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PRINCIPAL VERTEX OPERATORS AND SUPER HIROTA BILINEAR EQUATIONS FOR A NON-SIMPLY LACED AFFINE KAC-MOODY LIE ALGEBRA

N. Sthanumoorthy[1 §], C. Kiruba Bagirathi[2]

[1,2]Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Chennai, 600 005, INDIA
[1]e-mail: sthanun@yahoo.com
[2]e-mail: ckbhagirathi@rediffmail.com

Abstract: In this paper we construct the principal vertex operators and super Hirota bilinear equations for a non-simply laced affine Kac-Moody algebra $C_2^{(1)}$.

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1. Introduction

Date et al (see [1], [2], [3], [4]), using Boson-Fermion correspondence in 2-dimensional Q.F.T., developed a connection between soliton theory and classical affine Lie algebras. In Jimbo and Miwa [5], a detailed account of KP hierarchy and the corresponding Lie algebra $gl(\infty)$ was given and therein it was also shown how various types of soliton equations could be generated by considering suitable subalgebras of $gl(\infty)$ and their basic representations. In Frenkel and Kac [6], basic representations of affine Lie algebras were described. These basic representations can be constructed explicitly in terms of differential operators called vertex operators in infinitely many indeterminates. The vertex operator construction for $A_1^{(1)}$ was discovered by Lepowsky and Wilson [14]. This construction was generalized to affine Lie algebras of type $X_n^{(r)} (r = 1, 2, 3)$, where $X_N = A, D, E$, in principal picture, see Kac et al [8].

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§Correspondence author

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

In the same paper, the algebras $B_n^{(1)}, C_n^{(1)}, F_4^{(1)}$, and $G_2^{(1)}$ were realized as algebras of differential operators but without describing the basic modules in each case explicitly. The vertex operators with an additional fermionic field for $B_l^{(1)}$ in principal picture, using Z-algebras, were constructed in Mandia [15]. In Misra [16], explicit constructions of all level one standard modules of $C_n^{(1)}$, in particular, of $C_2^{(1)}$ and $C_3^{(1)}$ were given. In Peterson and Kac [17], the orbit of the highest weight vector in an integrable highest weight module of a group associated to a Lie algebra was described. This principle was used in Kac and Wakimoto [11] to develop a hierarchy of Hirota bilinear equations corresponding to an element τ of an integrable representation of an affine algebra with symmetric Cartan matrix. In Kac and Wakimoto [11], it was also shown that a vertex operator construction in the principal picture of the basic representation of $B_l^{(1)}$ involves an additional fermionic field and the corresponding system of Hirota equations turns out to be a hierarchy of super Hirota bilinear equations.

In Lepowsky and Primc [13], the basic modules for $B_l^{(1)}$ in homogeneous picture was constructed. Using this module description, Hirota bilinear equations were constructed in Sthanumoorthy and Kiruba Bagirathi [18] for $B_2^{(1)}$. In Sthanumoorthy and Kiruba Bagirathi [19], principal vertex operators for $B_3^{(1)}$ considering it as a subalgebra of $D_4^{(1)}$ were explicitly written using Kac et al [8], Mandia [15] and the corresponding hierarchy of super Hirota bilinear equations were constructed.

In this paper using Kac et al [8] and Misra [16], the principal vertex operators for $C_2^{(1)}$ were constructed by taking C_2 as a subalgebra of fixed points of an automorphism of A_3 and finally super Hirota bilinear equations for $C_2^{(1)}$ were also constructed.

2. Preliminaries

Let $\underline{\mathfrak{g}}$ be a finite dimensional Lie algebra of type A_3. Let $(x|y) = \frac{1}{2}\mathrm{tr}(xy)$ be a nondegenerate symmetric bilinear form defined on $\underline{\mathfrak{g}}$.

