t) + \sum_{j=1}^{n} q_j(t)x^{\alpha_j}(t) = 0. \tag{2}$$

The oscillatory behavior of equation (1.2) is discussed, for example, in [13, 15].

If $q_j(t) \equiv 0$ for $j = 1, 2, \ldots, n$, then equation (1.1) reduces to

$$(r(t)(z'(t))^\alpha)' + q(t)x^\alpha(\tau(t)) = 0 \tag{3}$$

whose oscillatory behavior has been studied in [1, 4, 5, 8, 11, 12, 16, 18, 17, 19, 20, 21, 22, 23].

If $p(t) \equiv 0$ and $n = 2$, then equation (1.1) reduces to

$$(r(t)(x'(t))^\alpha)' + q(t)x^\alpha(\tau(t)) + q_1(t)x^{\alpha_1}(\tau_1(t)) + q_2(t)x^{\alpha_2}(\tau_2(t)) = 0. \tag{4}$$

The oscillatory behavior of equation (1.4) and some of its particular cases has been studied in [2, 3, 6, 7, 14].

Motivated by the above observations, in this paper we extend and improve some of the above mentioned oscillation criteria to the more general neutral delay differential equation (1.1). In Section 2, we establish some new oscillation results for equation (1.1) using an arithmetic-geometric mean inequality, and in Section 3, we provide some examples to illustrate the results.

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2. Oscillation Results

In this section, we establish some new oscillation criteria for equation (1.1) in the cases where $0 \leq p(t) \leq p_1 < 1$ and $-1 < p \leq p(t) \leq 0$. These results can be seen to extend some of those in [16, 18, 17, 19, 20, 21, 22, 23]. We begin with the following lemma which is a generalization of Lemma 1 of Sun and Wong [15].

Lemma 2.1. *Let $\{\alpha_i\}$, $i = 1, 2, 3, \ldots, n$ be the n-tuple satisfying $\alpha_1 > \alpha_2 > \ldots > \alpha_m > \alpha > \alpha_{m+1} > \ldots > \alpha_n > 0$. Then there exist an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ satisfying $\sum_{i=1}^{n} \alpha_i \eta_i = \alpha$ and $\sum_{i=1}^{n} \eta_i = 1$.*

The proof of the next lemma can be found in [10].

Lemma 2.2. *If A and B are non-negative constants then*

$$A^\lambda - \lambda A B^{\lambda-1} + (\lambda - 1)B^\lambda \geq 0, \ \lambda > 1,$$

and equality holds if and only if $A = B$.

We first establish oscillation results for equation (1.1) in case

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)} ds = \infty \tag{5}$$

and

$$0 \leq p(t) \leq p_1 < 1. \tag{6}$$

Theorem 2.3. *Assume that (2.1) and (2.2) hold and that there is a positive and differentiable function $\rho(t)$ such that*

$$\int_{t_0}^{\infty} \left[\rho(t)Q(t) - \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \frac{r(\delta(t))}{(\delta'(t))^\alpha} \frac{(\rho'(t))^{\alpha+1}}{\rho^\alpha(t)} \right] dt = \infty, \tag{7}$$

where

$$Q(t) = q(t)(1 - p(\tau(t)))^\alpha + k_1 \prod_{j=1}^{n} (q_j(t)(1 - p(\tau_j(t)))^{\alpha_j})^{\eta_j}, \tag{8}$$

$k_1 = \prod_{i=1}^{n} \eta_i^{-\eta_i}$, and $\eta_1, \eta_2, \ldots, \eta_n$ are positive constants satisfying Lemma 2.1. Then all solutions of equation (1) are oscillatory.

Proof. Suppose that equation (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive; the proof is similar if $x(t)$ is eventually negative. Then there exists $T_0 \geq t_0$ such that $x(t) > 0$, $x(\sigma(t)) > 0$,

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and $x(\delta(t)) > 0$ for all $t \geq T_0$. It follows from (6) that $z(t) > 0$ and $z(\delta(t)) > 0$ for $t \geq T_0$.

