International Electronic Journal of Pure and Applied Mathematics

Volume 1 No. 4 2010, 445-458

HOPF BIFURCATION FOR A PREDATOR-PREY SYSTEM WITH DISCRETE AND DISTRIBUTED DELAYS

Guang-Ping Hu[1 §], Xiao-Ling Li[2]

[1,2]School of Mathematics and Physics
Nanjing University of Information and Technology
Nanjing, 210044, P.R. CHINA
[1]e-mail: hugp@nuist.edu.cn
[2]e-mail: lxl871@nuist.edu.cn

Abstract: This paper is concerned with a Lotka-Volterra two species predator-prey system with both discrete and distributed delays. By regarding the discrete time delay as the bifurcation parameter, the stability of positive equilibrium and Hopf bifurcations induced by time delay are investigated. Finally, to verify our theoretical predictions, some numerical simulations are also included.

AMS Subject Classification: 34C23, 34C25, 92D25
Key Words: prey-predator system, time delay, stability, Hopf bifurcation

1. Introduction

In 1973, May [3] first proposed and discussed briefly the dynamics of the following delayed Lotka-Volterra predator-prey system with a single discrete delay

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1 - a_{11}x(t-\tau) - a_{12}y(t)\right], \\ \dot{y}(t) = y(t)\left[-r_2 + a_{21}x(t) - a_{22}y(t)\right]. \end{cases} \tag{1}$$

Subsequently, by regarding the delay τ as the bifurcation parameter, Song, Han and Wei [4] investigated the effect of delay τ on the dynamics of system (1) and it was found that the increase of delay not only can destabilize the positive equilibrium and also can lead to the stability switch of positive equilibrium. By verifying the conditions of Hopf bifurcation theorem to FDEs, they also found that system (1) can undergo a Hopf bifurcation when the delay τ crosses through a sequence of critical values.

Received: June 28, 2010

§Correspondence author

Recently, by incorporating the delay τ into the predator density of the second equation of system (1), Yan and Li [7] considered the following system

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1 - a_{11}x(t-\tau) - a_{12}y(t)\right], \\ \dot{y}(t) = y(t)\left[-r_2 + a_{21}x(t) - a_{22}y(t-\tau)\right], \end{cases} \tag{2}$$

In this case, by taking the delay τ as the bifurcation parameter, the authors also studied the effect of delay τ on the dynamics of (2) and found that the increase of delay τ in system (2) can only destabilize the positive equilibrium. Motivated by the above works, in the present paper, we consider an analogue of system (2) containing both discrete and diagonal distributed delays, which takes the following form:

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1 - a_{11}\int_{-\infty}^{t} F(t-s)x(s)ds - a_{12}y(t-\tau)\right], \\ \dot{y}(t) = y(t)\left[-r_2 + a_{21}x(t-\tau) - a_{22}\int_{-\infty}^{t} F(t-s)y(s)ds\right], \end{cases} \tag{3}$$

where $F(\cdot)$ is nonnegative bounded delay kernel defined on $[0, +\infty)$ which reflect the influence of the past states on the current dynamics.

In this model, the presence of the distributed time delay must not affect the equilibrium values, so we normalize the kernel such that

$$\int_0^\infty F(s)ds = 1.$$

Following the ideas of Cushing et al [1], the weak kernel

$$F(s) = \alpha e^{-\alpha s}, \quad \alpha > 0, \tag{4}$$

and the strong kernel

$$F(s) = \alpha^2 s e^{-\alpha s}, \quad \alpha > 0, \tag{5}$$

are frequently encountered in the literature. Due to the accompanying analytical convenience, in this paper, we consider system (3) with the weak kernel (4), then system (3) should be modified as the following delayed predator-prey system:

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1 - a_{11}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}x(s)ds - a_{12}y(t-\tau)\right], \\ \dot{y}(t) = y(t)\left[-r_2 + a_{21}x(t-\tau) - a_{22}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}y(s)ds\right]. \end{cases} \tag{6}$$

Taking discrete delay τ as the bifurcation parameter, we shall investigate the effect of the delay τ on the dynamics of system (6). It is well known that the zero solution of system (6) is asymptotically stable if and only if all the roots of (6) have negative real parts. However, there are few explicit conditions that guarantee the asymptotic stability of the zero solution of (6) (also see [8, 2, 5]).

In the following of this paper, we shall study the stability of the positive equilibrium and the existence of Hopf bifurcation. In order to verify our theoretical prediction, some numerical simulations are also included.

2. Stability Analysis and Hopf Bifurcation

It is obvious that system (1) has a unique positive equilibrium $E_*(x^*, y^*)$ provided that the condition

(H1) $r_1a_{21} - r_2a_{11} > 0$

is satisfied, where

$$x^* = \frac{r_1a_{22} + r_2a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \; y^* = \frac{r_1a_{21} - r_2a_{11}}{a_{11}a_{22} + a_{12}a_{21}}.$$

Let

$$u(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} x(s)ds, \quad v(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} y(s)ds.$$

By the linear chain trick technique, system (6) can be transformed into the following system:

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1 - a_{11}u(t) - a_{12}y(t-\tau)\right], \\ \dot{y}(t) = y(t)\left[-r_2 + a_{21}x(t-\tau) - a_{22}v(t)\right], \\ \dot{u}(t) = \alpha x(t) - \alpha u(t), \\ \dot{v}(t) = \alpha y(t) - \alpha v(t). \end{cases} \tag{7}$$