Let $e_0 = e_{41}$, $e_1 = e_{12}$, $e_2 = e_{23}$, $e_3 = e_{34}$, $f_0 = e_{14}$, $f_1 = e_{21}, f_2 = e_{32}$, $f_3 = e_{43}$, $h_0 = e_{44} - e_{11}$, $h_1 = e_{11} - e_{22}$, $h_2 = e_{22} - e_{33}$ and $h_3 = e_{33} - e_{44}$ be a set of canonical generators for $\underline{\mathfrak{g}}$.

Let $\underline{\mathfrak{h}} = \mathrm{span}\{h_i,\ 1 \le i \le 3\}$ be a Cartan subalgebra of $\underline{\mathfrak{g}}$, Δ a simple root system. The bilinear form $(\cdot|\cdot)$ is normalized so that $(\alpha|\alpha) = 2$ for all $\alpha \in \Delta$. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of Δ.

Let θ be the Dynkin diagram automorphism of order 2 given by $\theta(\alpha_1) = \alpha_3$, $\theta(\alpha_2) = \alpha_2$ and $\theta(\alpha_3) = \alpha_1$. The automorphism θ is extended to $\underline{\mathfrak{g}}$ by defining $\theta(x_\alpha) = x_{\theta(\alpha)}$ for all $\alpha \in \Delta$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Hence we have the following decomposition of $\underline{\mathfrak{g}}$ with respect to θ:

$$\underline{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}/2\mathbb{Z}} \underline{\mathfrak{g}}_{[j]}, \text{ where } \underline{\mathfrak{g}}_{[j]} = \{x \in \underline{\mathfrak{g}} \mid \theta(x) = \epsilon^j x\} \text{ with } \epsilon = e^{\pi I},\ I = \sqrt{-1}.$$

We have

$$\begin{aligned}\underline{\mathfrak{g}}_{[0]} &= \mathbb{C}(e_{12}+e_{34}) \oplus \mathbb{C}e_{23} \oplus \mathbb{C}e_{14} \oplus \mathbb{C}e_{41} \oplus \mathbb{C}e_{32} \oplus \mathbb{C}(e_{21}+e_{43}) \\ &\quad \oplus \mathbb{C}(h_1+h_3) \oplus \mathbb{C}h_2 \oplus \mathbb{C}(e_{13}-e_{24}) \oplus \mathbb{C}(e_{31}-e_{42}) \text{ and} \\ \underline{\mathfrak{g}}_{[1]} &= \mathbb{C}(e_{12}-e_{34}) \oplus \mathbb{C}(e_{21}-e_{43}) \oplus \mathbb{C}(h_1-h_3) \\ &\quad \oplus \mathbb{C}(e_{13}+e_{24}) \oplus \mathbb{C}(e_{31}+e_{42}).\end{aligned}$$

The proof of the following proposition is similar to the result in Lepowsky and Primc [13].

Proposition 2.1. *i)* $\underline{\mathfrak{g}}_{[0]}$ *is of the type* C_2 *and* $\underline{\mathfrak{h}}_{[0]}$ *is a Cartan subalgebra of* $\underline{\mathfrak{g}}_{[0]}$.

ii) The set $\left\{\frac{\alpha+\theta\alpha}{2};\ \alpha \in \Delta\right\}$ *is a root system with respect to* $\underline{\mathfrak{h}}_{[0]}$ *with basis* $\left\{\frac{\alpha_i+\theta\alpha_i}{2};\ 1 \le i \le 2\right\}$.

The corresponding generating elements of $\underline{\mathfrak{g}}_{[0]} = C_2$ are $H_2 = h_2$, $E_1 = e_{12}+e_{34}$, $F_1 = e_{21}+e_{43}$, $E_2 = e_{23}$, $F_2 = e_{32}$ and $H_1 = h_1 + h_3$.

In order to construct principal vertex operators for C_2, we consider C_2 as the subalgebra of fixed points of the automorphism θ of A_3.