From equation (1), we have

$$(r(t)(z'(t))^\alpha)' \leq 0, \quad t \geq T_0.$$

Therefore, $r(t)(z'(t))^\alpha$ is a decreasing function. From condition (5), there exist $T \geq T_0$ such that

$$z(t) > 0, \quad z'(t) > 0, \quad \text{and} \quad (r(t)(z'(t))^\alpha)' \leq 0 \quad \text{for} \quad t \geq \delta(T). \tag{9}$$

Note that $r(t)(z'(t))^\alpha \leq r(\delta(t))(z'(\delta(t)))^\alpha$ from which follows that

$$z'(\delta(t)) \geq z'(t) \left(\frac{r(t)}{r(\delta(t))} \right)^{1/\alpha}. \tag{10}$$

Since $x(t) \leq z(t)$ and $z(t)$is nondecreasing, we see that

$$x(t) \geq (1 - p(t))z(t), \quad t \geq \delta(T). \tag{11}$$

From equation (1) and (11), we obtain

$$(r(t)(z'(t))^\alpha)' + q(t)(1 - p(\tau(t)))^\alpha z^\alpha(\delta(t)) + \sum_{j=1}^{n} q_j(t)(1 - p(\tau_j(t)))^{\alpha_j} z^{\alpha_j}(\delta(t)) \leq 0 \tag{12}$$

for $t \geq T_1$, for some $T_1 \geq \delta(T)$. Set

$$w(t) = \rho(t) \frac{r(t)(z'(t))^\alpha}{z^\alpha(\delta(t))}, \quad t \geq T_1. \tag{13}$$

Clearly $w(t) > 0$. Differentiating (13), using (12), (10), and the arithmetic-geometric mean inequality, we have

$$w'(t) \leq -\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha\delta'(t)}{(\rho(t)r(\delta(t)))^{1/\alpha}} w^{1+1/\alpha}(t). \tag{14}$$

Now if in Lemma 2.2 we take

$$A = (\alpha\delta'(t))^{\frac{\alpha}{\alpha+1}} \frac{w(t)}{(\rho(t)r(\delta(t)))^{\frac{1}{\alpha+1}}},$$

$$B = \frac{(\alpha)^{\frac{\alpha}{\alpha+1}}}{(\alpha+1)^\alpha} \left(\frac{\rho'(t)}{\rho(t)} \right)^\alpha \frac{(\rho(t)r(\delta(t)))^{\frac{\alpha}{\alpha+1}}}{(\delta'(t))^{\frac{\alpha^2}{\alpha+1}}}$$

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and $\lambda = 1 + 1/\alpha$, then inequality (14) becomes

$$w'(t) \le -\rho(t)Q(t) + \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \frac{(\rho'(t))^{\alpha+1} r(\delta(t))}{\rho^{\alpha}(t)(\delta'(t))^{\alpha}}.$$

Integrating from T_1 to t gives

$$0 < w(t) \le w(T_1) - \int_{T_1}^{t} \left[\rho(s)Q(s) - \left(\frac{1}{\alpha+1}\right)^{\alpha+1} \frac{(\rho'(s))^{\alpha+1} r(\delta(s))}{\rho^{\alpha}(s)(\delta'(s))^{\alpha}}\right] ds.$$

Letting $t \to \infty$ in the last inequality and applying (7) yields a contradiction. This completes the proof of the theorem. □

Letting $\rho(t) = R^{\alpha}(\delta(t))$ in Theorem 2.3, we obtain the following result.

Corollary 2.4. *Let* (5) *and* (6) *hold. If*

$$\int_{t_0}^{\infty} \left[Q(t)R^{\alpha}(\delta(t)) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\delta'(t)}{r^{1/\alpha}(\delta(t))R(\delta(t))}\right] dt = \infty, \tag{15}$$

then all solutions of equation (1) *are oscillatory.*

Based on Corollary 2.4 and the proof of Corollary 2.2 and Corollary 2.3 in [21] we can easily obtain the following corollaries for equation (1).

Corollary 2.5. *Assume that* (5) *and* (6) *hold. If*

$$\liminf_{t\to\infty} \frac{1}{\ln R(\delta(t))} \int_{t_1}^{t} Q(s)R^{\alpha}(\delta(s))ds > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{16}$$

for some $t_1 \ge t_0$, *then all solutions of equation* (1) *are oscillatory.*

Corollary 2.6. *Assume that* (5) *and* (6) *hold. If*

$$\liminf_{t\to\infty} \frac{Q(t)R^{\alpha+1}(\delta(t))r^{1/\alpha}(\delta(t))}{\delta'(t)} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{17}$$

then all solutions of equation (1) *are oscillatory.*

Remark 2.7. For $n = 2$, $\rho(t) = R^{\alpha}(\delta(t))$, and $p(t) = 0$, Theorem 2.3 reduces to Theorem 2 of Sun and Meng [14]. Also Theorem 2.3 and Corollaries 2.4 and 2.6 improve and generalize some of the results of Sun and Meng [14], Ye and Xu [22], Agarwal, Shieh and Yeh [3], and Dzurina and Stavroulakis [6].

