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UNITED EXPLICIT FORM FOR A GAME OF MONOTONE AND CHAOTIC MATRIX MEANS

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Abstract: In the present paper, we would like to give an united explicit form for a family of monotone and chaotic matrix means. Our approach includes some of the most familiar means and a class of means that appear to be new.

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1. Introduction

Scalar and matrix means arise in various areas and have recently extensive several developments and interesting applications. It has been proved to be an useful tool from the theoretical viewpoint as well as for practical purposes, see [1], [2], [6] and [7] for instance and the related references therein. For positive matrices (operators), the theory of matrix means has been constructed axiomatically by Kubo and Ando in [6]. In their sense, for a binary matrix mean m, the corresponding functional $x \longmapsto \Psi_m(x) := m(1, x)$ is matrix monotone in the sense

$$0 \leq A \leq B \implies \Psi_m(A) \leq \Psi_m(B),$$

and the correspondence $m \longmapsto \Psi_m$ is bijective. Precisely, if Ψ is a continuous

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positive matrix monotone functional on $[0,+\infty[$ with $\Psi(1) = 1$, then the binary map m such that

$$m(A,B) = A^{1/2}\Psi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

induces a matrix mean between positive matrices. The Kubo-Ando theory includes the arithmetic, harmonic and geometric monotone matrix means. However, other familiar matrix means introduced in the literature are not matrix monotone in the Kubo-Ando sense and so the matrix case, relatively to the scalar one, presents some profound difficulties. For more precisions concerning this latter point, let us sketch our aim more precisely. The following families maps,

$$\mathcal{B}_p : (a,b) \longmapsto \left(\frac{a^p + b^p}{2}\right)^{1/p},$$

$$\mathcal{D}_p : (a,b) \longmapsto \frac{p-1}{p}\frac{a^p - b^p}{a^{p-1} - b^{p-1}},$$

$$\mathcal{L}_p : (a,b) \longmapsto \left(\frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right)^{1/p},$$

called, respectively, the power binomial, difference and logarithmic means, occur a primordial importance as examples of binary scalar means. These families include some of the most familiar means, namely:

For $\mathcal{B}_p$, it is understood that

$$\mathcal{B}_{-\infty} := \lim_{p\to-\infty}\mathcal{B}_p(a,b) = \min(a,b),\ \ \mathcal{B}_{-1} = \frac{2ab}{a+b},\ \ \mathcal{B}_0 := \lim_{p\to 0}\mathcal{B}_p(a,b) = \sqrt{ab},$$

$$\mathcal{B}_1 = \frac{a+b}{2},\ \ \mathcal{B}_{1/2} = \frac{1}{2}\left(\frac{a+b}{2}\right) + \frac{1}{2}\sqrt{ab},\ \ \mathcal{B}_{+\infty} := \lim_{p\to+\infty}\mathcal{B}_p(a,b) = \max(a,b).$$

For $\mathcal{D}_p$, some special (limit) values of this mean are

$$\mathcal{D}_{-\infty} = \min(a,b),\ \ \mathcal{D}_{-1} = \frac{2ab}{a+b},\ \ \mathcal{D}_0 = ab\frac{\operatorname{Log}\,a - \operatorname{Log}\,b}{a-b},\ \ \mathcal{D}_{1/2} = \sqrt{ab},$$

$$\mathcal{D}_1 = \frac{a-b}{\operatorname{Log}\,a - \operatorname{Log}\,b},\ \ \mathcal{D}_2 = \frac{a+b}{2},\ \ \mathcal{D}_{+\infty} = \max(a,b).$$

For $\mathcal{L}_p$, the particular special values of p are understood as

$$\mathcal{L}_{-\infty} = \min(a,b),\ \mathcal{L}_{-2} = \sqrt{ab},\ \mathcal{L}_{-1} = \frac{a-b}{\operatorname{Log}\,a - \operatorname{Log}\,b},$$

$$\mathcal{L}_0 = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)},\ \mathcal{L}_1 = \frac{a+b}{2},\ \mathcal{L}_{+\infty} = \max(a,b).$$

As proved in [3], for some values of the parameter p, the three above means are non-matrix monotone and so the extension of the above means from the case that the variables are positive real numbers to the case that the variables are positive matrices is not obvious and appears to be interesting. Our fundamental goal in this paper is, first, to give an united form containing the three above means, as further other ones, differently introduced in the literature. Our approach, having a convex character, includes the monotone and chaotic matrix means whose certain of them appear us to be new. We also establish different matrix mean inequalities under a new angle and in a fast way, so proving the interest of our work and the generality of our approach.

2. Basic Notions

In this section we at first introduce some standard notions and results that are needed in the sequel. Let $n \geq 1$ be an integer and $\mathcal{M}_n$ the space of $n \times n$ matrices with real or complex entries. If we denote by $\mathcal{S}_n$ the subspace of $n \times n$ self-adjoint matrices, the subset of all (self-adjoint) positive semi-definite matrices, denoted by $\mathcal{S}_n^+$, is a closed convex cone of $\mathcal{S}_n$ and induces a partial ordering on $\mathcal{S}_n$: for A, B in $\mathcal{S}_n$, we write $A \leq B$ as usual if $B - A$ is positive semi-definite. Henceforth, whenever we consider an order relation $A \leq B$, it will be assumed that A and B belong to $\mathcal{S}_n$. By $\mathcal{S}_n^{+*}$, we denote the open convex cone of positive definite matrices. It is well known that $\mathcal{S}_n^{+*}$ is the topological interior of $\mathcal{S}_n^+$.