Let $E = e_0 + \sum_{i=1}^{3} e_i$, where e_0 is the lowest root vector of A_3 with respect to $\underline{\mathfrak{h}}$. Then $E = e_{41}+e_{12}+e_{23}+e_{34}$ is the cyclic element of A_3. Let $\underline{\mathfrak{a}} = \{x \in \underline{\mathfrak{g}} \mid [x,E] = 0\}$. Then $\underline{\mathfrak{a}}$ is the principal Cartan subalgebra of A_3. The basis elements of $\underline{\mathfrak{a}}$ are $E = e_{41}+e_{12}+e_{23}+e_{34}$, $F = e_{14}+e_{21}+e_{32}+e_{43}$ and $G = e_{13}+e_{24}+e_{31}+e_{42}$, with $\underline{\mathfrak{a}} = \mathbb{C}E \oplus \mathbb{C}F \oplus \mathbb{C}G$. The lowest root of C_2 with respect to $\underline{\mathfrak{h}}_{[0]}$ is $-(\alpha_1+\alpha_2+\alpha_3)$. The corresponding root vector is $E_0 = e_{41}$. Then $E_0 + E_1 + E_2$ is the cyclic element of $\underline{\mathfrak{g}}_{[0]}$.

Let $\underline{\eta} = \{x \in \underline{\mathfrak{g}}_{[0]} \mid [x,E] = 0\}$. Then $\underline{\eta}$ is the principal Cartan subalgebra of C_2. A basis of $\underline{\eta}$ is $\{E,\ F\}$. We omit the proof of following proposition which can be easily proved.

Proposition 2.2. $\underline{\eta}$ *is the subalgebra of* $\underline{\mathfrak{a}}$ *of* θ *fixed points.*

It is clear that Coxeter number in Kostant [12] of C_2 and A_3 is 4. Let $\nu : A_3 \to A_3$ be the principal automorphism defined by

$$\nu(e_j) = \omega e_j,\quad \nu(f_j) = \omega^{-1} f_j, \nu(h_j) = h_j,\ j = 1,2,3 \ \text{ and } \omega = e^{\frac{2\pi I}{4}}.$$

Then $\underline{\mathfrak{g}} = A_3$ is decomposed with respect to this automorphism. This is called the principal gradation of $\underline{\mathfrak{g}}$.

We have the following decomposition of $\underline{\mathfrak{g}}$, with respect to the principal grada-

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

tion:

$$\underline{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}/4\mathbb{Z}} \underline{\mathfrak{g}}_j, \text{ where } \underline{\mathfrak{g}}_j = \{x \in \underline{\mathfrak{g}} \mid \nu x = \epsilon^j x\},\ \epsilon = e^{2\pi I/4} \text{ and } j = 0,1,2,3,$$

with

$$\underline{\mathfrak{g}}_0 = \mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}h_3; \qquad \underline{\mathfrak{g}}_1 = \mathbb{C}e_{12} \oplus \mathbb{C}e_{23} \oplus \mathbb{C}e_{34} \oplus \mathbb{C}e_{41}$$
$$\underline{\mathfrak{g}}_2 = \mathbb{C}e_{13} \oplus \mathbb{C}e_{24} \oplus \mathbb{C}e_{31} \oplus \mathbb{C}e_{42}; \text{ and } \underline{\mathfrak{g}}_3 = \mathbb{C}e_{21} \oplus \mathbb{C}e_{32} \oplus \mathbb{C}e_{43} \oplus \mathbb{C}e_{14}.$$

The principal Cartan subalgebra $\underline{\mathfrak{a}}$ of A_3 is also graded with respect to the principal gradation as

$$\underline{\mathfrak{a}} = \bigoplus_{j\in\mathbb{Z}/4\mathbb{Z}} \underline{\mathfrak{a}}_j,\ \underline{\mathfrak{a}}_j = \underline{\mathfrak{a}} \cap \underline{\mathfrak{g}}_j,$$

where $\underline{\mathfrak{a}}_0 = \{0\},\ \underline{\mathfrak{a}}_1 = \mathbb{C}E,\ \underline{\mathfrak{a}}_2 = \mathbb{C}G$ and $\underline{\mathfrak{a}}_3 = \mathbb{C}F$.