In the sequel, we shall say that a function $H = H(t,s)$ belongs to the class of functions $\mathbb{J}$ if $H \in C(D, [0,\infty))$, where $D = \{(t,s) : -\infty < s \le t \le \infty\}$, and H satisfies

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(H_1) $H(t,t)=0$, $H(t,s)>0$ for $t>s$;

(H_2) H has partial derivatives $\frac{\partial H(t,s)}{\partial t}$ and $\frac{\partial H(t,s)}{\partial s}$ on D such that

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)\sqrt{H(t,s)}$$

and

$$\frac{\partial H(t,s)}{\partial s} = -h_2(t,s)\sqrt{H(t,s)},$$

where $h_1,\ h_2 \in L_{loc}(D,R)$.

The following lemma will be useful for establishing oscillation criteria for equation (1). Its proof can be found in [5].

Lemma 2.8. *Let A_0, A_1, $A_2 \in C([t_0,\infty),\mathbb{R})$ with $A_2>0$ and $w \in C^1([t_0,\infty),\mathbb{R})$. If there exists an interval $(a,b)\subset[t_0,\infty)$ such that*

$$w'(s) \le -A_0(s) + A_1(s)w(s) - A_2(s)(w(s))^{1+\frac{1}{\alpha}} \tag{18}$$

for $s\in(a,b)$, then for any $c\in(a,b)$,

$$\frac{1}{H(c,a)}\int_a^c \left[H(s,a)A_0(s) - \frac{k\alpha^\alpha}{(A_2(s))^\alpha}[\phi_1(s,a)]^{\alpha+1}\right]ds$$
$$+\frac{1}{H(b,c)}\int_c^b \left[H(b,s)A_0(s) - \frac{k\alpha^\alpha}{(A_2(s))^\alpha}[\phi_2(b,s)]^{\alpha+1}\right]ds \le 0$$

for every $H\in\mathbb{J}$, where

$$k=\left(\frac{1}{\alpha+1}\right)^{\alpha+1},$$

$$\phi_1(s,a) = \frac{h_1(s,a)\sqrt{H(s,a)} + A_1(s)H(s,a)}{(H(s,a))^{\frac{\alpha}{\alpha+1}}},$$

and

$$\phi_2(b,s) = \frac{-h_2(b,s)\sqrt{H(b,s)} + A_1(s)H(b,s)}{(H(b,s))^{\frac{\alpha}{\alpha+1}}}.$$

Theorem 2.9. *Let (5) and (6) hold and assume that there is an interval $(a,b)\subset[t_0,\infty)$, a constant $c\in(a,b)$, and functions $H\in\mathbb{J}$ and $\rho\in C^1([t_0,\infty),(0,\infty))$ such that $\rho'(t)>0$ and*

$$\frac{1}{H(c,a)}\int_a^c \rho(s)\left[H(s,a)Q_1(s) - \frac{kr(\delta(s))}{(\delta'(s))^\alpha}[\phi_1(s,a)]^{\alpha+1}\right]ds$$

$$+\frac{1}{H(b,c)}\int_c^b \rho(s)\left[H(b,s)Q_1(s)-\frac{kr(\delta(s))}{(\delta'(s))^\alpha}[\phi_2(b,s)]^{\alpha+1}\right]ds>0, \quad (19)$$

where

$$Q_1(t)=q(t)(1-p(\tau(t)))^\alpha+k_1\prod_{j=1}^n q_j^{\eta_j}(t)(1-p(\tau_j(t)))^{\alpha_j\eta_j},$$

$$k=\left(\frac{1}{\alpha+1}\right)^{\alpha+1}, \quad k_1=\prod_{i=1}^n \eta_i^{-\eta_i},$$

$$\phi_1(t,a)=\frac{h_1(t,a)\sqrt{H(t,a)}+(\rho'(t)/\rho(t))H(t,a)}{(H(t,a))^{\frac{\alpha}{\alpha+1}}},$$

and

$$\phi_2(b,t)=\frac{h_1(b,t)\sqrt{H(b,t)}+(\rho'(t)/\rho(t))H(b,t)}{(H(b,t))^{\frac{\alpha}{\alpha+1}}}.$$

Then every solution of equation (1) *has at least one zero in* (a,b).

