International Electronic Journal of Pure and Applied Mathematics

Volume 1 No. 4 2010, 459-467

SOME APPLICATIONS OF AN IMPORTANT WEIGHTED REFINEMENT OF DISCRETE JENSEN'S INEQUALITY

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Abstract: In this paper, using an important refinement of discrete Jensen's inequality via two suitable weight functions, we refine some important inequalities, such as AGM, Ky Fan, Sandor and entropy inequalities.

AMS Subject Classification: 26D15, 26A06
Key Words: convexity, weight function, mean

1. Introduction

Undoubtedly, Jensen's inequality is the most important inequality in analysis, because it implies at once the main part of the other classical inequalities (e.g. those by Hölder, Minkowski, Young, and the AGM inequality, etc.). Therefore, it is worth studying it thoroughly and refining it from different points of view. There are many refinements and variants of Jensen's inequality, see e.g. Mercer [3], Rooin [4], Rooin [5], Pečarić [8] and the references in them. In this paper, using an important refinement of discrete Jensen's inequality via two suitable weight functions described below, we give several important applications in sharpening of important inequalities, such as AGM, Ky Fan, Sandor and entropy inequalities.

Throughout this paper, we suppose that C is a convex subset of a real vector space, $x_1, \cdots, x_n \in C$ and $\varphi : C \to \mathbb{R}$ a convex mapping. Also, we suppose that $\mu = (\mu_1, \cdots, \mu_m)$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$ are two probability measures; i.e. $\mu_i,\ \lambda_j \geq$

Received: June 29, 2010

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0 $(1 \le i \le m,\ 1 \le j \le n)$ with

$$\sum_{i=1}^{m} \mu_i = 1 \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j = 1.$$

By a (discrete separately) weight function (with respect to μ and λ), we always mean a mapping

$$\omega : \{(i,j) :\ 1 \le i \le m,\ 1 \le j \le n\} \to [0, \infty),$$

such that

$$\sum_{i=1}^{m} \omega(i,j)\mu_i = 1 \qquad (j = 1, \cdots, n),$$

and

$$\sum_{j=1}^{n} \omega(i,j)\lambda_j = 1 \qquad (i = 1, \cdots, m).$$

For example, if $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_n)$ with $\|u\| = (\sum_{i=1}^{m} u_i^2)^{1/2} \le 1$ and $\|v\| = (\sum_{j=1}^{n} v_j^2)^{1/2} \le 1$ belong to $\mu^\perp$ and $\lambda^\perp$ respectively, then the function ω with $\omega(i,j) = 1 + u_i v_j$ $(1 \le i \le m,\ 1 \le j \le n)$ is a weight function.

Also, we say that a quadratic matrix $A = [a_{ij}]_{n\times n}$ with nonnegative entries is a double stochastic matrix if the sum of each of its rows and columns is unit; that is $\sum_{i=1}^{n} a_{ij} = 1$ $(j = 1, \cdots, n)$ and $\sum_{j=1}^{n} a_{ij} = 1$ $(i = 1, \cdots, n)$.

If ω_1 and ω_2 are two weight functions, we denote by ϕ_{ω_1,ω_2} the real-valued function

$$\phi_{\omega_1,\omega_2}(t) = \sum_{i=1}^{m} \mu_i \varphi\left(\sum_{j=1}^{n} [(1-t)\omega_1(i,j) + t\omega_2(i,j)]\lambda_j x_j\right) \tag{1}$$

$$(0 \le t \le 1).$$

In Rooin [5], the following refinement of discrete Jensen's inequality

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \le \sum_{j=1}^{n} \lambda_j \varphi(x_j) \tag{2}$$

is established:

Theorem A. *If ω_1 and ω_2 are two weight functions, then the function ϕ_{ω_1,ω_2} is convex and bounded on $[0,1]$, and we have*

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \le \int_0^1 \phi_{\omega_1,\omega_2}(t)dt \le \sum_{j=1}^{n} \lambda_j \varphi(x_j). \tag{3}$$