Now $\nu|_{\underline{\mathfrak{g}}_{[0]}}$ is the principal automorphism of C_2.

We have the decomposition of $C_2 = \underline{\mathfrak{g}}_{[0]}$ as follows:

$$\underline{\mathfrak{g}}_{[0]} = \bigoplus_{j\in\mathbb{Z}/4\mathbb{Z}} (\underline{\mathfrak{g}}_{[0]})_j,$$

where

$$(\underline{\mathfrak{g}}_{[0]})_0 = \mathbb{C}(h_1 + h_3) \oplus \mathbb{C}h_2;\ , \qquad (\underline{\mathfrak{g}}_{[0]})_1 = \mathbb{C}(e_{12} + e_{34}) \oplus \mathbb{C}e_{23} \oplus \mathbb{C}e_{41},$$
$$(\underline{\mathfrak{g}}_{[0]})_2 = \mathbb{C}(e_{13} - e_{24}) \oplus \mathbb{C}(e_{31} - e_{42}) \text{ and } (\underline{\mathfrak{g}}_{[0]})_3 = \mathbb{C}(e_{21} + e_{43}) \oplus \mathbb{C}e_{32} \oplus \mathbb{C}e_{14}.$$

The principal Cartan subalgebra $\underline{\eta}$ of C_2 is also graded with respect to the principal gradation:

$$\underline{\eta} = \bigoplus_{j\in\mathbb{Z}/4\mathbb{Z}} \underline{\eta}_j, \underline{\eta}_j = \eta \cap (\underline{\mathfrak{g}}_{[0]})_j \text{ where } \underline{\eta}_0 = \{0\}, \underline{\eta}_1 = \mathbb{C}E, \underline{\eta}_2 = \{0\} \text{ and } \underline{\eta}_3 = \mathbb{C}F.$$

Let $\Delta_{\underline{\mathfrak{a}}}$ be the set of roots for A_3 with respect to $\underline{\mathfrak{a}}$.

The following result, similar to one in Mandia [15], is also easy to be verified.

Theorem 2.3. *The set $\Delta_{\underline{\eta}}$ of roots of C_2 with respect to η is*

$$\Delta_{\underline{\eta}} = \left\{ \underline{\beta} = \frac{\beta + \theta\beta}{2}\ ;\ \beta \in \Delta_{\underline{\mathfrak{a}}} \right\}.$$

We also omit the proof of the following proposition which can be directly established:

Proposition 2.4. *Let $\{\gamma_1, \gamma_2, \gamma_3\}$ be the set of simple roots of A_3 with respect to $\Delta_{\underline{\mathfrak{a}}}$ such that*

$$\langle \gamma_i, \gamma_i \rangle = 2,\ \langle \gamma_1, \gamma_2 \rangle = \langle \gamma_2, \gamma_3 \rangle = -1 \text{ and } \langle \gamma_1, \gamma_3 \rangle = 0.$$

Then $X_{\pm\gamma_i}$, $(1 \le i \le 3)$ are the root vectors for the simple roots $\pm\gamma_i$, $(1 \le i \le 3)$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

satisfying the following relations:

$$[E, X_{\pm\gamma_1}] = \pm(1+I)X_{\pm\gamma_1},\ [F, X_{\pm\gamma_1}] = \pm(1-I)X_{\pm\gamma_1},\ [G, X_{\pm\gamma_1}] = \mp 2X_{\pm\gamma_1},$$

$$[E, X_{\pm\gamma_2}] = \pm(-I+1)X_{\pm\gamma_2},\ [F, X_{\pm\gamma_2}] = \pm(I+1)X_{\pm\gamma_2},\ [G, X_{\pm\gamma_2}] = \pm 2X_{\pm\gamma_2},$$

$$[E, X_{\pm\gamma_3}] = \mp(1+I)X_{\pm\gamma_3},\ [F, X_{\pm\gamma_3}] = \pm(I-1)X_{\gamma_3} \text{ and } [G, X_{\pm\gamma_3}] = \mp 2X_{\pm\gamma_3}.$$