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that there exists $t_1\geq t_0$ such that

$$x(t)>0,\ x(\sigma(t))>0,\ x(\delta(t))>0,\ t\geq t_1.$$

Then following the proof of Theorem 2.3, we have

$$w'(t)\leq-\rho(t)Q_1(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\alpha(w(t))^{1+\frac{1}{\alpha}}\delta'(t)}{[p(t)r(\delta(t))]^{\frac{1}{\alpha}}}, \quad t\geq T_0 \quad (20)$$

for some $T_0\geq t_1$. Comparing the above inequality and (18) we see that $A_0(t)=\rho(t)Q_1(t)$, $A_1(t)=\dfrac{\rho'(t)}{\rho(t)}$, and $A_2(t)=\dfrac{\alpha\delta'(t)}{[p(t)r(\delta(t))]^{\frac{1}{\alpha}}}$. Applying Lemma 2.8 to (20) yields a contradiction to (19). Hence, $x(t)$ has at least one zero in (a,b). The proof for the case $x(t)<0$ is similar. □

If the conditions of Theorem 2.9 hold for increasing divergent sequences of positive numbers $\{a_j\}$, $\{b_j\}$, $\{c_j\}$ with $T_0\leq a_j<c_j<b_j$, then we may conclude that equation (1) is oscillatory. That is, we have the following theorem.

Theorem 2.10. *For each* $T\geq t_0$, *if there exist functions* $H\in\mathbb{J}$ *and* $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ *and sequences* $\{a_j\}$, $\{b_j\}$, $\{c_j\}$ *in* $\mathbb{R}^+$ *with* $T\leq a_j<c_j<b_j$ *such that the hypotheses of Theorem 2.9 are satisfied for each* j, *then equation* (1) *is oscillatory.*

With appropriate choices of the functions H, a number of oscillation criteria can be derived from Theorem 2.9. We will denote the class of functions in $\mathbb{J}$ with $h_1(t-s)=h_2(t-s)=h(t-s)$ by $\mathcal{J}_0$. If we apply Theorem 2.9 to $\mathcal{J}_0$, we obtain the following result.

Theorem 2.11. *Assume that* (5) *and* (6) *hold. If for each* $T \geq t_0$, *there exist* $H \in \mathcal{J}_0$, $\rho \in C^1([t_0,\infty),(0,\infty))$, *and constants* a, $c \in \mathbb{R}$ *such that* $T \leq a < c$ *and*

$$\int_a^c H(t-s)[\rho(s)Q_1(s)+\rho(2c-s)Q_1(2c-s)]ds > k\theta(a,c),$$

where

$$\theta(a,c) = \int_a^c \frac{r(\delta(s))}{(\delta'(s))^\alpha (H(t-s))^\alpha}\Bigg\{\rho(s)\Bigg[h(s-a)\sqrt{H(s-a)} + \frac{\rho'(s)}{\rho(s)}H(s-a)\Bigg]^{\alpha+1} + \rho(2c-s)\Bigg[-h(s-a)\sqrt{H(s-a)} + \frac{\rho'(2c-s)}{\rho(2c-s)}H(s-a)\Bigg]^{\alpha+1}\Bigg\}ds,$$

then equation (1) *is oscillatory.*

In [5, 17, 22], the authors consider particular cases of equation (1) and prove that under conditions (5) and

$$-1 < p = p(t) < 0, \tag{21}$$

all solutions of equation (1) are oscillatory. There is a mistake in the proof of those theorems in that the authors neglected to consider the case $z(t) < 0$ in their proofs. Here we wish to correct their mistake by proving the following result. In order to do this, we need to introduce the following class of functions previously used in [17]. Let $D_0 = \{(t,s) \in \mathbb{R}^2 : t > s \geq t_0\}$ and $D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\}$. We say that the function $H \in C(D,\mathbb{R})$ belongs to the class $\mathcal{J}_1$ if

(H_3) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on $(t,s) \in D_0$,

(H_4) H has a continuous and non-positive partial derivative with respect to the second variable on D_0 such that

$$\frac{\partial}{\partial s}H(t,s) = -h(t,s)H(t,s), (t,s) \in D_0, \tag{22}$$

where $h \in C(D,\mathbb{R})$.