Let $\mathcal{C}$ be a nonempty subset of $\mathcal{M}_n$ and a map $\Phi : \mathcal{C} \longrightarrow \mathcal{S}_n$. We say that Φ is matrix increasing (or order-preserving) if $A \leq B$ implies $\Phi(A) \leq \Phi(B)$. If in addition $\mathcal{C}$ is convex, we say that Φ is matrix convex if for all A and B in $\mathcal{C}$ and all real number $t \in [0, 1]$ one has

$$\Phi\left((1-t)A + tB\right) \leq (1-t)\Phi(A) + t\Phi(B).$$

The map Φ is said to be matrix concave (resp. matrix decreasing) if $-\Phi$ is matrix convex (resp. matrix increasing), and it is called matrix monotone if it is matrix increasing or matrix decreasing. As examples, it is well known that the map $X \longmapsto X^{-1}$, from $\mathcal{S}_n^{+*}$ into itself, is matrix convex-decreasing while the map $X \longmapsto \text{Log } X$, from $\mathcal{S}_n^{+*}$ into $\mathcal{S}_n$, is matrix concave-increasing. As proved in [4], a positive map is matrix concave if and only if it is matrix increasing. For instance, if α is a real number such that $0 < \alpha < 1$ then the map $X \longmapsto X^\alpha$, from $\mathcal{S}_n^{+*}$ into itself, is matrix concave-increasing. It is also well-known that the map $X \longmapsto X^2$, from $\mathcal{S}_n^+$ into $\mathcal{S}_n^+$, is matrix convex but not matrix monotone, while $X \longmapsto \exp\ X$ is not matrix convex neither matrix monotone. Equipping $\mathcal{S}_n^{+*}$ with the chaotic partial ordering,

$$A \preceq B \iff \text{Log } A \leq \text{Log } B,$$

then the map $X \longmapsto \exp\ X$ remains chaotic convex in the sense

$$\exp\left((1-t)A+tB\right) \preceq (1-t)\exp\ A + t\exp\ B,$$

for every $A, B \in \mathcal{S}_n$ and $t \in [0,1]$.

Now, we precise some notions about matrix mean. In what follows, we adopt the following definition.

Definition 2.1. A binary map $(A,B) \longmapsto m(A,B)$ from $\mathcal{S}_n^+$ into itself will be called matrix mean if the following three axioms are both satisfied:

(A_1) $m(A,A) = A$ for all $A \in \mathcal{S}_n^+$,

(A_2) $m(\alpha A, \alpha B) = \alpha m(A,B)$ for all $A, B \in \mathcal{S}_n^+$ and $\alpha > 0$,

(A_3) The maps $A \longmapsto m(A,B)$ and $B \longmapsto m(A,B)$ are continuous.

The map m is a monotone matrix mean if moreover

(A_4) The maps $A \longmapsto m(A,B)$ and $B \longmapsto m(A,B)$ are matrix increasing.

It is clear that the set of all matrix means is convex. Every matrix mean can be studied in $\mathcal{S}_n^{+*}$ and extended to $\mathcal{S}_n^+$ by continuity. Otherwise, we notice that the symmetry axiom of a mean, i.e. $m(A,B) = m(B,A)$, is not needed here because there are matrix means, as

$$m(A,B) = (1-\alpha)A + \alpha B,$$

$$m(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{\alpha}A^{1/2},$$

$$m(A,B) = A(\alpha A + (1-\alpha)B)^{-1}B,$$

with $0 < \alpha < 1$, which are not symmetric. Further, from every non-symmetric matrix mean m we can derive various symmetric ones m_s by symmetrizing, as

$$m_s(A,B) = \mathcal{M}\left(m(A,B), m(B,A)\right),$$

where $\mathcal{M}$ stands for arbitrary known symmetric mean.

Throughout what follows, the monotonicity for a mean m will not be always assumed since there are some non-monotone matrix means introduced in the literature, as in [7]

$$m(A,B) = \exp\left((1-t)\mathrm{Log}\ A + t\mathrm{Log}\ B\right), \quad 0 < t < 1.$$

For the same reason, the transformer inequality for a matrix mean m, i.e.

$$T^* m(A,B) T \leq m(T^*AT, T^*BT), \quad \text{for all matrix } T,$$

which remains an inequality if T is invertible, will not be needed since several means, as the above chaotic one, did not satisfied such inequality. In fact, this transformer inequality for m implies that

$$m(A,B) = A^{-1/2} m\left(I, A^{-1/2}BA^{-1/2}\right) A^{1/2},$$

with $A^{-1} = \lim_{\epsilon\downarrow 0}(A+\epsilon I)^{-1}$ for the sake of convenience. In this case the study of such matrix mean is then reduced to that of scalar case, by virtue of the Gelfand (or spectral) representation.

Let m be a matrix mean and set

$$m^*(A,B) = \left(m(A^{-1},B^{-1})\right)^{-1}.$$

It is easy to see that m^* is also a matrix mean called the dual of m. The mean m is called self-dual if $m^* = m$. A matrix mean m is called arithmetic if, for all A_1, A_2, B_1, B_2 there holds

$$m(A_1+A_2, B_1+B_2) = m(A_1,B_1) + m(A_2,B_2),$$

which in the symmetric case yields $m(A,B) = (1/2)A + (1/2)B$. A matrix mean is called harmonic if it is the dual of a certain arithmetic mean. Hence, an arithmetic matrix mean and its associate harmonic mean are mutually dual. The classical monotone and chaotic geometric matrix means are self-dual. For further details concerning these notions introduced in a general context, we indicate a recent paper [14].

3. Generalized Matrix Mean

As already pointed, this section is devoted to introduce our general concept, namely the generalized matrix mean. We first precise some additional notations. Let $\Phi : \mathcal{S}_n^+ \longrightarrow \mathcal{S}_n$, $f : \mathcal{S}_n \longrightarrow \mathcal{S}_n$ and $F : \mathcal{S}_n \longrightarrow \mathcal{S}_n^+$ be three maps, and state the following assumptions:

(H_1) F, f and Φ are bijective, with $F \circ f \circ \Phi = Id$,

(H_2) F, f and Φ are homogenous of real degrees p, q and r, with $pqr = 1$,

(H_3) F, f and Φ are continuous,

(H_4) One the three maps F, f, Φ is monotone increasing and the two other are both monotone increasing or monotone decreasing.