It is easy to see that $(x^*, y^*, u^*, v^*) = (x^*, y^*, x^*, y^*)$ is an equilibrium of system (7) under the hypothesis (H1). Let $u_1(t) = x(t) - x^*$, $u_2(t) = y(t) - y^*$, $u_3(t) = u(t) - u^*$, $u_4(t) = v(t) - v^*$, then system (7) can be rewritten as the following equivalent system

$$\begin{cases} \dot{u}_1(t) = (u_1(t) + x^*)\left[-a_{11}u_3(t) - a_{12}u_2(t-\tau)\right], \\ \dot{u}_2(t) = (u_2(t) + y^*)\left[a_{21}u_1(t-\tau) - a_{22}u_4(t)\right], \\ \dot{u}_3(t) = \alpha u_1(t) - \alpha u_3(t), \\ \dot{u}_4(t) = \alpha u_2(t) - \alpha u_4(t). \end{cases} \tag{8}$$

Thus, the positive equilibrium (x^*, y^*, u^*, v^*) of system (7) is transformed into the trivial equilibrium $(0, 0, 0, 0)$ of system (8). Linearizing system (8) about the origin $(0, 0, 0, 0)$ yields the following linear system

$$\begin{cases} \dot{u}_1(t) = -a_{11}x^*u_3(t) - a_{12}x^*u_2(t-\tau), \\ \dot{u}_2(t) = a_{21}y^*u_1(t-\tau) - a_{22}y^*u_4(t), \\ \dot{u}_3(t) = \alpha u_1(t) - \alpha u_3(t), \\ \dot{u}_4(t) = \alpha u_2(t) - \alpha u_4(t), \end{cases} \tag{9}$$

the associated characteristic equation of system (9) is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-2\lambda\tau} = 0, \tag{10}$$

where

$$a_3 = 2\alpha, \; a_2 = \alpha(a_{11}x^* + a_{22}y^* + \alpha), \; a_1 = \alpha^2(a_{11}x^* + a_{22}y^*), \; a_0 = \alpha^2 a_{11}a_{22}x^*y^*,$$

$$b_2 = a_{12}a_{21}x^*y^*, \quad b_1 = 2\alpha a_{12}a_{21}x^*y^*, \quad b_0 = \alpha^2 a_{12}a_{21}x^*y^*.$$

The stability of the origin $(0,0,0,0)$ of system (8) depends on the locations on the complex plane of the roots of the characteristic equation (10). When all roots of (10) locate on the left half-plane of complex plane, the origin $(0,0,0,0)$ of system (10) is stable; otherwise, it is unstable.

When $\tau = 0$, (10) becomes

$$\lambda^4 + a_3\lambda^3 + (a_2+b_2)\lambda^2 + (a_1+b_1)\lambda + a_0 + b_0 = 0. \tag{11}$$

Note that $\alpha > 0$, by the Routh-Hurwitz criterion, a set of the necessary and sufficient conditions for all roots of (10) to have negative real parts are given by

$$D_1 = \det\begin{pmatrix} a_3 & a_1+b_1 \\ 1 & a_2+b_2 \end{pmatrix} = a_3(a_2+b_2) - (a_1+b_1) > 0,$$

$$D_2 = \det\begin{pmatrix} a_3 & a_1+b_1 & 0 \\ 1 & a_2+b_2 & a_0+b_0 \\ 0 & a_3 & a_1+b_1 \end{pmatrix} = a_3[(a_2+b_2)(a_1+b_1) - a_3(a_0+b_0)] - (a_1+b_1)^2 > 0,$$

$$D_3 = \det\begin{pmatrix} a_3 & a_1+b_1 & 0 & 0 \\ 1 & a_2+b_2 & a_0+b_0 & 0 \\ 0 & a_3 & a_1+b_1 & 0 \\ 0 & 1 & a_2+b_2 & a_0+b_0 \end{pmatrix} = a_0 + b_0 > 0.$$

Thus, one can immediately obtain the following result.

Lemma 1. *Assume that*

(H) $D_1 > 0, D_2 > 0, D_3 > 0.$

Then all the roots of equation (10) *with* $\tau = 0$ *have always negative real parts.*

Next, we shall investigate the distribution of roots of equation (10) with $\tau > 0$. Obviously, $i\omega(\omega > 0)$ is a root of equation (10) if and only if ω satisfies the following equation

$$\omega^4 - ia_3\omega^3 - a_2\omega^2 + a_0 + (-b_2\omega^2 + ib_1\omega + b_0)(\cos 2\omega\tau - i\sin 2\omega\tau) = 0. \tag{12}$$

Separating the real and imaginary parts of equation (12) gives the following equations

$$\begin{cases} \omega^4 - a_2\omega^2 + a_0 = (b_2\omega^2 - b_0)\sin 2\omega\tau + b_1\omega\cos 2\omega\tau, \\ a_3\omega^3 - a_1\omega = (b_2\omega^2 - b_0)\cos 2\omega\tau - b_1\omega\sin 2\omega\tau. \end{cases} \tag{13}$$

Adding up the squares of the corresponding sides of the above equations leads to

$$(\omega^4 - a_2\omega^2 + a_0)^2 + (a_3\omega^3 - a_1\omega)^2 = (b_2\omega^2 - b_0)^2(b_1\omega)^2. \tag{14}$$