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In particular if C is an interval of $\mathbb{R}$, then

$$\varphi\left(\sum_{j=1}^{n}\lambda_j x_j\right) \leq \sum_{i=1}^{m}\mu_i A\left(\varphi;\sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j,\sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j\right) \leq \sum_{j=1}^{n}\lambda_j\varphi(x_j), \quad (4)$$

where the arithmetic mean A is defined for an integrable function f over an interval with end points a and b, by

$$A(f;a,b)=\frac{1}{b-a}\int_a^b f(x)dx. \quad (5)$$

(*We set $A(f;a,a)=f(a)$.*)

In the following section, using Theorem A, we give some refinements of important inequalities, such as AGM, Ky Fan, Sandor and entropy, which in turn extend the results of Rooin [6] and Rooin [7] as well.

2. Refinements

Throughout this section, we use the terminologies and results of the above section, and as before, we suppose that ω_1 and ω_2 are two weight functions, and $B=[b_{ij}]_{n\times n}$ and $C=[c_{ij}]_{n\times n}$ are two double stochastic matrices.

We recall that the p-logarithmic, identric and logarithmic means of $a,b>0$ are defined respectively by

$$L_p(a,b)=\begin{cases} a & \text{if } a=b,\\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } a\neq b,\end{cases} \quad (6)$$

$$I(a,b)=\begin{cases} a & \text{if } a=b,\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b,\end{cases} \quad (7)$$

$$L(a,b)=\begin{cases} a & \text{if } a=b,\\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b.\end{cases} \quad (8)$$

Note that

$$\lim_{p\to 0}L_p(a,b)=I(a,b) \quad \text{and} \quad \lim_{p\to -1}L_p(a,b)=L(a,b).$$

So, we can take $L_0 = I$ and $L_{-1} = L$.

Theorem 2.1. *If $x_1, x_2, \cdots, x_n$ are n positive real numbers and $p \geq 1$ or $p \leq 0$, then*

$$\left(\sum_{j=1}^{n} \lambda_j x_j\right)^p \leq \sum_{i=1}^{m} \mu_i L_p^p \left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j x_j^p. \quad (9)$$

In particular,

$$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} L_p^p \left(\sum_{j=1}^{n} b_{ij} x_j, \sum_{j=1}^{n} c_{ij} x_j\right) \leq \frac{x_1^p + x_2^p + \cdots + x_n^p}{n}, \quad (10)$$

and

$$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} L_p^p\left(x_i, x_{n+1-i}\right) \leq \frac{x_1^p + x_2^p + \cdots + x_n^p}{n}. \quad (11)$$

Moreover if p is a positive integer, then

$$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^p \leq \frac{1}{n(p+1)}\sum_{i=1}^{n}\sum_{k=0}^{p} x_i^k x_{n+1-i}^{p-k} \leq \frac{x_1^p + x_2^p + \cdots + x_n^p}{n}, \quad (12)$$

$$\left(\frac{x_1 + x_2}{2}\right)^p \leq \frac{1}{p+1}\sum_{k=0}^{p} x_1^k x_2^{p-k} \leq \frac{x_1^p + x_2^p}{2}. \quad (13)$$

Proof. The function $\varphi(x) = x^p$ $(x > 0)$ is convex and $A(\varphi; a, b) = L_p^p(a, b)$ $(a, b > 0)$. Thus, (9) follows from (4).

In particular, (10) follows from (9) by taking $m = n$, $\mu_i = \lambda_j = \frac{1}{n}$, $\omega_1(i,j) = nb_{ij}$, $\omega_2(i,j) = nc_{ij}$ $(i, j = 1, \cdots, n)$.

Also, (11) follows from (10) by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$ $(i, j = 1, \cdots, n)$, where δ_{ij} is the Kronecker delta.

Finally, if p is an positive integer, (12) follows from (11) by taking into account that $L_p^p(a, b) = \frac{1}{p+1}\sum_{k=0}^{p} a^k b^{p-k}$; and (13) follows from (12) by taking $n = 2$. □

Remark 2.2. (i) If $0 < p < 1$, the function $\varphi(x) = x^p$ $(x > 0)$ is concave and so, all inequalities in (9), (10) and (11) are valid in the reversed order.