Let T be a nonempty subset of $I\!R$, $d\nu$ a probability measure on T and θ a map from T into $[0,1]$. We now define the map $(A,B) \longmapsto m(A,B)$ by the central relationship:

$$F^{-1}(m(A,B)) = \int_T f\left((1-\theta(t))\Phi(A) + \theta(t)\Phi(B)\right) d\nu(t). \tag{1}$$

Of course, we can take instead of (1) the following

$$F^{-1}(m(A,B)) = \int_T f\left((1-t)\Phi(A) + t\Phi(B)\right) d\mu(t),$$

where $d\mu$ is also a probability measure on T depending on θ and $d\nu$. However, for the sake of presentation for some explicit matrix means presented below, we adopt (1) throughout the following.

Proposition 3.1. *Assume that the above hypotheses* $(H_1), (H_2)$ *and* (H_3) *are satisfied. Then the map* m *defined by (1) is a matrix mean. If in addition* (H_4) *holds then* m *is a monotone matrix mean.*

Proof. It is a simple exercise for the reader. □

Precisely, assumption (H_1) is a necessary and sufficient condition to have the normalization axiom (A_1), i.e. $m(A, A) = A$. However, the assumption (H_2) is just a sufficient condition to ensure the homogeneity axiom (A_2). Let us observe in the following example that (H_2) is not necessary for such homogeneity.

Example 3.1. Let $\Phi(X) = X,\ f(X) = \text{Log } X$ and $F(X) = \exp\ X$. The maps f and F are not homogenous, but it is easy to see that the corresponding matrix mean defined by (1) satisfies the homogeneity axiom. For more details about this, see Example 5.2 below.

Definition 3.1. If the binary map $(A, B) \longmapsto m(A, B)$ defined by (1) is a (monotone) matrix mean, it will be called the generalized (monotone) matrix mean of A and B.

If the maps F, f, Φ are both linear then m is a arithmetic mean and it is given by

$$m(A, B) := \mathcal{A}(A, B) = \int_T ((1-\theta(t))A + \theta(t)B)\, d\nu(t),$$

which, in the symmetric case, yields $\mathcal{A}(A, B) = (1/2)A+(1/2)B$. As already pointed, the associate harmonic matrix mean can be immediately obtained by duality, i.e. with

$$\Phi(X) = X^{-1},\ f(X) = X,\ F(X) = X^{-1}.$$

The matrix mean m will be called chaotic mean if one of the corresponding maps is $X \longmapsto \exp\ X$, the second is $X \longmapsto \text{Log } X$ and so the third is the identity map. This corresponds to one of the following three cases

$$\Phi(X) = \text{Log } X,\ f(X) = X,\ F(X) = \exp\ X,$$

$$\Phi(X) = \text{Log } X,\ f(X) = \exp\ X,\ F(X) = X,$$

$$\Phi(X) = X,\ f(X) = \text{Log } X,\ F(X) = \exp\ X.$$

See Example 5.2 for more details.

Let F, f, Φ as in the above and m be defined by (1). In the case where F, f and Φ are matrix convex (resp. concave) and matrix monotone, some matrix inequalities can be then deduced. For instance, we may state the following result.

Proposition 3.2. *Assume that the two maps* F *and* $F \circ f$ *are matrix convex (resp. concave) then*

$$m(A, B) \leq (\geq) \int_T ((1-\theta(t))A + \theta(t)B)\, d\nu(t),$$

i.e. $m(A,B)$ *is upper (resp. lower) bounded by its associate arithmetic matrix mean.*

Proof. If F is matrix convex then relationship (1) implies

$$m(A,B) \leq \int_T F \circ f\left((1-\theta(t))\Phi(A) + \theta(t)\Phi(B)\right) d\nu(t),$$

which, with the fact that $F \circ f$ is matrix convex and the hypothesis (H_1), implies the desired inequality.

We left the reader to deduce some other matrix inequalities by considering the matrix convexity (resp. concavity) and monotonicity for the maps F, f, Φ. □

4. Scalar Means

As we shall see later, choosing convenient F, f, Φ in a part and $T, d\nu, \theta$ in another part, we will obtain various monotone and non-monotone matrix means. For the sake of presentation for the reader, we start by stating our approach for the scalar case, i.e. for $n = 1$. Let us first remark that every homogenous functional Ψ of degree p can be written $\Psi(x) = c.x^p$, with $c = \Psi(1)$, for all $x \geq 0$. For this, let a and b be two positive reals numbers, we state the following examples.

Example 4.1. (Arithmetic, Harmonic, Geometric, Logarithmic, Exponential Means) In this example, we take

$$T = [0,1],\ d\nu(t) = dt,\ \theta(t) = t,$$

where dt denotes the Lebesgue probability measure.

(i) Let $\Phi(x) = x,\ f(x) = x,\ F(x) = x$. A small calculation gives

$$m(a,b) := \mathcal{A}(a,b) = \frac{a+b}{2},$$

as the arithmetic mean of a and b.

(ii) Take $\Phi(x) = 1/x,\ f(x) = x,\ F(x) = 1/x$. Computing, we obtain

$$m(a,b) := \mathcal{H}(a,b) = \frac{2ab}{a+b},$$

which is the harmonic mean of a and b.

(iii) Let us choose $\Phi(x) = \text{Log } x,\ f(x) = x,\ F(x) = e^x$. A simple calculation yields

$$m(a,b) := \mathcal{G}(a,b) = \sqrt{ab},$$

geometric mean of a and b.

It is interesting to notice that this geometric mean can be also obtained by another choice (see Example 4.2 and Example 4.3 below).

(iv) Setting $\Phi(x) = x,\ f(x) = 1/x,\ F(x) = 1/x$, we obtain the logarithmic mean of a and b given by

$$m(a,b) := \mathcal{L}(a,b) = \frac{a-b}{\text{Log } a - \text{Log } b} \text{ if } a \neq b, \quad \mathcal{L}(a,a) = a.$$

It is interesting to notice that this logarithmic mean can be also obtained by choosing $\Phi(x) = \text{Log } x,\ f(x) = e^x,\ F(x) = x$.

(v) If we take $\Phi(x) = 1/x,\ f(x) = 1/x,\ F(x) = x$, an elementary computation gives

$$m(a,b) := \mathcal{L}^\star(a,b) = ab\frac{\text{Log } a - \text{Log } b}{a-b}, \text{ with } \mathcal{L}^\star(a,a) = a,$$

which is the dual logarithmic mean of a and b.