It follows from (14) that

$$\omega^8 + a\omega^6 + b\omega^4 + c\omega^2 + d = 0, \tag{15}$$

where

$$a = a_3^2 - 2a_2,\ b = a_2^2 + 2a_0 - 2a_1a_3 - b_2^2,\ c = a_1^2 - 2a_0a_2 + 2b_0b_1 - b_1^2,\ d = a_0^2 - b_0^2.$$

Let $z = \omega^2$, then (15) becomes

$$h(z) := z^4 + az^3 + bz^2 + cz + d = 0. \tag{16}$$

From (16), we have

$$g(z) := \frac{dh(z)}{dz} = 4z^3 + 3az^2 + 2bz + c. \tag{17}$$

Let $y = z + \frac{a}{4}$, then the equation $g(z) = 0$ becomes

$$y^3 + my + n = 0,$$

where

$$m = \frac{8b - 3a^2}{16},\ n = \frac{a^3 - 4ab + 8c}{32}.$$

Define

$$\begin{aligned} &D = \frac{n^2}{4} + \frac{m^3}{27},\quad \sigma = \frac{-1 + i\sqrt{3}}{2},\quad z_i = y_i - \frac{a}{4},\ i = 1,2,3,\\ &y_1 = \sqrt[3]{-\frac{n}{2} + \sqrt{D}} + \sqrt[3]{-\frac{n}{2} - \sqrt{D}},\\ &y_2 = \sqrt[3]{-\frac{n}{2} + \sqrt{D}}\sigma + \sqrt[3]{-\frac{n}{2} - \sqrt{D}}\sigma^2,\\ &y_3 = \sqrt[3]{-\frac{n}{2} + \sqrt{D}}\sigma^2 + \sqrt[3]{-\frac{n}{2} - \sqrt{D}}\sigma. \end{aligned}$$

Assume that $D > 0$, then from the Cardano's formulae for the third-degree algebra equation we know that the equation $g(z) = 0$ has only one real root $z_1^* = z_1$. If $D = 0$, then the equation $g(z) = 0$ have three real roots z_1, z_2 and z_3 (where $z_2 = z_3$), and in this case we define z_2^* by $\max\{z_1, z_2\}$. If $D < 0$, we know that the equation $g(z) = 0$ has three different real roots z_1, z_2 and z_3. In the last case, we define z_3^* by $z_3^* = \max\{z_1, z_2, z_3\}$.

According to Lemma 2.1 in Yan and Li [7], we have the following

Lemma 2. **(i)** *If* $d < 0$, *then* (16) *has at least one positive root.*

(ii) *If* $d \geq 0$, *then* (16) *has no positive root if one of the following conditions holds:*

(a) $D > 0$ *and* $z_1^* \leq 0$; (b) $D = 0$ *and* $z_2^* \leq 0$; (c) $D < 0$ *and* $z_3^* \leq 0$.

(iii) *If* $d \geq 0$, *then* (16) *has at least a positive root if one of the following conditions holds:*

(a) $D > 0$, $z_1^* > 0$ *and* $h(z_1^*) < 0$; (b) $D = 0$, $z_2^* > 0$ *and* $h(z_2^*) < 0$; (c) $D < 0$, $z_3^* \leq 0$ *and* $h(z_3^*) < 0$.

Without loss of generality, suppose that equation (16) has four positive real roots, denoted by z_1, z_2, z_3, z_4, respectively. Then (15) should also have four positive real roots

$$\omega_1 = \sqrt{z_1},\ \omega_2 = \sqrt{z_2}\ ,\ \omega_3 = \sqrt{z_3}\ ,\ \omega_4 = \sqrt{z_4}\ .$$

From the first equation of (13), we have

$$\begin{cases} \sin 2\omega_k\tau = \dfrac{(a_3\omega_k^3 - a_1\omega_k)(b_2\omega_k^2 - b_0) - b_1\omega_k(\omega_k^4 - a_2\omega_k^2 + a_0)}{(b_2\omega_k^2 - b_0)^2 + b_1^2\omega_k^2}, \\ \cos 2\omega_k\tau = \dfrac{(\omega_k^4 - a_2\omega_k^2 + a_0)(b_2\omega_k^2 - b_0) + b_1\omega_k(a_3\omega_k^2 - a_1\omega_k)}{(b_2\omega_k^2 - b_0)^2 + b_1^2\omega_k^2}. \end{cases} \tag{18}$$

Define

$$\tau_k^j = \frac{1}{2\omega_k} \times \left[\arctan\left(\frac{(a_3\omega_k^3 - a_1\omega_k)(b_2\omega_k^2 - b_0) - b_1\omega_k(\omega_k^4 - a_2\omega_k^2 + a_0)}{(\omega_k^4 - a_2\omega_k^2 + a_0)(b_2\omega_k^2 - b_0) + b_1\omega_k(a_3\omega_k^2 - a_1\omega_k)}\right) + j\pi\right], \tag{19}$$

where $k = 1, 2, 3, 4,\ j = 0, 1, 2, \ldots$. Then (τ_k^j, ω_k) are solutions of equation (12) and $\lambda = \pm i\omega_k$ are a pair of purely imaginary roots of (10) with $\tau = \tau_k^j$. Define

$$\tau_0 = \tau_{k_0}^0 = \min_{1\leq k\leq 4}\{\tau_k^0\},\ \omega_0 = \omega_{k_0},$$

where $k_0 \in \{1, 2, 3, 4\}$. Then τ_0 is the first value of τ such that (10) have purely imaginary roots.

Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the root of equation (10) near $\tau = \tau_k^j$ satisfying $\alpha(\tau_k^j) = 0$, $\omega(\tau_k^j) = \omega_0 (j = 0, 1, 2, \ldots)$.

Lemma 3. *Suppose* $h'(z_k) \neq 0$, *where* $h(z)$ *is defined by* (16), *then the following transversality conditions are satisfied:*

$$\left.\frac{dRe\lambda(\tau)}{d\tau}\right|_{\tau=\tau_k^j} \neq 0.$$

Moreover, the sign of $\left.\frac{dRe\lambda(\tau)}{d\tau}\right|_{\tau=\tau_k^j}$ *is consistent with that of* $h'(z_k)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Proof. Differentiating the two sides of equation (10) regarding τ and noticing that λ is a function of τ, we can obtain

$$(4\lambda^3 + 3a_3^2\lambda^2 + 2a_2\lambda + a_1)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + (2b_2\lambda + b_1)e^{-2\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} - 2(b_2\lambda^2 + b_1\lambda + b_0)e^{-2\lambda\tau}\left(\tau\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} + \lambda\right) = 0.$$

This gives

$$\begin{aligned}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} &= \frac{(4\lambda^3 + 3a_3^2\lambda^2 + 2a_2\lambda + a_1) + (2b_2\lambda + b_1)e^{-2\lambda\tau} - 2(b_2\lambda^2 + b_1\lambda + b_0)\tau e^{-2\lambda\tau}}{2(b_2\lambda^2 + b_1\lambda + b_0)\lambda e^{-2\lambda\tau}} \\ &= \frac{(4\lambda^3 + 3a_3^2\lambda^2 + 2a_2\lambda + a_1)e^{2\lambda\tau}}{2(b_2\lambda^2 + b_1\lambda + b_0)\lambda} + \frac{(2b_2\lambda + b_1)}{2(b_2\lambda^2 + b_1\lambda + b_0)\lambda} - \frac{\tau}{\lambda}.\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{sign}\left[\frac{\mathrm{d}(\mathsf{Re}\lambda)}{\mathrm{d}\tau}\right]_{\tau=\tau_k^j} &= \operatorname{sign}\left[\mathsf{Re}\left(\frac{(4\lambda^3 + 3a_3^2\lambda^2 + 2a_2\lambda + a_1)e^{2\lambda\tau}}{2(b_2\lambda^2 + b_1\lambda + b_0)\lambda} + \frac{(2b_2\lambda + b_1)}{2(b_2\lambda^2 + b_1\lambda + b_0)\lambda} - \frac{\tau}{\lambda}\right)^{-1}\right]_{\tau=\tau_k^j} \\ &= \operatorname{sign}\mathsf{Re}\left[\frac{[a_1 - 3a_3^2\omega_0^2 + (2a_2\omega_0 - 4\omega_0^3)i](\cos 2\omega_0\tau_k^j + i\sin 2\omega_0\tau_k^j)}{-2b_1\omega_0^2 + 2(-b_2\omega_0^2 + b_0)i\omega_0} + \frac{b_1 + 2b_2\omega_0 i}{-2b_1\omega_0^2 + 2(-b_2\omega_0^2 + b_0)i\omega_0}\right] \\ &= \operatorname{sign}\frac{1}{R_0}\left[\{(a_1 - a_3^2\omega_0^2)\cos 2\omega_0\tau_k^j - (2a_2\omega_0 - 4\omega_0^3)\sin 2\omega_0\tau_k^j\}(-2b_1\omega_0^2) + 2\{(a_1 - 3a_3^2\omega_0^2)\sin 2\omega_0\tau_k^j + (2a_2\omega_0 - 4\omega_0^3)\cos 2\omega_0\tau_k^j\}(-b_2\omega_0^2 + b_0)\omega_0 - 2b_1^2\omega_0^2 + 4b_2\omega_0^2(-b_2\omega_0^2 + b_0)\right] \\ &= \operatorname{sign}\frac{1}{R_0}\left[2\omega_0(a_3^2\omega_0^2 - a_1)\{b_1\omega_0\cos\omega_0\tau_k^j + (b_2\omega_0^2 - b_0)\sin\omega_0\tau_k^j\} + 4(2\omega_0^4 - a_2\omega_0^2)\{(b_2\omega_0^2 - b_0)\cos\omega_0\tau_k^j - b_1\omega_0\sin\omega_0\tau_k^j\} - 2b_1^2\omega_0^2 + 4b_2\omega_0^2(-b_2\omega_0^2 + b_0)\right] \\ &= \operatorname{sign}\frac{1}{R_0}\left[2\omega_0(a_3^2\omega_0^2 - a_1)\{\omega_0^4 - a_2\omega_0^2 + a_0\} + 4(2\omega_0^4 - a_2\omega_0^2)\right.\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

$$\begin{aligned}
&\{a_3\omega_0^3 - a_1\omega_0\} - 2b_1^2\omega_0^2 + 4b_2\omega_0^2(-b_2\omega_0^2 + b_0)\big] \\
&= \text{sign}\frac{1}{R_0}\left[(4\omega_0^6 + 3(a_3^2 - 2a_2)\omega_0^4 + 2(a_2^2 + 2a_0 - 2a_1a_3 - b_2^2)\omega_0^2\right. \\
&\qquad \left. + a_1^2 - 2a_0a_2 + 2b_0b_1 - b_1^2\right] \\
&= \text{sign}\frac{1}{R_0}\left[4\omega_0^6 + 3a\omega_0^4 + 2b\omega_0^2 + c\right] = \text{sign}\frac{\omega_0^2}{R_0}\left[h'(\omega_0^2)\right] = \text{sign}\left[h'(\omega_0^2)\right],
\end{aligned}$$

where $R_0 = 4b_1^2\omega_0^4 + 4\left(b_0 - b_2\omega_0^2\right)^2$. It follows from the hypothesis $h'(z_k) \neq 0$ that $h'(\omega_0^2) \neq 0$ and therefore the transversality conditions are satisfied. □