Theorem 2.12. *Assume that conditions* (5) *and* (21) *hold,* $\sigma(t) = t-\kappa$, $\kappa > 0$, *and that there exist a function* $H \in \mathcal{J}_1$ *and a positive, nondecreasing differentiable function* ρ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)\rho(s)\left[Q_2(s) - \frac{k_2 r(\delta(s))\lambda^{1+\alpha}(t,s)}{(\delta'(s))^\alpha}\right]ds = \infty, \tag{23}$$

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where

$$Q_2(t) = q(t) + k_1 \prod_{i=1}^{n} q_i^{\eta_i}(t), \quad \lambda(t,s) = \left| \frac{\rho'(t)}{\rho(t)} - h(t,s) \right|,$$

and $k_2 = 1/(\alpha+1)^{\alpha+1}$. *Then a solution of equation* (1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say

$$x(t) > 0, \ x(t-\kappa) > 0, \ \text{ and } \ x(\delta(t)) > 0 \ \text{ for } \ t \geq t_1, \tag{24}$$

for some $t_1 \geq t_0$. From the definition of $z(t)$ and (21), we see that $z(t) < x(t)$ but the sign of $z(t)$ is not known. However, equation (1) implies that $(r(t)(z'(t))^\alpha)' \leq 0$ for $t \geq t_1$, so eventually $z'(t)$ and $z(t)$ have fixed signs, say for $t \geq T_0 \geq t_1$.

First, suppose that $z(t) > 0$ for $t \geq T_0$. Then (9) holds, and since $z(t) \leq x(t)$, equation (1) can be rewritten as

$$(r(t)(z'(t))^\alpha)' + q(t)z^\alpha(\delta(t)) + \sum_{j=1}^{n} q_j(t) z^{\alpha_j}(\delta(t)) \leq 0. \tag{25}$$

Define

$$w(t) = \rho(t)\frac{r(t)(z'(t))^\alpha}{z^\alpha(\delta(t))}.$$

Then, as in the proof of Theorem 2.3, we have

$$w'(t) \leq -\rho(t)Q_2(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha w^{1+\frac{1}{\alpha}}(t)\delta'(t)}{[\rho(t) r(\delta(t))]^{\frac{1}{\alpha}}}, \quad t \geq T_0. \tag{26}$$

Multiplying by $H(t,s)$, and integrating from T to t with $T \geq T_0$, we have

$$\begin{aligned} &\int_T^t H(t,s)\rho(s)Q_2(s)ds \\ &\quad \leq H(t,T)w(T) + \int_T^t H(t,s)\lambda(t,s)w(s)ds - \alpha \int_T^t \frac{H(t,s)\delta'(s)}{[\rho(s)r(\delta(s))]^{\frac{1}{\alpha}}} w^{1+\frac{1}{\alpha}}(s)ds \\ &\quad = H(t,T)w(T) - \int_T^t H(t,s)\left[\frac{\alpha\delta'(s)}{[\rho(s)r(\delta(s))]^{\frac{1}{\alpha}}} w^{1+\frac{1}{\alpha}}(s) - \lambda(t,s)w(s)\right]ds \end{aligned} \tag{27}$$

for $t \geq T \geq T_0$.

Taking

$$A = (\alpha\delta'(t))^{\frac{\alpha}{\alpha+1}} \frac{w(t)}{(\rho(t) r(\delta(t)))^{\frac{1}{\alpha+1}}},$$

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$$B = \frac{(\alpha)^{\frac{\alpha}{\alpha+1}}}{(\alpha+1)^{\alpha}}(\lambda(t,s))^{\alpha}\frac{(\rho(t)r(\delta(t)))^{\frac{\alpha}{\alpha+1}}}{(\delta'(t))^{\frac{\alpha^2}{\alpha+1}}},$$

and $\lambda = 1 + 1/\alpha$ in Lemma 2.2, (27) yields

$$\int_T^t H(t,s)\rho(s)Q_2(s) \le H(t,T)w(T) + k_2\int_T^t \frac{H(t,s)\rho(s)r(\delta(s))}{(\delta'(t))^{\alpha}}\lambda^{1+\alpha}(t,s)ds.$$

With $T = T_0$, this becomes

$$\int_{T_0}^t H(t,s)\rho(s)\left[Q_2(s) - k_2\frac{r(\delta(s)}{(\delta'(t))^{\alpha}}\lambda^{1+\alpha}(t,s)\right]ds \le H(t,T_0)w(T_0).$$

Hence,

$$\int_{t_0}^t H(t,s)\rho(s)\left[Q_2(s) - k_2\frac{r(\delta(s)}{(\delta'(t))^{\alpha}}\lambda^{1+\alpha}(t,s)\right]ds$$
$$\le H(t,t_0)\left[\int_{t_0}^{T_0}\rho(s)\left[Q_2(s) - k_2\frac{r(\delta(s))\lambda^{1+\alpha}(t,s)}{(\delta'(s))^{\alpha}}\right]ds + |w(T_0)|\right]. \quad (28)$$

Dividing (28) by $H(t,t_0)$ and taking the lim sup as $t \to \infty$, we obtain a contradiction to (23). Therefore, $z(t) < 0$ for $t \ge T_0$.