(vi) We now set $\Phi(x) = x,\ f(x) = \text{Log } x,\ F(x) = e^x$, it is not hard to see that

$$m(a,b) := \mathcal{E}(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}, \ \mathcal{E}(a,a) = a,$$

called the exponential (or identric) mean of a and b.

(vii) Finally, with $\Phi(x) = 1/x,\ f(x) = \text{Log } x,\ F(x) = e^{-x}$, we obtain, after a routine computation, the dual exponential mean of a and b given by

$$\mathcal{E}^*(a,b) = e\left(\frac{a^b}{b^a}\right)^{1/b-a}, \ \mathcal{E}^*(a,a) = a.$$

The reader can easily see that, by Proposition 3.2, all above scalar means are upper bounded by the associate arithmetic mean. Precisely, we have the following inequalities whose certain of them are differently established in the literature.

Proposition 4.1. *For all $a, b > 0$, the following inequalities are met*

$$\mathcal{H}(a,b) \leq \mathcal{E}^*(a,b) \leq \mathcal{L}^*(a,b) \leq \mathcal{G}(a,b) \leq \mathcal{L}(a,b) \leq \mathcal{E}(a,b) \leq \mathcal{A}(a,b).$$

Proof. Since $\mathcal{G}(a,b)$ is self-dual then it is sufficient to show the three right inequalities because the other ones can be immediately deduced by duality. To not lengthen this proof, we limit our attention to state how to prove one of these inequalities by using our convex approach. Let us show $\mathcal{L}(a,b) \leq \mathcal{E}(a,b)$ for example. We have

$$\text{Log } \mathcal{L}(a,b) = -\text{Log} \int_0^1 ((1-t)a + tb)^{-1}\, dt,$$

which, by convexity of $x \longmapsto -\text{Log } x = \text{Log } x^{-1}$, becomes

$$\text{Log } \mathcal{L}(a,b) \leq \int_0^1 \text{Log } ((1-t)a + tb)\, dt,$$

so proving the desired inequality. We left the reader to show the rest of inequalities by displaying the convex integral character of the corresponding means.

Example 4.2. (Intermediary Special Means) Let us take

$$T = [0, \pi/2],\ d\nu(t) = \frac{2}{\pi} dt,\ \theta(t) = \sin^2 t.$$

(i) Setting $\Phi(x) = x^2,\ f(x) = 1/\sqrt{x},\ F(x) = 1/x$, we immediately obtain the expression

$$m(a,b) := \mathcal{AG}(a,b) = \left(\frac{2}{\pi} \int_0^{\pi/2} \left(a^2 co^2 t + b^2 \sin^2 t\right)^{-1/2} dt \right)^{-1}.$$

It is well known that this quantity, inverse of an elliptic integral, is equal to the arithmetic-geometric mean of a and b defined as the same limit of the adjacent sequences (u_n) and (v_n) such that

$$u_{n+1} = \frac{u_n + v_n}{2},\ \ v_{n+1} = \sqrt{u_n v_n},\quad u_0 = a,\ v_0 = b.$$

(ii) Analogously, with $\Phi(x) = 1/x^2, ; f(x) = 1/\sqrt{x},\ F(x) = x$ we find

$$m(a,b) := \mathcal{GH}(a,b) = \frac{2}{\pi} \int_0^{\pi/2} \left(a^{-2} \cos^2 t + b^{-2} \sin^2 t\right)^{-1/2} dt,$$

geometric-harmonic of a and b, and limit of the adjacent sequences

$$u_{n+1} = \sqrt{u_n v_n},\ \ v_{n+1} = \frac{2u_n v_n}{u_n + v_n},\quad u_0 = a,\ v_0 = b.$$

(iii) Now, let $\Phi(x) = 1/x,\ f(x) = 1/x,\ F(x) = x$, and as previous we obtain

$$m(a,b) = \frac{2}{\pi} \int_0^{\pi/2} \left(a^{-1} \cos^2 t + b^{-1} \sin^2 t\right)^{-1} dt,$$

which after a simple computation yields

$$m(a,b) := \mathcal{G}(a,b) = \sqrt{ab} = \mathcal{AH}(a,b),$$

geometric mean of a and b which, as well known, coincides with the arithmetic-harmonic mean of a and b common limit of the sequences

$$u_{n+1} = \frac{u_n + v_n}{2},\ \ v_{n+1} = \frac{2u_n v_n}{u_n + v_n},\quad u_0 = a,\ v_0 = b.$$

The following inequalities are well-known

$$\mathcal{H}(a,b) \le \mathcal{GH}(a,b) \le \mathcal{G}(a,b) = \mathcal{AH}(a,b) \le \mathcal{AG}(a,b) \le \mathcal{A}(a,b).$$

Example 4.3. (Parameterized Arithmetic, Harmonic, Geometric Means) (i) Let p be a positive real parameter and set

$$T = [0, 1],\ d\nu(t) = dt,\ \theta(t) = t^p.$$

a. With $\Phi(x) = x,\ f(x) = x,\ F(x) = x$ we obtain, with $\alpha = 1/(p + 1)$

$$m(a, b) = \frac{p}{p + 1}a + \frac{1}{p + 1}b = (1 - \alpha)a + \alpha b,\ 0 \le \alpha \le 1,$$

which is the non-symmetric parameterized arithmetic mean of a and b.

b. Analogously, letting $\Phi(x) = 1/x,\ f(x) = x,\ F(x) = 1/x$ we obtain the non-symmetric parameterized harmonic mean of a and b given by

$$m(a, b) = \left((1 - \alpha)a^{-1} + \alpha b^{-1}\right)^{-1} = a\left(\alpha a + (1 - \alpha)b\right)^{-1} b,\ \ 0 \le \alpha \le 1.$$