Since the multiplicity of roots with positive real parts of equation (10) can change only if a root appears on or crosses the imaginary axis as time delay τ varies, similar to the proof of the lemma of Wei and Ruan [6], by Lemma 3, we have the following result.

Lemma 4. *Suppose that equation* (16) *has at least one positive root. If* $\tau \in (\tau_k^j, \tau_k^{j+1})$, *then equation* (10) *has at least* $2(j+1)(j = 0, 1, 2, \ldots)$ *roots with positive real part.*

By Lemmas 1-4, we have the following result regarding on stability and bifurcation of system (1).

Theorem 5. *Suppose that (H) holds, we have the following:*

(i) *All roots of* (10) *have negative real parts and the zero solution of system* (8) *is absolutely stable if* $d \geq 0$ *and one of the following conditions is satisfied:*

(a) $D > 0$ *and* $z_1^* \leq 0$; (b) $D = 0$ *and* $z_2^* \leq 0$; (c) $D < 0$ *and* $z_3^* \leq 0$.

(ii) *All roots of* (10) *have negative real parts and the zero solution of system* (8) *is asymptotically stable for* $\tau \in [0, \tau_0)$ *if* $d < 0$ *or* $d \geq 0$ *and one of the following conditions is satisfied:*

(a) $D > 0$, $z_1^* > 0$ *and* $h(z_1^*) < 0$; (b) $D = 0$, $z_2^* > 0$ *and* $h(z_2^*) < 0$; (c) $D < 0$, $z_3^* \leq 0$ *and* $h(z_3^*) < 0$.

(iii) *if the conditions as stated in (ii) are satisfied, and* $h'(z_k) \neq 0$, *then system* (8) *undergoes Hopf bifurcations at* $\tau = \tau_k^j (k = 1, 2, 3, 4, j = 0, 1, 2, \ldots)$.

3. A Numerical Example

In this section, we give some numerical simulations to illustrate our results. As an example, we consider system (6) with $a_{11} = 0.5$, $a_{12} = 1.8$, $a_{21} = 1.3$, $a_{22} =$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