(ii) If we take pth root from each side of (9) and then let p tends to zero, we get the following known refinement of AGM inequality $\prod_{j=1}^{n} x_j^{\lambda_j} \leq \sum_{j=1}^{n} \lambda_j x_j$, as:

$$\prod_{j=1}^{n} x_j^{\lambda_j} \leq \prod_{i=1}^{m}\left[I\left(\sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right)\right]^{\mu_i} \leq \sum_{j=1}^{n} \lambda_j x_j. \quad (14)$$

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See Rooin [5].

Theorem 2.3. *If $x_j > 0$ $(j = 1, \cdots, n)$, then*

$$\prod_{j=1}^{n} x_j^{\lambda_j} \leq \sum_{i=1}^{m} \mu_i L\left(\prod_{j=1}^{n} x_j^{\omega_1(i,j)\lambda_j}, \prod_{j=1}^{n} x_j^{\omega_2(i,j)\lambda_j}\right) \leq \sum_{j=1}^{n} \lambda_j x_j, \tag{15}$$

which is a new refinement of AGM inequality.

In particular, we have

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{1}{n} \sum_{i=1}^{n} L\left(\prod_{j=1}^{n} x_j^{b_{ij}}, \prod_{j=1}^{n} x_j^{c_{ij}}\right) \leq \frac{x_1 + \cdots + x_n}{n}, \tag{16}$$

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{1}{n} \sum_{i=1}^{n} L\left(x_i, x_{n+1-i}\right) \leq \frac{x_1 + \cdots + x_n}{n}, \tag{17}$$

$$\sqrt{x_1 x_2} \leq L\left(x_1, x_2\right) \leq \frac{x_1 + x_2}{2}. \tag{18}$$

Proof. The function $\varphi(x) = e^x$ is convex on $\mathbb{R}$ and $A(\varphi; a, b) = L(e^a, e^b)$. Now, substituting $\ln x_j$ instead of x_j $(1 \leq j \leq n)$ in (4) we get (15).

In particular, (16) follows from (15) by taking $m = n,\ \mu_i = \lambda_j = \frac{1}{n},\ \omega_1(i,j) = nb_{ij},\ \omega_2(i,j) = nc_{ij}$ $(i, j = 1, \cdots, n)$.

Also, (17) follows from (16) by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$ $(i, j = 1, \cdots, n)$, where δ_{ij} is the Kronecker delta.

Finally, (18) is an special case of (17), taking $n = 2$. □

Theorem 2.4. *If $x_j \in (0, 1/2]$ $(j = 1, \cdots, n)$, and $A_n = \sum_{j=1}^{n} \lambda_j x_j$ and $G_n = \prod_{j=1}^{n} x_j^{\lambda_j}$ (also, $A'_n = \sum_{j=1}^{n} \lambda_j (1 - x_j)$ and $G'_n = \prod_{j=1}^{n} (1 - x_j)^{\lambda_j}$) are the arithmetic and geometric means of $x_1, \cdots, x_n$ (of $1 - x_1, \cdots, 1 - x_n$) respectively, then we have the following refinement of Ky Fan's inequality $\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}$, see Beckenbach et al* [1], *as:*

$$\begin{aligned} \frac{A'_n}{G'_n} &\leq \frac{1}{G'_n} \sum_{i=1}^{m} \mu_i \left(1 - \frac{L(u_i, v_i)}{L(1 + u_i, 1 + v_i)}\right) \leq \frac{1}{G_n + G'_n} \\ &\leq \frac{1}{G_n} \sum_{i=1}^{m} \mu_i \frac{L(u_i, v_i)}{L(1 + u_i, 1 + v_i)} \leq \frac{A_n}{G_n}, \end{aligned} \tag{19}$$

where

$$u_i = \prod_{j=1}^{n} \left(\frac{x_j}{1 - x_j}\right)^{\omega_1(i,j)\lambda_j} \quad \textit{and} \quad v_i = \prod_{j=1}^{n} \left(\frac{x_j}{1 - x_j}\right)^{\omega_2(i,j)\lambda_j}.$$

