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COMPOSITION OPERATORS ON SEQUENCE SPACES OF ENTIRE FUNCTIONS

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Abstract: In this paper we characterize composition operators on sequence spaces of entire functions and we also make an attempt to characterize Fredholmness, isometry, closed range, invertibility of these operators.

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1. Introduction

Let (X, T) be a Hausdorff locally convex space and $U(x)$ be the fundamental system of balanced, convex and absorbing neighbourhoods of origin. For $u \in U(x)$, set $P_u = \sup\{|f(x)| : f \in u^o\}$, where u^o is the polar of u. Then P_u is a seminorm. Moreover, $p_u = \inf\{\alpha > 0 : x \in \alpha u\}$. Let $D = \{p_u : u \in U(x)\}$. Then D is the family of continuous seminorms generating the topology T on X. We denote by $E(X)$, the class of all sequences $x = \{x_n\}$ where $x_n \in X$, and $\{p_u(x_n)\}^{\frac{1}{n}}$ tends to zero as n tends to infinity for each $u \in U(x)$. For each $x = \{x_n\} \subset E(X)$ we define $P_u(x) = \sup(p_u(x_n))^{\frac{1}{n}}$. Then P_u satisfies the following properties:

(i) $P_u(x) \geq 0$,

(ii) $P_u(x + y) \leq P_u(x) + P_u(y)$,

(iii) $P_u(\alpha x) \leq A(\alpha)P_u(x)$.

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Here $x, y \in E(X)$, α is a scalar and $A(\alpha) = \max\{|\alpha|, 1\}$. Srivastava [4] proved in his paper that P_u is a paranorm on $E(X)$ and $(E(X), P_u)$ is a paranormed space for each $u \in U(x)$. If we consider VP_u the superemum of the topologies induced by all paranorms $P_u, u \in U(x)$, then $(E(X), VP_u)$ is a topological vector space. It is proved in [4] that if X is complete, then $(E(X), VP_u)$ is also complete.

Let $v : X \to X$ be a mapping. Then we define a composition transformation $T_v : E(X) \to E(X)$ as $T_v(f) = fov$, for every $f \in E(X)$. In case T_v is bounded, we call it a composition operator. By $B(E(X))$, we denote the set of all bounded linear operators from $E(X)$ into itself.

A linear transformation A defined on a locally convex space (X, T_P), where P is the family of seminorms on X, is an isometry if for every $p \in P$, $p(Af) = p(f)$, for every $f \in X$. For more details about sequence spaces one can refer to [4], [5]. Whereas the study of Composition operators on some function spaces are considered by [1], [2], [3].

In this paper we plan to study the composition operators on sequence spaces of entire functions.

Theorem. *Let $v : N \to N$ be a mapping. Then $T_v : E(X) \to E(X)$ is continuous iff there exists $M > 0$ such that*

$$\frac{n}{v(n)} < M \quad \text{for all} \quad n \in N. \tag{i}$$

Proof. Suppose that the condition (i) is true. Let $f \in E(X)$. Then for given $\epsilon, 0 < \epsilon < 1$ and a balanced convex neighbourhood, u of zero, there exist a positive integer n_o such that $p_u(f(n))^{\frac{1}{n}} < \epsilon$ for all $n \geq n_o$. Choose $m_o = Mn_o$. Then from the given condition (i) for $n \geq m_o$, we have

$$v(n) > \frac{n}{M} \geq \frac{m_o}{M} = n_o.$$

Therefore

$$(p_u(f(v(n))))^{\frac{1}{v(n)}} < \epsilon \quad \text{for all} \quad n \geq m_o.$$

Now,

$$(p_u(f(v(n))))^{\frac{1}{n}} = (p_u(f(v(n)))^{\frac{1}{v(n)}})^{\frac{v(n)}{n}} < \epsilon^{\frac{1}{M}},$$

for all $n \geq m_o$. Hence $fov \in E(X)$. It is easy to show that the topology of pointwise convergence is weaker than the topology of strong convergence of $E(X)$. Suppose $T_v f_n \to g$ and $f_n \to f$. Now $\lim_{n\to\infty} f_n(v(k)) = g(k)$ and $\lim_{n\to\infty} f_n(m) = f(m)$ for every m so that $\lim_{n\to\infty} f_n(v(k)) = f(v(k))$. Thus $g(k) = f(v(k))$. Hence T_v is continuous by the closed graph theorem.