(ii) Now, let $\alpha \in]0, 1[$ be a parameter and set

$$T = [0, \pi/2],\ \ d\nu(t) = \frac{\sin(\alpha\pi)}{2\pi} tan^{2\alpha} t.dt,\ \ \theta(t) = \sin^2 t.$$

a. Let $\Phi(x) = x,\ f(x) = x,\ F(x) = x$. By (1) with a computation, we obtain again

$$m(a, b) := \mathcal{A}_\alpha(a, b) = (1 - \alpha)a + \alpha b,$$

the (non-symmetric) parameterized arithmetic mean of a and b.

b. By duality, i.e. setting $\phi(x) = 1/x,\ f(x) = x,\ 1/x$, we obtain the (non-symmetric) parameterized harmonic mean of a and b:

$$m(a, b) := \mathcal{H}_\alpha(a, b) = \left((1 - \alpha)a^{-1} + \alpha b^{-1}\right)^{-1}.$$

c. Now, with $\Phi(x) = 1/x,\ f(x) = 1/x,\ F(x) = x$ we obtain, after a computation (see [10] for instance), the (non-symmetric) parameterized geometric mean of a and b, i.e.

$$m(a, b) := \mathcal{G}_\alpha(a, b) = a^{1-\alpha} b^\alpha.$$

It is well-known that, for all $a, b > 0$, we have

$$\mathcal{H}_\alpha(a, b) \le \mathcal{G}_\alpha(a, b) \le \mathcal{A}_\alpha(a, b).$$

Example 4.4. (Power Binomial, Logarithmic, Difference Means) This example presents some families of means including various particular ones. The power means cited in the above introduction are here obtained as particular cases. Let p be a real parameter and take

$$T = [0, 1],\ d\nu(t) = dt,\ \theta(t) = t.$$

Then, we obtain the following situations.

(i) Let $\Phi(x) = x^p,\ f(x) = x,\ F(x) = x^{1/p}$. A simple calculation gives

$$m(a,b) := \mathcal{B}_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p},$$

which is the known power binomial mean of a and b.

(ii) Setting $\Phi(x) = x,\ f(x) = x^p,\ F(x) = x^{1/p}$ and computing we obtain, with the variable change $(1-t)a + tb = s$

$$m(a,b) := \mathcal{L}_p(a,b) = \left(\frac{1}{b-a}\int_a^b t^p dt\right)^{1/p} = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p},$$

called the (first) power logarithmic mean of a and b.

(iii) If we take $\Phi(x) = x^p,\ f(x) = x^{-1/p},\ F(x) = 1/x$, a simple computation yields the power difference mean given by

$$m(a,b) := \mathcal{D}_p(a,b) = \frac{p-1}{p}\frac{a^p - b^p}{a^{p-1} - b^{p-1}}.$$

(iv) Now, setting $\Phi(x) = x^p,\ f(x) = x^{1/p},\ F(x) = x$, the reader can easily verify that we find again the class of difference means, namely

$$m(a,b) := \mathcal{D}_{p+1}(a,b) = \frac{p}{p+1}\frac{a^{p+1} - b^{p+1}}{a^p - b^p}.$$

By convexity of $x \longmapsto 1/x$ on $]0, +\infty[$ and using the integral expressions of $\mathcal{D}_p(a,b)$ and $\mathcal{D}_{p+1}(a,b)$ relatively to the above (iii) and (iv), we deduce

$$\mathcal{D}_p(a,b) \le \mathcal{D}_{p+1}(a,b),$$

for all real number p and all $a, b > 0$.

(v) Consider $\Phi(x) = x^p,\ f(x) = 1/x,\ F(x) = x^{-1/p}$. A simple calculation in (1) yields

$$m(a,b) := \mathcal{L}_p^2(a,b) = \left(\frac{b^p - a^p}{p(\mathrm{Log}\ b - \mathrm{Log}\ a)}\right)^{1/p}.$$

It is easy to see that for $p = 1$ and $p = -1$ we obtain, respectively, the logarithm mean and its dual. For $p = 2$ we obtain the geometric mean of the arithmetic and logarithmic means. The mean $\mathcal{L}_p^2$ will be called here the second power logarithmic mean. For the limit value $p = 0$, a routine computation gives $\sqrt{ab}$ geometric mean of a and b. For $p = -\infty$ and $p = +\infty$ we obtain $\min(a,b)$ and $\max(a,b)$ respectively.

Proposition 4.2. *With the above, we have the following:*
1. If $p \le 1$ then $\mathcal{B}_p(a,b) \le \mathcal{L}_p(a,b)$,
2. If $p \ge 1$ then $\mathcal{L}_p(a,b) \le \mathcal{B}_p(a,b)$.

Proof. If we write that the map $x \longmapsto x^p$ is concave (resp. increasing) if and only if $0 < p < 1$ (resp. $p > 0$), the rest of the proof is immediate via the integral expressions of $\mathcal{B}_p(a,b)$ and $\mathcal{L}_p(a,b)$. We omit the details here. □

Example 4.5. (Power Integral Means) In this example, we present some other power means whose certain of them appear us to be new. Let p be a real parameter and set

$$T = [0, \pi/2],\ d\nu(t) = \frac{2}{\pi}dt,\ \theta(t) = \sin^2 t.$$

(i) With $\Phi(x) = x^p,\ f(x) = x,\ F(x) = x^{1/p}$ and after a simple calculation we find

$$m(a,b) := \mathcal{B}_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p},$$

the power binomial mean already obtained in the previous example.

(ii) Setting $\Phi(x) = x^p,\ f(x) = x^{1/p},\ F(x) = x$ we obtain

$$m(a,b) := \mathcal{P}_p(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \left(a^p co^2 t + b^p \sin^2 t\right)^{1/p} dt,$$

which, for $p = -2$, coincides with the elliptic integral characterizing the geometric-harmonic mean of a and b previously stated.

(iii) Similarly, choosing $\Phi(x) = x^p,\ f(x) = x^{-1/p},\ F(x) = 1/x$, we obtain the following mean

$$m(a,b) := \mathcal{Q}_p(a,b) = \left(\frac{2}{\pi}\int_0^{\pi/2} \left(a^p \cos^2 t + b^p \sin^2 t\right)^{-1/p} dt\right)^{-1},$$

which for $p = 2$ is the above expression of the arithmetic-geometric mean of a and b.