In particular,

$$\frac{A'_n}{G'_n} \le \frac{1}{G'_n}\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{L\left(\frac{x_i}{1-x_i},\frac{x_{n+1-i}}{1-x_{n+1-i}}\right)}{L\left(\frac{1}{1-x_i},\frac{1}{1-x_{n+1-i}}\right)}\right) \le \frac{1}{G_n+G'_n} \tag{20}$$

$$\le \frac{1}{G_n}\frac{1}{n}\sum_{i=1}^{n}\frac{L\left(\frac{x_i}{1-x_i},\frac{x_{n+1-i}}{1-x_{n+1-i}}\right)}{L\left(\frac{1}{1-x_i},\frac{1}{1-x_{n+1-i}}\right)} \le \frac{A_n}{G_n},$$

where $\lambda_j = \frac{1}{n}$ $(j=1,\cdots,n)$.

Proof. Consider the function $\varphi(x) = \frac{1}{1+e^x}$. Since $\varphi''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3}$, φ is convex on $[0,\infty)$ and concave on $(-\infty,0]$. Now the last two inequalities in (19) follow from (4) by taking $\ln\frac{1-x_j}{x_j}$ instead of x_j, taking into account that

$$A(\varphi;a,b) = \frac{L(e^{-a},e^{-b})}{L(1+e^{-a},1+e^{-b})} \qquad (a,b\in\mathbb{R}) \tag{21}$$

and dividing each side by G_n.

Similarly, the first two inequalities in (19) follow from the concavity of φ on $(-\infty,0]$, by taking $\ln\frac{x_j}{1-x_j}$ instead of x_j in the reversed order form of (4) and taking into account that

$$\frac{L(a^{-1},b^{-1})}{L(1+a^{-1},1+b^{-1})} = 1-\frac{L(a,b)}{L(1+a,1+b)} \qquad (a,b>0). \tag{22}$$

In particular, (20) follows from (19) by taking $m=n$, $\mu_i=\lambda_j=1/n$, $\omega_1(i,j) = n\delta_{ij}$, and $\omega_2(i,j) = n\delta_{i,n+1-j}$ $(i,j=1,\cdots,n)$. □

Theorem 2.5. *If* $x_j\in(0,1/2]$ $(j=1,\cdots,n)$, *and* $A_n=\sum_{j=1}^{n}\lambda_j x_j$ *and* $H_n = \left(\sum_{j=1}^{n}\lambda_j x_j^{-1}\right)^{-1}$ *(also,* $A'_n=\sum_{j=1}^{n}\lambda_j(1-x_j)$ *and* $H'_n = (\sum_{j=1}^{n}\lambda_j(1-x_j)^{-1})^{-1}$ *) are the arithmetic and harmonic means of* $x_1,\cdots,x_n$ *(of* $1-x_1,\cdots,1-x_n$ *) respectively, then we have the following refinement of Sandor's inequality* $\frac{1}{A_n}-\frac{1}{A'_n}\le\frac{1}{H_n}-\frac{1}{H'_n}$, *see Sandor* [9], *as:*

$$\frac{1}{A_n}-\frac{1}{A'_n} \le \sum_{i=1}^{m}\frac{\mu_i}{L(\sum_{j=1}^{n}\omega_1(i,j)\lambda_j x_j,\sum_{j=1}^{n}\omega_2(i,j)\lambda_j x_j)} \tag{23}$$

$$-\sum_{i=1}^{m}\frac{\mu_i}{L(\sum_{j=1}^{n}\omega_1(i,j)\lambda_j(1-x_j),\sum_{j=1}^{n}\omega_2(i,j)\lambda_j(1-x_j))}$$

$$\le \frac{1}{H_n}-\frac{1}{H'_n}.$$

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In particular, see Rooin [7],

$$\frac{1}{A_n} - \frac{1}{A'_n} \le \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{L(\sum_{j=1}^{n} b_{ij}x_j, \sum_{j=1}^{n} c_{ij}x_j)} - \frac{1}{L(\sum_{j=1}^{n} b_{ij}(1-x_j), \sum_{j=1}^{n} c_{ij}(1-x_j))}\Big] \le \frac{1}{H_n} - \frac{1}{H'_n}, \tag{24}$$

where $\lambda_j = \frac{1}{n}$ $(j = 1, \cdots, n)$.