Conversely, if the condition of the theorem is not true, then for every $k \in N$, we can find $n_k \in N$ such that $\frac{n_k}{v(n_k)} > k$. Let $0 < R < 1$ and $x \in u$ such that $p_u(x) \neq g_o$. Consider the sequence $f_{n_k} = R^{n_k}.x\chi_{v(n_k)}$. Then

$$P_u(f_{n_k}(v(n_k))) = \sup\{p_u(f_{n_k}(m)^{\frac{1}{m}} : m \geq 1\} = p_u(R^{n_k}x)^{\frac{1}{v(n_k)}} = R^{\frac{n_k}{v(n_k)}}.p_u(x)^{\frac{1}{v(n_k)}}$$
$$< R^{\frac{n_k}{v(n_k)}} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence $f_{n_k} \to 0$ as $k \to \infty$. But

$$p_u(T_v f_{n_k}) = P_u(f_{n_k} ov) = \sup\{(p_u(f_{n_k} v(m)))^{\frac{1}{m}} : m \geq 1\} \geq p_u(R^{n_k}.x)^{\frac{1}{n_k}}$$
$$= RP_u(x)^{\frac{1}{n_k}}.$$

Thus $P_u(T_v f_{n_k})$ does not go to zero as k approaches infinity, which contradicts the continuity of T_v. Hence the condition of the theorem must be true. □

Corollary. *If $T_v \in B(E(X))$, then # ran $v = \infty$.*

Theorem. *Let $T_v \in B(E(X))$. Then T_v is a Fredholm operator if and only if:*
(i) $N \backslash v(N)$ is a finite set.
(ii) $E = \{n : \#v^{-1}(v(n)) > 1\}$ is a finite set.
(iii) $\frac{n}{v(n)} \geq \epsilon$ for all but finitely many values of n.

Proof. We first assume that T_v is a Fredholm operator. If the condition (i) fails, then for a neighbourhood $u \in U(x)$, taking $0 \neq x \in u$ we see that $f_{x,n} \in \ker T_v$ for every $n \in N \backslash v(N)$, where $f_{x,n}$ is defined as

$$f_{x,n}(m) = \begin{cases} x, & \text{if } m = n; \\ 0, & \text{elsewhere.} \end{cases}$$

Also $\{f_{x,n} : n \in N \backslash v(N)\}$ is an infinite independent set contained in ker T_v, which contradicts that ker T_v is finite dimensional. Hence condition (i) must hold.

Further if the condition (ii) fails then the set $\{n : \#v^{-1}(v(n)) > 1\}$ is an infinite set. We can choose an infinite number of pairs (x_k, y_k) such that $v(x_k) = v(y_k)$.

Let $D_{x_k,y_k} : E(X) \to C$ be defined as $D_{x_k,y_k}(f) = f(x_k) - f(y_k)$. Then $D_{x_k,y_k} \in (E(X))^*$. Clearly

$$T_v^*(D_{x_k,y_k})(f) = D_{x_k,y_k}(T_v f) = f(v(x_k)) - f(v(y_k)) = 0.$$

But $\{D_{x_k,y_k} : k \in N\}$ is an infinite linearly independent set in $[E(X)]^*$ which is contained in ker T_v^*. This contradicts the fact that T_v^* is finite dimensional. Hence the condition (ii) must be true.

The condition (iii) is equivalent to the fact that the composition operator has closed range. □

The converse of the theorem is easy to prove.

Theorem. *Let $T_v \in B(E(X))$. Then T_v is an isometry if and only if v is the identity map.*

Proof. We first suppose that T_v is an isometry. Define

$$f_{x,n}(m) = \begin{cases} x, & \text{if } m = n; \\ 0, & \text{elsewhere.} \end{cases}$$

Then $P_u(T_v f_{x,n}) = P_u(f_{x,n})$, implies that

$$\sup\{(p_u(f_{x,n}(v(m))))^{\frac{1}{m}}, n \geq 1\} = \sup\{(p_u(f_{x,n}(m)))^{\frac{1}{m}}, m \geq 1\} = (p_u(x))^{\frac{1}{n}}.$$

Equivalently:

$$\sup_{m \in v^{-1}(n)} (p_u(x))^{\frac{1}{m}} = (p_u(x))^{\frac{1}{n}}, \qquad (i)$$

since $v^{-1}(n)$ is a finite set so there exists $m_o \in v^{-1}(n)$ such that

$$(p_u(x))^{\frac{1}{m_o}} = \sup_{m \in v^{-1}(n)} (p_u(x))^{\frac{1}{m}} = (p_u(x))^{\frac{1}{m}}. \qquad (ii)$$

From (i) and (ii) we infer that $n = v(n)$. Since $v^{-1}(N) = N$, we have $v = 1$.

Conversely, if the given condition is satisfied, then

$$P_u(T_v f) = \sup\{(p_u(f(v(n))))^{\frac{1}{n}}, n \geq 1\} = P_u(f)$$

for every $f \in E(X)$.