Proposition 4.3. *With the above, the following assertions are met:*
1. *For all real number p, $\mathcal{Q}_p(a,b) \le \mathcal{P}_p(a,b)$.*
2. *If $p \le 1$ then $\mathcal{B}_p(a,b) \le \mathcal{P}_p(a,b)$ and, if $p \ge 1$ the above inequality is reversed.*
3. *If $p \ge -1$ then $\mathcal{Q}_p(a,b) \le \mathcal{B}_p(a,b)$ and, if $p \le -1$ the inequality is reversed.*
4. *If $-1 \le p \le 1$ then $\mathcal{Q}_p(a,b) \le \mathcal{B}_p(a,b) \le \mathcal{P}_p(a,b)$.*

Proof. It is a simple exercise for the reader. We omit the details here. □

We invite the reader to construct some other scalar means.

5. Monotone and Chaotic Matrix Means

In this section we discuss the above scalar examples for matrix case. If the considered mean is matrix monotone, its computation is simple from the scalar case. In the non-monotone case, such computation remains generally not always possible. Let us sketch our aim more precisely.

Example 5.1. (Arithmetic, Harmonic, Logarithmic Monotone Matrix Means) Letting

$$T = [0,1],\ d\nu(t) = dt,\ \theta(t) = t,$$

as above, we obtain the following monotone matrix means.

(i) Take $\Phi(X) = X,\ f(X) = X,\ F(X) = X$. Analogously, we obtain

$$m(A,B) := \mathcal{A}(A,B) = \frac{A+B}{2},$$

arithmetic matrix mean of A and B.

(ii) Let $\Phi(X) = X^{-1},\ f(X) = X,\ F(X) = X^{-1}$. After a computation we find the harmonic matrix mean of A and B given by

$$m(A,B) := \mathcal{H}(A,B) = 2A(A+B)^{-1}B.$$

(iii) Now, set $\Phi(X) = X,\ f(X) = X^{-1},\ F(X) = X^{-1}$ to obtain

$$m(A,B) := \mathcal{L}(A,B) = \left(\int_0^1 ((1-t).A + t.B)^{-1}\, dt\right)^{-1},$$

the logarithmic matrix mean of A and B whose the explicit form is given by

$$\mathcal{L}(A,B) = A^{1/2}\Psi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},\ \text{ with }\ \Psi(x) = \frac{x-1}{\operatorname{Log}\, x}.$$

(iv) As the scalar case, setting $\Phi(X) = X^{-1},\ f(X) = X,\ F(X) = X^{-1}$ we obtain the dual logarithmic matrix mean of A and B given by the relationship

$$\mathcal{L}^*(A,B) = A^{1/2}\Psi^*\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},\ \text{ with }\ \Psi^*(x) = \frac{x.\operatorname{Log}\, x}{x-1}.$$

Proposition 5.1. *With the above, we have*

$$\mathcal{H}(A,B) \le \mathcal{L}^*(A,B) \le \mathcal{L}(A,B) \le \mathcal{A}(A,B).$$

Proof. All the above means are matrix monotone and satisfy the transformer inequality. By Gelfand (or spectral) representation, the proof is then reduced to that of scalar case already given in the above. □

Example 5.2. (Chaotic Geometric, Logarithmic, Exponential Matrix Means) Again with

$$T = [0,1],\ d\nu(t) = dt,\ \theta(t) = t,$$

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we state the following non-monotone matrix means.

(i) Let $\Phi(X) = \mathrm{Log}\ X,\ f(X) = X,\ F(X) = \exp\ X$. It is easy to see that

$$m(A,B) := \mathcal{CG}(A,B) = \exp \frac{\mathrm{Log}\ A + \mathrm{Log}\ B}{2},$$

which is the known chaotic geometric matrix mean of A and B. It is clear that $\mathcal{CG}$ is self-dual and satisfies the chaotic inequality

$$\mathcal{H}(A,B) \preceq \mathcal{CG}(A,B) \preceq \mathcal{A}(A,B).$$

For the monotone geometric matrix mean, see Example 5.3 below.

(ii) Choosing $\Phi(X) = \mathrm{Log}\ X,\ f(X) = \exp\ X,\ F(X) = X$, we obtain by (1)

$$m(A,B) := \mathcal{CL}(A,B) = \int_0^1 \exp\left((1-t)\mathrm{Log}\ A + t\mathrm{Log}\ B\right) dt,$$

which, by analogy with the standard case, we propose here as the chaotic logarithmic matrix mean of A and B.

(iii) We now set $\Phi(X) = X,\ f(X) = \mathrm{Log}\ X,\ F(X) = \exp\ X$ to obtain

$$m(A,B) := \mathcal{CE}(A,B) = \exp \int_0^1 \mathrm{Log}\ \left((1-t)A + tB\right) dt,$$

which we propose here as the chaotic exponential (or identric) matrix mean of A and B. The dual matrix mean $\mathcal{CE}^*$ of $\mathcal{CE}$ is immediate by duality. Otherwise, we think that an explicit computation of $\mathcal{CL}(A,B)$ or $\mathcal{CE}(A,B)$ is not possible when the matrices A and B are not commuting.

Proposition 5.2. *With the above, the following chaotic inequalities hold*

$$\mathcal{CE}^*(A,B) \preceq \mathcal{CG}(A,B) \preceq \mathcal{CE}(A,B),$$

$$\mathcal{CL}^*(A,B) \preceq \mathcal{CG}(A,B) \preceq \mathcal{CL}(A,B).$$

Proof. Let us prove the first double inequality. Since the map $X \longmapsto \mathrm{Log}\ X$ is matrix concave we can write

$$\mathrm{Log}\ \mathcal{CE}(A,B) \geq \int_0^1 \left((1-t)\mathrm{Log}\ A + t\mathrm{Log}\ B\right)dt = \frac{\mathrm{Log}\ A + \mathrm{Log}\ B}{2},$$

i.e $\mathcal{CG}(A,B) \preceq \mathcal{CE}(A,B)$. By duality, the proof of the first double inequality is complete. The other one can be showed by a similar manner and the details are left to the reader. □

The above proposition gives a comparison, with respect to the chaotic order, between $\mathcal{CG}(A,B)$ and one of $\mathcal{CE}(A,B)$ and $\mathcal{CL}(A,B)$. However, we do not see how to compare chaotically $\mathcal{CE}(A,B)$ and $\mathcal{CL}(A,B)$.