Here

$$\begin{aligned}
X_{\gamma_1} &= e_{12} - Ie_{23} - e_{34} - e_{13} + Ie_{24} + e_{14} - h_1 + (-1+I)h_2 \\
&\quad + Ih_3 - Ie_{21} - e_{32} + Ie_{43} + e_{31} - Ie_{42} + Ie_{41}, \\
X_{-\gamma_1} &= e_{12} + Ie_{23} - e_{34} - Ie_{13} + e_{24} - e_{14} + Ih_1 + (-1+I)h_2 \\
&\quad - h_3 - Ie_{21} + e_{32} + Ie_{43} - e_{42} + Ie_{31} - Ie_{41}, \\
X_{\gamma_2} &= Ie_{12} + e_{23} - Ie_{34} + e_{13} - Ie_{24} - Ie_{14} - h_1 + (-1+I)h_2 \\
&\quad + Ih_3 - e_{21} + Ie_{32} + e_{43} - e_{31} + Ie_{42} - e_{41}, \\
X_{-\gamma_2} &= Ie_{12} - e_{23} - Ie_{34} + Ie_{13} - e_{24} + Ie_{14} + Ih_1 + (-1+I)h_2 \\
&\quad - h_3 - e_{21} - Ie_{32} + e_{43} + e_{42} - Ie_{31} + e_{41}, \\
X_{\gamma_3} &= -e_{12} + Ie_{23} + e_{34} - e_{13} + Ie_{24} - e_{14} - h_1 + (-1+I)h_2 \\
&\quad + Ih_3 + Ie_{21} + e_{32} - Ie_{43} + e_{31} - Ie_{42} - Ie_{41}, \\
X_{-\gamma_3} &= -e_{12} - Ie_{23} + e_{34} + e_{24} + I(-e_{13}) + e_{14} + Ih_1 + (-1+I)h_2 \\
&\quad - h_3 + Ie_{21} - e_{32} - Ie_{43} - e_{42} + Ie_{31} + Ie_{41},\ \textit{with } i = \sqrt{-1}.
\end{aligned}$$

3. Principal Vertex Operators for $C_2^{(1)}$

Decomposing $X_{\gamma_i} \in \underline{\mathfrak{g}}$, $(1 \le i \le 3)$ with respect to the principal gradation, we get, $X_{\gamma_i} = \sum_{j=0}^{3} X_{\gamma_{i,j}}$, $X_{\gamma_{i,j}} \in \underline{\mathfrak{g}}_j$, we observe that the elements $\{E_i, X_{\gamma_{i,j}},\ 1 \le i \le 3;\ 0 \le j \le 3\}$ form a basis of A_3. Now the corresponding affine algebra is defined by $\hat{\underline{\mathfrak{g}}}' = \bigoplus_{j\in\mathbb{Z}} t^j \otimes \underline{\mathfrak{g}}_{j \bmod 4} \oplus \mathbb{C}K$, where K is the canonical central element and the generators for $A_3^{(1)}$ are given by

$$\hat{e}_j = 1 \otimes e_j, \hat{e}_0 = t \otimes e_0, \hat{f}_j = 1 \otimes f_j,\ \hat{f}_0 = t^{-1} \otimes f_0, \hat{h}_j = \alpha_j \quad \text{and } \hat{h}_0 = \frac{2}{(\theta|\theta)}K - \theta^\vee.$$

Here $j \in \{1, 2, 3\}$ and $\theta^\vee$ is the coroot corresponding to the highest root θ of A_3.