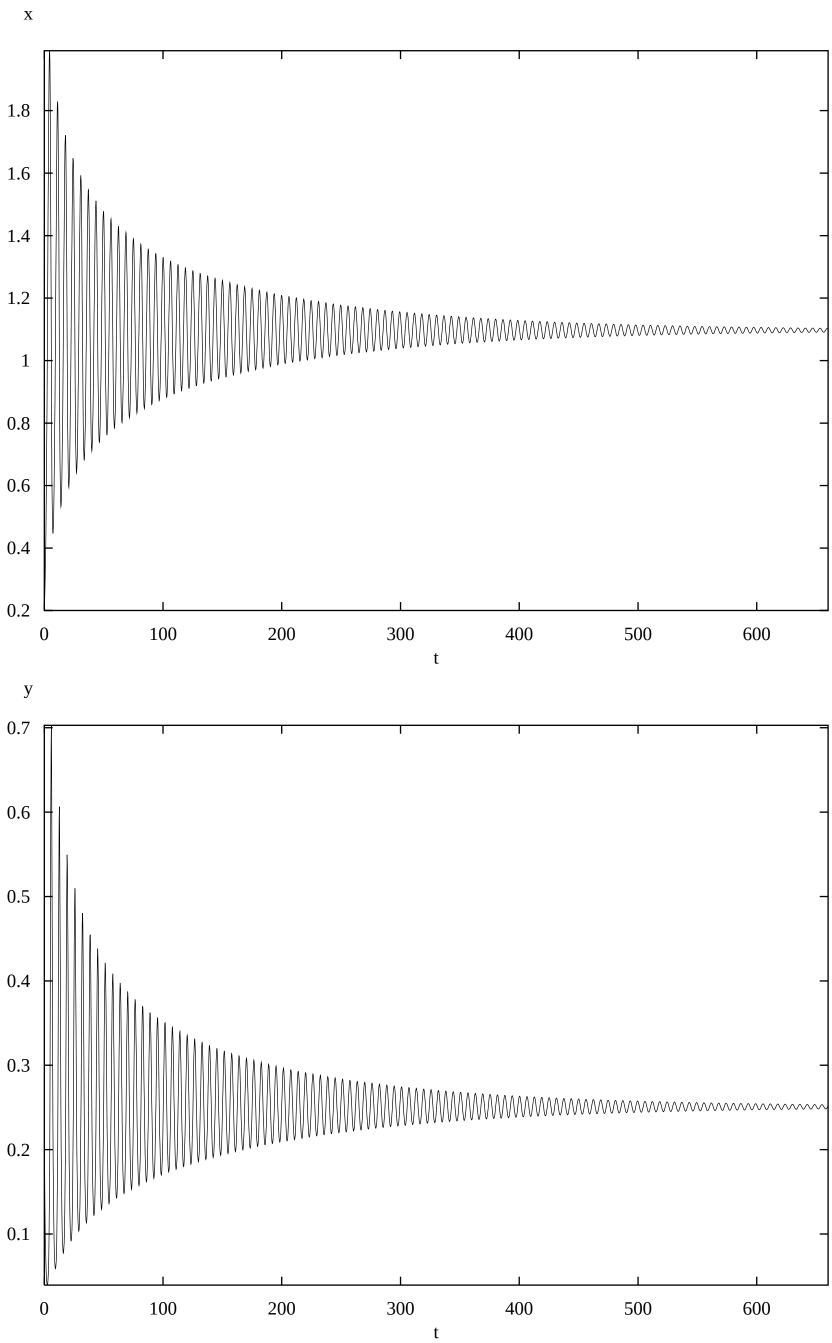

Figure 1: The trajectory graph of system (20) with $\tau = 0.179$ and initial data $x(t) = y(t) = 0.2, t \in [-0.179, 0]$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

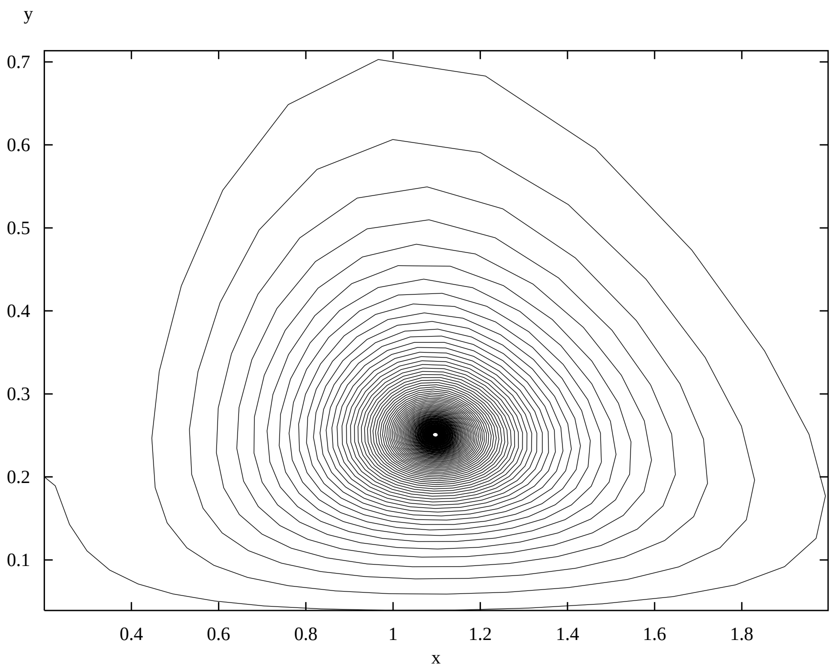

Figure 2: The phase graph of system (20) with $\tau = 0.179$ and initial data $x(t) = 0.2, y(t) = 0.2, t \in [-0.179, 0]$ in the $x - y$ plane

1.7, $r_1 = r_2 = 1$, and kernel $F(s) = e^{-s}$, i,e., $\alpha = 1$. In this case, the form of system (6) is

$$\begin{cases} \dot{x}(t) = x(t)\left[1 - 0.5\int_{-\infty}^{t} \alpha e^{-(t-s)} x(s)ds - 1.8y(t-\tau)\right], \\ \dot{y}(t) = y(t)\left[-1 + 1.3x(t-\tau) - 1.7\int_{-\infty}^{t} \alpha e^{-(t-s)} y(s)ds\right]. \end{cases} \tag{20}$$

From the theoretical analysis in Section 2, then (20) becomes the following system

$$\begin{cases} \dot{x}(t) = x(t)\left[1 - 0.5u(t) - 1.8y(t-\tau)\right], \\ \dot{y}(t) = y(t)\left[-1 + 1.3x(t-\tau) - 1.7v(t)\right], \\ \dot{u}(t) = x(t) - u(t), \\ \dot{v}(t) = y(t) - v(t), \end{cases} \tag{21}$$

which has a steady state $(0, 0, 0, 0)$. It is easy to verify that hypotheses (H1) and (H) are hold. By directly calculate, we can obtain that $\tau_0 = 0.19$, $h'(\omega_0^2) = 17.6614 \neq 0$. Thus from Theorem 5 we know that the positive equilibrium of system (20) is asymptotically stable when $0 < \tau < \tau_0 = 0.19$ (see Figure 1 and Figure 2).

In addition, we also know from Theorem 5 that the positive equilibrium of system (20) is unstable when $\tau > \tau^0 = 0.19$ and when $0 < \tau - \tau^0 \ll 1$, a nonconstant periodic solution can be bifurcated from the positive equilibrium of system (20), see Figures 3-5.