Now from the definition of $z(t)$ we have $x(t) < -p(t)x(\sigma(t-\tau)) < x(\sigma(t-\tau))$ with $\sigma(t) < t$. Hence $x(t)$ is bounded and therefore $z(t)$ is also bounded. Suppose $z'(t) < 0$ for $t \ge T_1 \ge T_0$. Then there exists a positive constant $M > 0$ such that $z'(t) \le \left(-\frac{M}{r(t)}\right)^{\frac{1}{\alpha}}$ for all $t \ge T_1$. Integrating from T_1 to t, we obtain

$$z(t) \le z(T_1) - M^{\frac{1}{\alpha}}\int_{T_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)}ds.$$

Letting $t \to \infty$ and using (11) we obtain $z(t) \to -\infty$ which contradicts the fact that $z(t)$ is bounded. Hence, $z'(t) > 0$ for $t \ge T_1$.

Finally, since $z(t)$ is negative, bounded, and $z'(t) > 0$, we have $\lim_{t\to\infty} z(t) = c$, $-\infty < c \le 0$. Since $x(t)$ is positive and bounded and $\lim_{t\to\infty} z(t) = c$ exists, by Corollary 1.5.1 of Györi and Ladas [9], we have $\lim_{t\to\infty} x(t) = a \ge 0$ exists. If $a > 0$, then we have $0 \ge c = \lim_{t\to\infty} z(t) = \lim_{t\to\infty}(x(t) + px(t-\tau)) = (1+p)a > 0$, which is a

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contradiction. Hence, $a = 0$, that is, $\lim_{t\to\infty} x(t) = 0$, and this completes the proof of the theorem. □

As an immediate consequence of Theorem 2.12, we have the following corollary.

Corollary 2.13. *Let the hypothesis of Theorem 2.12 hold. Then a solution $x(t)$ of equation (1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$ provided*

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)Q_2(s)ds = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)\frac{r(\delta(s))}{(\delta'(s))^{\alpha}}\lambda^{1+\alpha}(t,s)ds < \infty.$$

Next we assume that

$$\lim_{t\to\infty} R(t) < \infty. \tag{29}$$

In [22], the authors obtained oscillation criteria for the equation (1) when $q_j(t) = 0$ for $j = 1, 2, \ldots, n$ and condition (29) holds. But the conclusion of Theorems 2.3 and 2.4 in [22] is wrong. In the proof of these theorems, the authors used the inequality $x(t) \geq (1 - p(t))z(t)$ which is not true when $z'(t) \leq 0$. We wish to correct this mistake by proving the following theorem.

Theorem 2.14. *Assume that (29) holds, $\sigma(t) = t - \kappa$, $\kappa > 0$,*

$$0 \leq p(t) = p_1 < 1, \tag{30}$$

and there exists a positive nondecreasing function $\rho(t)$ such that

$$\int^{\infty} \rho(t)\overline{Q}(t)dt = \infty \tag{31}$$

and

$$\int^{\infty} \left(\frac{1}{r(t)\rho(t)} \int^{t} \rho(s)\overline{Q}(s)ds\right)^{1/\alpha} dt = \infty, \tag{32}$$

where $\overline{Q}(t) = q(t) + \sum_{j=1}^{n} q_j(t)$. Then every solution $x(t)$ of equation (1) either oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an eventually positive solution of equation (1). As in the proof of Theorem 2.3, we have $z(t) > 0$ and $z(\delta(t)) > 0$ for all $t \geq T_0$ for some $T_0 \geq t_0$. There are two possible cases for the sign of $z'(t)$.

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If $z'(t) > 0$ for sufficiently large t, then we are back to the case of Theorem 2.3, and using similar arguments, we will obtain a contradiction to (31).