The Lie bracket in $\hat{\underline{\mathfrak{g}}}'$ is given by

$$[p_1(t) \otimes g_1 \oplus \lambda_1 K, p_2(t) \otimes g_2 \oplus \lambda_2 K] = p_1(t)p_2(t) \otimes [g_1, g_2] \oplus \frac{1}{4}\mathrm{Res}\left(\frac{dp_1(t)}{dt}p_2(t)\right)(g_1|g_2)K,$$

for all $p_1(t),\ p_2(t) \in \mathbb{C}[t, t^{-1}]$ and $g_1, g_2 \in \underline{\mathfrak{g}}$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Explicitly,

$$\underline{\hat{\mathfrak{g}}'} = \bigoplus_{j\in\mathbb{Z}} t^j \otimes \underline{\mathfrak{g}}_{j\text{mod}4} \oplus \mathbb{C}K = t^0\ \mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{g}}_0 \oplus t\ \mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{g}}_1 \oplus t^2\ \mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{g}}_2 \oplus t^3\ \mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{g}}_3 \oplus \mathbb{C}K.$$

The principal Cartan subalgebra $\underline{\mathfrak{a}}$ of $\underline{\mathfrak{g}}$ has $\{E,F,G\}$ as basis elements. Let us rename these basis elements as $E_1,\ E_2,\ E_3$, where $E_1 = E, E_2 = G, E_3 = F$, so that $E_i \in \underline{\mathfrak{a}_i}$ and $(E_i \mid E_{4-j}) = 4\delta_{ij}$ for $j = 1,3$.

Now $\mathfrak{a} = \bigoplus_{j\in\mathbb{Z}}(t^j \otimes \underline{\mathfrak{a}}_{j\text{mod}4}) \oplus \mathbb{C}K = t\mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{a}}_1 \oplus t^2\mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{a}}_2 \oplus t^3\mathbb{C}[t^4,t^{-4}] \otimes \underline{\mathfrak{a}}_3 \oplus \mathbb{C}K$.

This is the principal Heisenberg-Cartan subalgebra of $\underline{\hat{\mathfrak{g}}'}$ and we observe that $E_+ = \{n \in \mathbb{Z}_+ \mid n \neq 0 \text{ mod}4\} = \{1,2,3,5,6,7,9,10\cdots\}$ is the set of exponents of $A_3^{(1)}$. Its principal subalgebra $\mathfrak{a}$ has basis elements $K,\ p_i = t^i \otimes E_{i'},\ q_i = \frac{1}{i}t^{-i} \otimes E_{4-i'}$, where $i \in E_+$ and $i' \in \{1,2,3\}$ such that $i' = i$ mod4. These generators satisfy the relation $[p_i,\ q_j] = K\delta_{ij}$.

The principal Cartan subalgebra of C_2 is $\underline{\eta}$ with $\{E,F\}$ as a basis. Let us rename the basis elements of $\underline{\eta}$ as $E_1,\ E_3$, where $E_1 = E,\quad E_3 = F$ so that $E_i \in \underline{\eta}_i$ and $(E_i|E_{4-j}) = 4\delta_{ij}$ for $i,j = 1,3$.

Let γ_i $(1 \le i \le 3)$ be the simple roots of A_3 with respect to $\underline{\mathfrak{a}}$. The root system of C_2 with respect to the Cartan subalgebra $\underline{\eta}$ of C_2 is

$$\Delta_{\underline{\eta}} = \pm\{\bar{\gamma_1}, \bar{\gamma_2}, \bar{\gamma_1} + \bar{\gamma_2}, 2\bar{\gamma_1} + \bar{\gamma_2}\},$$

with $(\Delta_{\underline{\eta}})_l^+ = \{2\bar{\gamma_1} + \bar{\gamma_2}, \bar{\gamma_2}\}$ as the simple root of the system of long roots, and $(\Delta_{\underline{\eta}})_s^+ = \{\bar{\gamma_1}, \bar{\gamma_1} + \bar{\gamma_2}\}$ as the simple root of the system of short roots.

Here $\bar{\gamma_1} = \frac{1}{2}(\gamma_1 - \gamma_3)$, $\bar{\gamma_2} = (\gamma_2 + \gamma_3)$ are the simple roots and $X_{\bar{\gamma_i}}$ are the corresponding root vectors of C_2 and they are found out by using the relations $\bar{\gamma_i} = \frac{1}{2}(\gamma_i + \theta\gamma_i)$; $\quad X_{\bar{\gamma_i}} = X_{\gamma_i} + X_{\theta\gamma_i}\quad (i = 1,2)$.