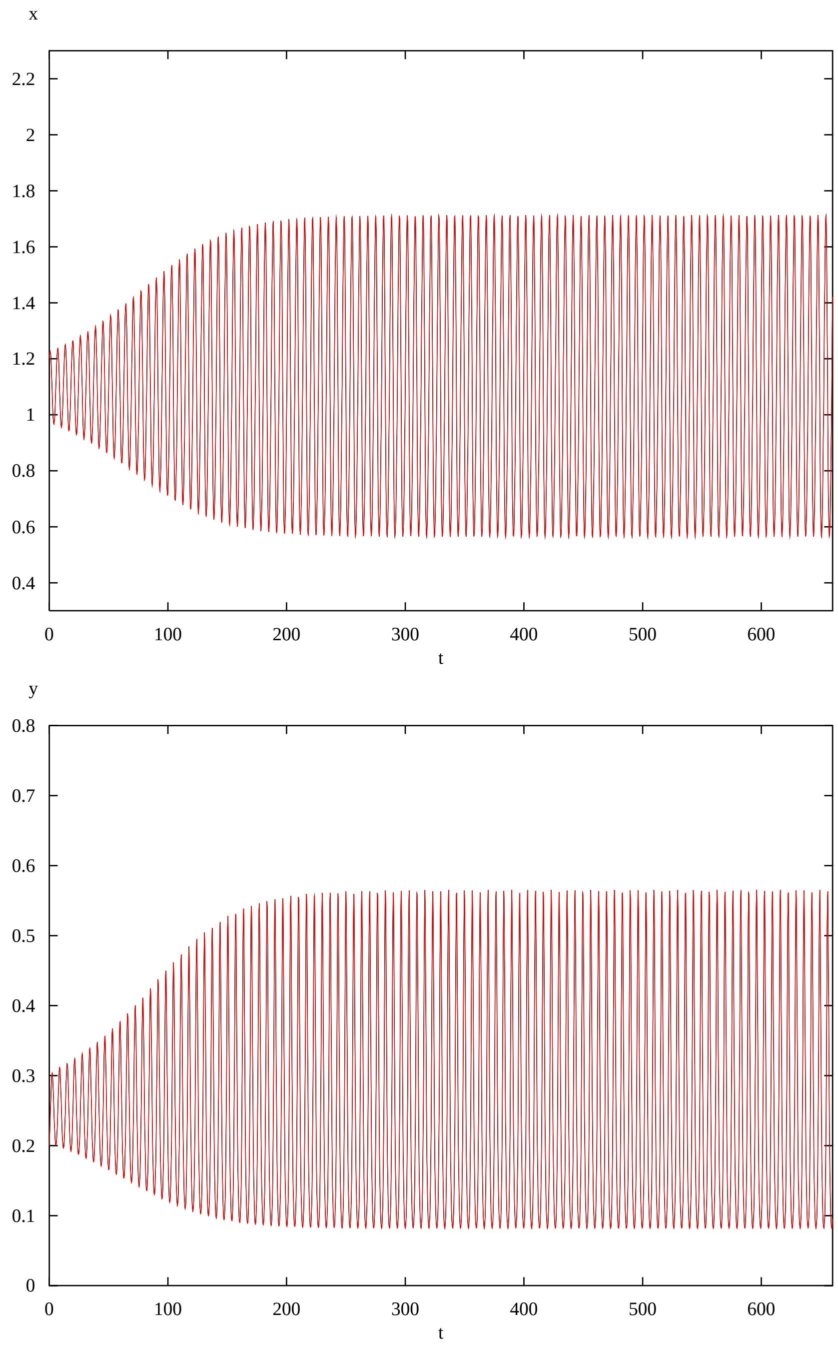

Figure 3: The trajectory graph of system (20) with $\tau = 0.215$ and initial data $x(t) = 1.2,\ y(t) = 0.2, t \in [-0.215, 0]$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