If $z'(t) \le 0$ for sufficiently large t, then $\lim_{t\to\infty} z(t) = a \ge 0$. Suppose $a > 0$. Then by Corollary 1.5.1 in [9], $x(t)$ has a limit, so

$$0 < \lim_{t\to\infty} z(t) = a \le \lim_{t\to\infty} x(t) + p_1 \lim_{t\to\infty} x(\sigma(t)) \le (1+p_1) \lim_{t\to\infty} x(t).$$

Hence, $\lim_{t\to\infty} x(t) \ge \dfrac{a}{1+p_1} > 0$. Choose a constant $M > 0$ such that $x^{\alpha}(\tau(t)) \ge M$ and $x^{\alpha_j}(\tau_j(t)) \ge M$ for all $t \ge t_1$ and $j = 1, 2, \dots, n$. From (1), we have

$$\begin{aligned}(r(t)(-z'(t))^{\alpha})' &= q(t)x^{\alpha}(\tau(t)) + \sum_{j=1}^{n} q_j(t)x^{\alpha_j}(\tau_j(t)) \\ &\ge M\left[q(t) + \sum_{j=1}^{n} q_j(t)\right] = M\overline{Q}(t)\end{aligned}$$

for $t \ge t_1$. Define $V(t) = \rho(t)r(t)(-z'(t))^{\alpha}$; then $V(t) \ge 0$ and

$$V'(t) = \rho(t)(r(t)(-z'(t))^{\alpha})' + \frac{\rho'(t)}{\rho(t)}V(t) \ge M\rho(t)\overline{Q}(t).$$

Integrating from t_1 to t, we obtain

$$V(t) \ge V(t_1) + M\int_{t_1}^{t} \rho(s)\overline{Q}(s)ds.$$

In view of (31), it is possible to choose a sufficiently large t_2 so that

$$V(t) \ge \frac{M}{2}\int_{t_1}^{t} \rho(s)\overline{Q}(s)ds$$

for all $t \ge t_2$. Then,

$$-z'(t) \ge \left(\frac{M}{2}\right)^{1/\alpha}\left(\frac{1}{\rho(t)r(t)}\int_{t_1}^{t} \rho(s)\overline{Q}(s)ds\right)^{1/\alpha}.$$

Integrating again from t_2 to t gives

$$z(t) \le z(t_2) - \left(\frac{M}{2}\right)^{1/\alpha}\int_{t_2}^{t}\left(\frac{1}{\rho(s)r(s)}\int_{t_1}^{s} \rho(u)\overline{Q}(u)du\right)^{1/\alpha} ds \to -\infty$$

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as $t \to \infty$ by (32). This contradicts the fact that $z(t) > 0$ for $t \geq t_1$. Hence, $a = \lim_{t\to\infty} z(t) = 0$, and therefore $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem. □

Similar to Corollaries 2.5 and 2.6 and Theorem 2.2 we can obtain the following results.

Corollary 2.15. *Assume that* (16), (29), *and* (30) *hold,* $\sigma(t) = t-\kappa$, $\kappa > 0$, *and there is a continuously differentiable function* $\rho(t)$ *such that* $\rho(t) > 0$ *and* $\rho'(t) \geq 0$ *for* $t \geq t_0$ *and* (32) *holds. Then every solution* $x(t)$ *of equation* (1) *either oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.16. *Assume that* (17), (29), *and* (30) *hold,* $\sigma(t) = t-\kappa$, $\kappa > 0$, *and there is a continuously differentiable function* $\rho(t)$ *such that* $\rho(t) > 0$ *and* $\rho'(t) \geq 0$ *for* $t \geq t_0$ *and* (32) *holds. Then every solution* $x(t)$ *of equation (1.1) either oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the differential equation

$$(t(z'(t))^3)' + tx^5(t-2\pi) + t^{1/2}x(t-3\pi) = 0, \quad t \geq 4\pi, \tag{33}$$

where $z(t) = x(t) + \frac{1}{2}x(t-\pi)$. Here $r(t) = t$, $p(t) = \frac{1}{2}$, $\alpha = 3$, $\alpha_1 = 5$, $\alpha_2 = 1$, $\delta(t) = \pi$, $q(t) = 0$, $q_1(t) = t$, and $q_2(t) = t^{1/2}$. Choosing $\rho(t) = 1$, we see that all the conditions of Theorem 2.3 are satisfied and so all solutions of equation (33) are oscillatory.

Example 3.2. Consider the differential equation

$$((z'(t))^3)' + e^{-6}x^3(t-2) + e^{2t-5}x^5(t-1) + e^{-2t-2}x(t-2) = 0, \quad t \geq 3, \tag{34}$$

where $z(t) = x(t) - \frac{2}{e}x(t-1)$. Choosing $H(t,s) = (t-s)^4$, $h(t,s) = 4/(t-s)$ and $\rho(t) = 1$, it is easy to see that all conditions of Theorem 2.12 are satisfied. Hence, any solution $x(t)$ of (34) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Here, $x(t) = e^{-t}$ is a solution of the equation.