Example 5.3. (Power Monotone Arithmetic, Harmonic, Geometric Matrix Means) In this example we shall find again the power arithmetic and harmonic matrix means already obtained, together with the monotone geometric matrix mean. We begin by the symmetric case.

(i) Let

$$T = [0, +\infty[,\ d\nu(t) = \frac{1}{\pi\sqrt{t}(1+t)}dt,\ \theta(t) = \frac{t}{1+t}.$$

a. Let $\Phi(X) = X,\ f(X) = X,\ F(X) = X$, by the variable change $tan\ s = \sqrt{t}$ in (1), an elementary computation gives the monotone arithmetic matrix mean of A and B:

$$m(A, B) := \mathcal{A}(A, B) = \frac{A + B}{2}.$$

b. If we set $\Phi(X) = X^{-1},\ f(X) = X,\ F(X) = X^{-1}$, we obtain the dual of the above case, i.e

$$m(A, B) := \mathcal{H}(A, B) = 2A(A + B)^{-1}B,$$

the monotone harmonic matrix mean of A and B.

c. Now, with $\Phi(X) = X^{-1},\ f(X) = X^{-1},\ F(X) = X,$ (1) yields (after a simple reduction)

$$m(A, B) := \mathcal{G}(A, B) = \frac{1}{\pi}\int_0^{+\infty} \frac{A(tA + B)^{-1}B}{\sqrt{t}}dt,$$

which is the integral form of the monotone geometric mean of A and B differently in the literature. Its explicit form is known by

$$\mathcal{G}(A, B) = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$

(ii) Here we extend the above example to find the power corresponding matrix means. For this, let $\alpha \in]0, 1[$ and

$$T = [0, +\infty[,\quad d\nu(t) = \frac{\sin\ \alpha\pi}{\pi}\frac{t^{\alpha-1}}{1+t}dt,\quad \theta(t) = \frac{t}{1+t}.$$

a. Let $\Phi(X) = X,\ f(X) = X,\ F(X) = X$, putting $t = tan^2 s$ in (1) we obtain again the power monotone arithmetic matrix mean of A and B:

$$m(A, B) := \mathcal{A}_\alpha(A, B) = (1 - \alpha)A + \alpha B$$

b. By duality, setting $\Phi(X) = X^{-1},\ f(X) = X,\ F(X) = X^{-1}$, we find again

$$m(A, B) = \mathcal{H}_\alpha(A, B) = \left((1 - \alpha)A^{-1} + \alpha B^{-1}\right)^{-1},$$

the power monotone harmonic matrix mean of A and B.

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c. With $\Phi(X) = X^{-1},\ f(X) = X^{-1},\ F(X) = X,$ (1) gives after an elementary reduction

$$m(A,B) := \mathcal{G}_\alpha(A,B) = \frac{\sin\ \alpha\pi}{\pi}\int_0^{+\infty} t^{\alpha-1}A\,(tA+B)^{-1}Bdt,$$

which is the power geometric matrix mean of A and B introduced in [10] by terms of convex analysis. As proved in [10] for example, the explicit form of $\mathcal{G}_\alpha(A,B)$ is given by

$$\mathcal{G}_\alpha(A,B) = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}.$$

Proposition 5.3. *With the above notations, for all positive matrices A and B and all real $\alpha \in [0,1]$, one has*

$$\mathcal{H}_\alpha A,B) \le \mathcal{G}_\alpha(A,B) \le \mathcal{A}_\alpha(A,B).$$

Proof. It is sufficient to establish the right inequality and the left one is deduced by duality, since $\mathcal{G}_\alpha$ is a self-dual matrix mean. Since these means are monotone satisfying the transformer inequality then the proof is reduced to that of scalar one which is other than the Young inequality. It is interesting to notice that another proof is possible by using the convex character of our approach. □

Example 5.4. (Power Binomial, Logarithmic, Difference Matrix Means) Let p be a real parameter. Let us put

$$T = [0,1],\ d\nu(t) = dt,\ \theta(t) = t,$$

we obtain the following power matrix means extending some monotone and chaotic ones.

(i) Let setting $\Phi(X) = X^p,\ f(X) = X,\ F(X) = X^{1/p}$ and computing to obtain

$$m(A,B) := \mathcal{B}_p(A,B) = \left(\frac{A^p + B^p}{2}\right)^{1/p},$$

the power binomial matrix mean of A and B. As established in [3], $\mathcal{B}_p$ is matrix monotone if and only if $-1 \le p \le +1$.

(ii) Setting $\Phi(X) = X,\ f(X) = X^p,\ F(X) = X^{1/p}$, relation (1) gives

$$m(A,B) := \mathcal{L}_p^1(A,B) = \left(\int_0^1 ((1-t)A + tB)^p\,dt\right)^{1/p},$$

which, by analogy to the scalar case, we propose here as the first power logarithmic matrix mean of A and B.

(iii) With $\Phi(X) = X^p,\ f(X) = X^{-1},\ F(X) = X^{-1/p}$ we obtain by (1) the following matrix mean

$$m(A,B) := \mathcal{L}_p^2(A,B) = \left(\int_0^1 ((1-t)A^p + tB^p)^{-1} dt\right)^{-1/p},$$

which, as already pointed for the scalar case, we propose as a second power logarithmic matrix mean. We notice that $\mathcal{L}_p^1(A,B)$ and $\mathcal{L}_p^2(A,B)$ are different as in the scalar case.

(iv) Let us choose $\Phi(X) = X^p,\ f(X) = X^{1/p},\ F(X) = X$ to obtain

$$m(A,B) := \mathcal{D}_p^1(A,B) = \int_0^1 ((1-t)A^p + tB^p)^{1/p}\, dt,$$

which, by analogy with the standard case, will be called the first power difference matrix mean of A and B.