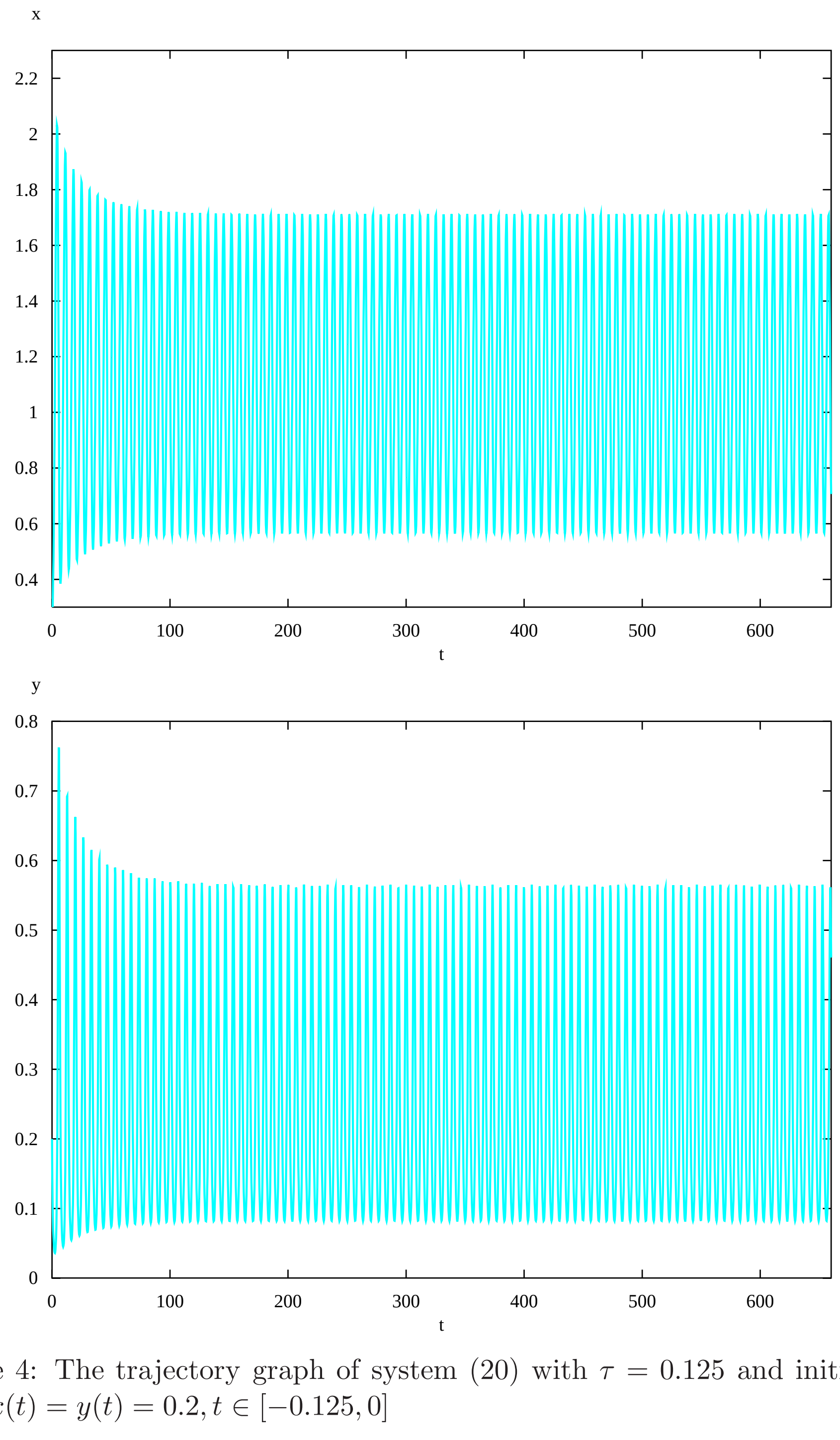

Figure 4: The trajectory graph of system (20) with $\tau = 0.125$ and initial data $x(t) = y(t) = 0.2, t \in [-0.125, 0]$

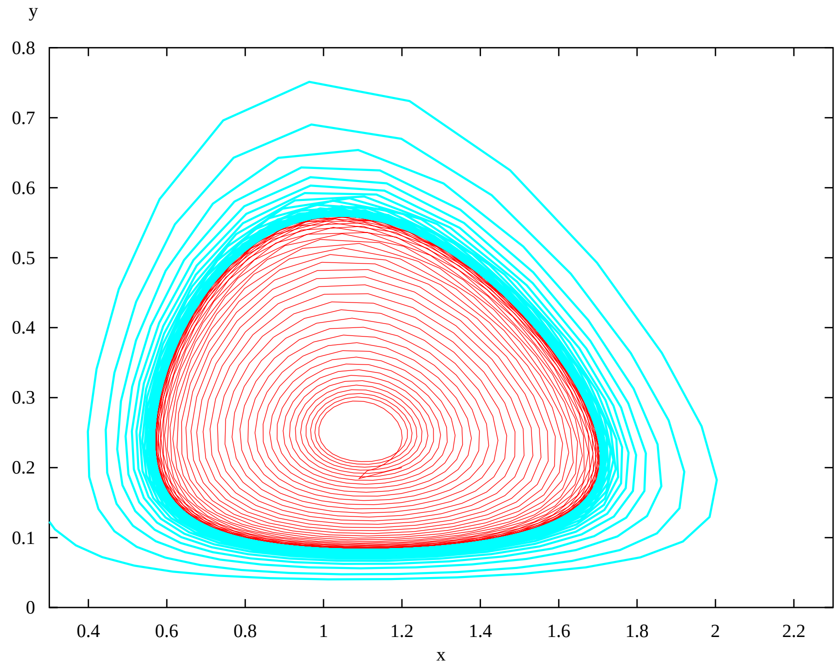

Figure 5: The phase graph of system (20) with $\tau = 0.215$

Acknowledgments

This work was supported by the Foundation of Nanjing University of Information and Technology.

References

[1] J.M. Cushing, *Integro-Differential Equations and Delay Models in Population Dynamics*, Springer, Heidelberg (1977).

[2] Hideaki Matsunaga, Stability swithches in a system of linear differential equations with diagonal delay, *Appl. Math. Comput.*, **212** (2009), 145-152.

[3] R.M. May, Time delay versus stability in population models with two and three trophic levels, *Ecology*, **4** (1973), 315-325.

[4] Y. Song, J. Wei, Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system, *J. Math. Anal. Appl.*, **301** (2005), 1-21.

[5] Y. Song, S. Yuan, Bifurcation analysis in a predator-prey system with time delay, *Nonlinear Anal.*, **7** (2006), 265-284.

[6] J. Wei, S. Ruan, Stability and bifurcation in a neural network model with two delays, *Physica D.*, **130** (1999), 255-272.

[7] X.P. Yan, W.T. Li, Hopf bifurcation and global periodic solutions in a delayed predator-prey system, *Appl. Math. Comput.*, **177** (2006), 427-445.

[8] X.P. Yan, C. H. Zhang, Hopf bifurcation in a delayed Lokta-Volterra predator-prey system, *Nonliear Analysis* RWA, **9** (2008), 114-127.