Proof. The function $\varphi(x) = \frac{1}{x} - \frac{1}{1-x}$ is convex on $(0, 1/2]$ and

$$A(\varphi; a, b) = \frac{1}{L(a,b)} - \frac{1}{L(1-a, 1-b)} \qquad (0 < a, b < 1).$$

Thus, (23) follows from (4) by taking into account that

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) = \frac{1}{A_n} - \frac{1}{A'_n}, \qquad \sum_{j=1}^{n} \lambda_j \varphi(x_j) = \frac{1}{H_n} - \frac{1}{H'_n}.$$

In particular, (24) follows from (23) by taking $m = n$, $\mu_i = \lambda_j = \frac{1}{n}$, $\omega_1(i,j) = nb_{ij}$ and $\omega_2(i,j) = nc_{ij}$ $(i, j = 1, \cdots, n)$. □

Theorem 2.6. *If* $x_1, x_2, \cdots, x_n > 0$, *then*

$$\left(\sum_{j=1}^{n} \lambda_j x_j\right)^{\sum_{j=1}^{n} \lambda_j x_j} \le \sqrt{\prod_{i=1}^{m} I\left(\left[\sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j\right]^2, \left[\sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right]^2\right)^{\mu_i \sum_{j=1}^{n} \frac{\omega_1(i,j)+\omega_2(i,j)}{2}\lambda_j x_j}} \le \prod_{j=1}^{n} (x_j^{x_j})^{\lambda_j}. \tag{25}$$

In particular, if $p_1, p_2, \cdots, p_n > 0$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$\frac{1}{n} \le \sqrt{\prod_{i=1}^{n} I\left(\left[\sum_{j=1}^{n} b_{ij}p_j\right]^2, \left[\sum_{j=1}^{n} c_{ij}p_j\right]^2\right)^{\sum_{j=1}^{n} \frac{b_{ij}+c_{ij}}{2}p_j}} \le \prod_{j=1}^{n} p_j^{p_j}, \tag{26}$$

which gives a refinement of entropy inequality $-\sum_{j=1}^{n} p_j \ln p_j \leq \ln n$*; see Jones et al* [2]. *Moreover,*

$$\frac{1}{n} \leq \sqrt{\prod_{i=1}^{n} I\left(p_i^2, p_{n+1-i}^2\right)^{\frac{p_i+p_{n+1-i}}{2}}} \leq \prod_{j=1}^{n} p_j^{p_j}, \tag{27}$$

and

$$\frac{1}{2} \leq \sqrt{I\left(p^2, q^2\right)} \leq p^p q^q \qquad (p, q > 0,\ p+q=1). \tag{28}$$

Proof. The function $\varphi(x) = x \ln x$ is convex on $(0, +\infty)$ and

$$A(\varphi; a, b) = \frac{a+b}{4} \ln I(a^2, b^2) \qquad (a, b > 0). \tag{29}$$

Thus, the inequalities in (25) follow from (4).

In particular, (26) follows from (25) by taking p_j instead of x_j, $m = n$, $\mu_i = \lambda_j = \frac{1}{n}$, $\omega_1(i, j) = nb_{ij}$ and $\omega_2(i, j) = nc_{ij}$ $(i, j = 1, \cdots, n)$.

Also, (27) follows from (26) by taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$ $(i, j = 1, 2, \cdots, n)$, where δ_{ij} is the Kronecker delta.

Finally, (28) is an special case of (27), taking $n = 2$, $p_1 = p$ and $p_2 = q$. □

Acknowledgments

The author is supported in part by the Institute for Advanced Studies in Basic Sciences, Zanjan, IRAN.

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