Thus T_v is an isometry. □

Theorem. *Let $T_v \in B(E(X))$. Then T_v has closed range if and only if there exists $b > 0$ such that $\frac{n}{v(n)} \geq b$ for all but finitely many values of n.*

Proof. Suppose first that the condition of the theorem is true. We prove that T_v has closed range. Let $f \in \text{ran}T_v$. Then there exists a sequence $\{f_n\}$ such that $T_v f_n \to f$. Then for given ϵ, $0 < \epsilon < 1$, $(p_u(f_n(v(k)) - f_m(v(k)))^{\frac{1}{k}} < \epsilon$ for all $k \geq k_o$. This implies that

$$(p_u(f_n(v(k)) - f_n(v(k))))^{\frac{1}{k}} < \epsilon^{\frac{k}{v(k)}} < \epsilon^b, \qquad (i)$$

for all k and for all $n, m \geq n_o$. Set

$$f_n^{\sim}(m) = \begin{cases} f_n(m), & \text{if } m \in \text{ran}\, v; \\ 0, & \text{if } m \notin \text{ran}\, v. \end{cases}$$

Then $\{f_n^{\sim}\}$ is a Cauchy sequence in $E(X)$. But $E(X)$ is complete. Therefore there exists $g \in E(X)$ is such that $\lim_n f_n^{\sim} = g$. By continuity of T_v we get $\lim T_v f_n^{\sim} = T_v g$. This implies that $\lim T_v f_n = T_v g$, which shows $f = T_v g$ and $f \in \text{ran}T_v$. Thus ran T_v is closed.

Conversely, if the condition of theorem is not satisfied, then we can find a sequence $\{n_k\}$ of positive integers such that $\frac{n_k}{v(n_k)} < \frac{1}{k}$. Suppose $R > 1$ and $o \neq x \in u$. Take $f_{n_k} = R^{-n_k} x \chi_{v(n_k)}$. Then

$$P_u(f_{n_k}) = \sup\{p_u(R^{-n_k}x))^{\frac{1}{v(n_k)}}, n \geq 1\} = \sup\{(R^{-1}(p_u(x))^{\frac{1}{v(n_k)}}, n \geq 1\} \leq (\frac{1}{R})^{\frac{n_k}{v(n_k)}}.$$

But

$$P_u(T_v f_{n_k}) = \sup\{(p_u(R^{-n_k} x \chi_{v(n_k ov)}(m)))^{\frac{1}{m}}\} = (p_u(R^{-n_k}x))^{\frac{1}{n_k}} = R^{-1}(p_u(x))^{\frac{1}{n_k}},$$

which does not go to zero. This implies that T_v is not bounded away zero on ker T_v, which is a contradiction and hence the theorem. □

Theorem. *Let $T_v \in B(E(X))$. Then T_v is not compact.*

Proof. Since T_v is continuous so $\#\ v(n) = \infty$. Let $\{v(n_k)\}$ be an infinite sequence in $v(N)$. Let $u \in U(X)$ and $0 \neq x \in u$ be such that $p_u(x) = 1$. Consider the sequence $\{f_{x,n_k}\}$. Then $P_u(T_v f_{x,v(n_k)}) \geq (p_u(x))^{\frac{1}{n_k}} = 1$. That is $T_v f_{x,v(n_k)}$ does not go to zero. But $f_{x,v(n_k)}$ is a weakly bounded sequence. Hence T_v is not compact. □

Theorem. *Let $T_v \in B(E(X))$. Then T_v is invertible if and only if v is invertible. There exists $m > 0$ such that $m < \frac{n}{v(n)}$.*

Proof. Suppose the condition (i) and(ii) are true. Then by the Theorem 1 $v^{-1} : N \to N$ induces a composition operator and $T_{v^{-1}} = T_v^{-1}$. Therefore T_v is invertible.

Conversely, suppose that T_v is invertible. If v is not surjective, then $f_{x,n} \in \ker T_v$ and $n \notin v(N)$ so that T_v has non-trivial kernel. Thus v must be surjective. Next if v is not injective, then for two distinct positive integers n_1 and n_2, we have $v(n_1) = v(n_2)$. We can infer that $f \in E(X)$ such that $f(x_1) \neq f(x_2)$, does not belongs to ran T_v, which is a contradiction. Thus v must be injective. This proves that v is invertible.

Next the condition $m < \frac{n}{v(n)}$ implies that $\frac{v(n)}{n} < \frac{1}{m}$ or $\frac{n}{w(n)} < m$, where $w = v^{-1}$. Thus in view of theorem (1) T_w is a bounded composition operator. Clearly $T_w T_v = 1 = T_v T_w$. Thus T_v is invertible with $T_v^{-1} = T_w$. □

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