(v) If we take $\Phi(X) = X^p,\ f(X) = X^{-1/p},\ F(X) = X^{-1}$, we obtain by (1)

$$m(A,B) := \mathcal{D}_p^2(A,B) = \left(\int_0^1 ((1-t)A^p + tB^p)^{-1/p}\, dt\right)^{-1},$$

which we call the second power difference matrix mean of A and B.

We notice that, contrary to $\mathcal{L}_p^1(A,B)$ and $\mathcal{L}_p^2(A,B)$, $\mathcal{D}_p^1(A,B)$ and $\mathcal{D}_p^2(A,B)$ define the same class of means in the scalar case and so in the commuting matrix case. In the general case, by convexity of $X \longmapsto X^{-1}$ it is clear that

$$\mathcal{D}_p^2(A,B) \le \mathcal{D}_p^1(A,B),$$

for all positive matrices A, B and all real number p. Some other matrix inequalities between the above matrix means can be obtained by discussing the matrix monotonicity and convexity of the map $X \longmapsto X^p$ following the real values of p. We omit the details here.

We left the reader to construct other matrix means by choosing convenient T, ν, θ in a part and Φ, f, F in a second part. In particular, an analogue of Example 4.5 can be stated for matrix case.

Finally, we may state the following remark.

Remark 5.1. With some precautions, the above matrix approach can be stated for operator mean. However, the extension of this approach from operator variables to functional variables, in the same sense as in [2], [5], [8], [9], [10], [11], [12], is not obvious and appears to be very interesting. For a partial answer of this latter point, we can consult [13].

6. Conclusion and Motivation

By Löwner's theory, Kubo and Ando [6] showed that every monotone operator mean m corresponds bijectively to the integral representation

$$m(A,B) = aA + bB + \int_0^{+\infty} (tA) : B \frac{1+t}{t} d\mu_m(t), \tag{2}$$

where $a = \mu_m(\{0\}), b = \mu_m(\{\infty\})$ and $A : B = (A^+ + B^+)^+$ with $A^+ = \lim_{\epsilon \downarrow 0}(A + \epsilon I)^{-1}$, for all positive operators A and B.

Let us present a comparative study between our approach, based on the explicit form (1), and that of Kubo-Ando represented by (2):

• Representation (2) concerns only monotone operator means, while our approach includes monotone and non-monotone (chaotic) operator means as explained by some examples in the above. For instance, our approach permitted us to construct (monotone and chaotic) geometric, logarithmic, exponential operator means.

• Operator means derived from representation (2) satisfy some mean inequalities for the Löwner partial ordering,

$$A \le B \Longleftrightarrow B - A \text{ is positive semi-definite,}$$

while (1) permitted us to obtain the same as (2) and more other mean inequalities for the chaotic order,

$$A \preceq B \Longleftrightarrow \text{Log } A \le \text{Log } B.$$

• Our representation (1) makes to appear the convex character in the integral representation (as in (2)) and also (contrary to (2)) in the integrand expression. This allowed us to simplify some known operator mean inequalities and to obtain other new ones in a fast way.

• Representation (2) is manipulable in the theoretical context. So, starting from a monotone operator mean m, it is not obvious to find explicitly the associate probability measure $d\mu_m$. While (1) is manipulable in the theoretical viewpoint as well as for practical purposes, as proved by some examples throughout the paper.

In conclusion, we think that our approach stems its importance in an intuitive aspect derived from a convex character explaining an idea for obtaining some known results, in a simple and fast way, for the monotone operator means, together with some fresh examples for monotone and non-monotone operator means.

References

[1] T. Ando, C.K. Li, R. Mathias, Geometric means, *Linear Algebra and Its Applications*, **385** (2004), 305-334.

[2] M. Atteia, M. Raïssouli, Self dual operator on convex functionals, geometric mean and square root of convex functionals, *Journal of Convex Analysis*, **8**, No. 1 (2001), 223-240.

[3] F. Hiai, H. Kosaki, Means for matrices and comparison of their norms, *Indiana University Mathematics Journal*, **48**, No. 3 (1999), 899-936.

[4] F. Hansen, G.K. Pedersen, Jensen's inequality for operators and Löwner's theorem, *Math. Ann.*, **258** (1982), 511-516.

[5] J.I. Fujii, Kubo-Ando theory of convex functional means, *Scientiae Mathematicae Japonicae*, **7** (2002), 299-311.

[6] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.*, **246** (1980), 205-224.

[7] R.D. Nussbaum, J.E. Cohen, The arithmetic-geometric mean and its generalizations for non-commuting linear operators, *Ann. Sci. Norm. Sup. Sci IV*, **15**, No. 2 (1989), 239-308.

[8] M. Raïssouli, M. Chergui, Arithmetico-geometric and geometrico-harmonic means of two convex functionals, *Scientiae Mathematicae Japonicae*, **55**, No. 3 (2002), 485-492.

[9] M. Raïssouli, H. Bouziane, Functional logarithm in the sense of convex analysis, *Journal of Convex Analysis*, **10**, No. 1 (2003), 229-244.

[10] M. Raïssouli, H. Bouziane, Arithmetico-geometrico-harmonic functional mean in the sense of convex analysis, *Annales des Sciences Mathématiques du Québec*, **30**, No. 1 (2006), 79-107.

[11] M. Raïssouli, Functional logarithmic mean in convex analysis, *Journal of Inequalities in Pure and Applied Mathematics*, **10**, No. 4 (2009), Article 102.

[12] M. Raïssouli, Tsallis relative entropy for convex functionals, *International Journal of Pure and Applied Mathematics*, **51**, No. 4 (2009), 555-563.

[13] M. Raïssouli, Discrete operator and functional means can be reduced to the continuous arithmetic mean, *International Journal of Open Problems in Computer Science and Mathematics*, **3**, No. 2 (2010), 186-199.

[14] M. Raïssouli, Duality and means, *In Preparation.*