Example 3.3. Consider the differential equation

$$(e^{4t}(z'(t))^3)' + \frac{e^{-3}}{2}e^{4t}x^3(t-1) + \frac{5e^{-10}}{4}e^{6t}x^5(t-2) + \frac{13e^{-3}}{8}e^{2t}x(t-3) = 0, \quad t \geq 4, \tag{35}$$

where $z(t) = x(t) + \frac{1}{2e}x(t-1)$. By choosing $\rho(t) = 1$, we see that all conditions of Theorem 2.14 are satisfied. Hence every solution $x(t)$ of equation (35) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. In fact $x(t) = e^{-t}$ is one such solution of equation (35).

References

[1] R.P. Agarwal, S.R. Grace, D.O. Regan, *Oscillation Theory for Second Order Linear, Half linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic, Dordrchet (2002).

[2] R.P. Agarwal, S.R. Grace, D.O. Regan, *Oscillation Theory for Second Order Dynamic Equations*, Taylor and Francis, London (2003).

[3] R.P. Agarwal, S.L. Shich, C.C. Yeh, Oscillation criteria for second order retarded differential equations, *Math. Comput. Modelling*, **26** (1997), 1-11.

[4] D.D. Bainov, D.P. Mishev, *Oscillation Theory for Neutral Equations with Delay*, Adam Hilger, Bristol (1991).

[5] M. Chen, Z. Xu, Interval oscillation of second order Emden-Fowler neutral delay differential equations, *Electron. J. Differential Equations*, **2007**, No. 58 (2007), 1-9.

[6] J. Dzurina, I.P. Stavroulakis, Oscillation criteria for second order delay differential equations, *Appl. Math. Comput.*, **140** (2003), 445-453.

[7] L.H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York (1995).

[8] S.R. Grace, B.S. Lalli, Oscillation of nonlinear second order neutral differential equations, *Rad. Math.*, **3** (1987), 77-84.

[9] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford (1991).

[10] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge (1952).

[11] D. Lackova, Oscillation criteria for second order nonlinear retarded differential equations, *Electron. J. Qual. Theory Differ. Equ., Proc. 8'th Coll. Qualitative Theory of Diff. Equ.*, **2007**, No. 12 (2007), 1-13.

[12] X. Lin, Oscillatory solutions of second order nonlinear neutral differential equations, *J. Math. Anal. Appl.*, **309** (2005), 442-452.

[13] Y.G. Sun, F.W. Meng, Interval criteria for oscillation of second order differential equations with mixed nonlinearities, *Appl. Math. Comput.*, **198** (2008), 375-381.

[14] Y.G. Sun, F.W. Meng, Oscillation of second order delay differential equations with mixed nonlinearities, *Appl. Math. Comput.*, **207** (2009), 135-139.

[15] Y.G. Sun, J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, *J. Math. Anal. Appl.*, **334** (2007), 549-560.

[16] X. Wang, F. Meng, Oscillation criteria of second order quasilinear neutral delay differential equations, *Math. Comput. Modelling*, **46** (2007), 415-421.

[17] Z. Xu, X. Liu, Philos-type oscillation criteria for Emden-Fowler neutral delay differential equations, *J. Comput. Appl. Math.*, **206** (2007), 1116-1126.

[18] R. Xu, F. Meng, Some new oscillation criteria for second order quasilinear neutral delay differential equations, *Appl. Math. Comput.*, **182** (2006), 789-802.

[19] R. Xu, F. Meng, Oscillation criteria for second order quasi linear neutral delay differential equations, *Appl. Math. Comput.*, **192** (2007), 216-222.

[20] R. Xu, F. Meng, New Kamenev-type oscillation criteria for second order neutral nonlinear differential equations, *Appl. Math. Comput.*, **188** (2007), 1364-1370.

[21] Q. Yang, L. Yang, S. Zhu, Interval criteria for oscillation of second order nonlinear neutral differential equations, *Comput. Math. Appl.*, **46** (2006), 903-918.

[22] L. Ye, Z. Xu, Oscillation criteria for second order quasilinear neutral differential equations, *Appl. Math. Comput.*, **207** (2009), 388-396.

[23] R.K. Zhuang, W.T. Li, Interval oscillation criteria for second order neutral nonlinear differential equations, *Appl. Math. Comput.*, **157** (2004), 39-51.