The corresponding root vectors satisfy the following relations:

$$\begin{aligned} [E_i, X_{\bar{\gamma}_1}] &= (1+I^i)X_{\bar{\gamma}_1} \quad [E_i, X_{-\bar{\gamma}_1}] = -(1+I^i)X_{\bar{\gamma}_1}, \\ [E_i, X_{\bar{\gamma}_2}] &= (-2)I^i X_{\bar{\gamma}_2} \quad [E_i, X_{-\bar{\gamma}_2}] = 2I^i X_{-\bar{\gamma}_2} \quad \text{where } i = 1,2. \end{aligned}$$

Decomposing $X_{\bar{\gamma_i}} \in \underline{\mathfrak{g}}$, $(1 \le i \le 2)$ with respect to the principal gradation of C_2, we get, $X_{\bar{\gamma_i}} = \sum_{j=0}^{3} X_{\bar{\gamma}_{i,j}},\quad X_{\bar{\gamma}_{i,j}} \in (\underline{\mathfrak{g}}_{[0]})_j$. We observe that the elements $\{E_i, X_{\bar{\gamma}_i,j},\ 1 \le i \le 2;\ 0 \le j \le 3\}$ form a basis of C_2. The corresponding affine algebra is given by, $\underline{\hat{\mathfrak{g}}}'_{[0]} = \bigoplus_{j\in\mathbb{Z}} t^j \otimes (\underline{\mathfrak{g}}_{[0]})_{j\text{mod}4} \oplus \mathbb{C}K$, with generators $\hat{E}_0 = \hat{e}_0,\ \hat{E}_1 = \hat{e}_1 + \hat{e}_3,\ \hat{E}_2 = \hat{e}_2, \hat{F}_0 = \hat{f}_0, \hat{F}_1 = \hat{f}_1 + \hat{f}_3, \hat{F}_2 = \hat{f}_2, \hat{H}_0 = \hat{h}_0,\ \hat{H}_1 = \hat{h}_1 + \hat{h}_3$ and $\hat{H}_2 = \hat{h}_2$.

Let $\underline{\hat{\mathfrak{g}}} = \underline{\hat{\mathfrak{g}}'} \oplus \sum_{j\in\mathbb{Z}} \mathbb{C}d_j$ and $u(j) = u \otimes t^j$ *for* $u \in \mathfrak{g}$ with brackets defined by $[d_n, u(j)] = -ju(j+n)$, $[d_n, K] = 0$ and $[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}K$.

Similarly we can define, $\underline{\hat{\mathfrak{g}}}_{[0]} = \underline{\hat{\mathfrak{g}}'}_{[0]} \bigoplus \sum_{j\in\mathbb{Z}} \mathbb{C}d_j$ with brackets defined as above.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Let $X_{\bar{\gamma}}(z) = \sum_{j\in\mathbb{Z}} z^{-j}(t^j \otimes X_{\bar{\gamma},j\text{mod}4})$. Then $[d_n, X_{\bar{\gamma}}(z)] = z^n \left(n + z\frac{d}{dz}\right) X_{\bar{\gamma}}(z)$.

Let us consider the principal Cartan subalgebra $\underline{\eta}$ of C_2 and its basis $\{E_1,\ E_3\}$.

Let $\eta = \oplus_{j\in\mathbb{Z}}(t^j \otimes \underline{\eta}_{j\text{mod}4}) \oplus \mathbb{C}K$. This is the principal subalgebra of $\hat{\underline{\mathfrak{g}}}'_{[0]}$.

Here we observe that $\mathbb{Z}_+^{\text{odd}}$ is the set of exponents for $C_2^{(1)}$, which we denote by $\underline{E}_+$. We also note that $\underline{E}_+ \subset E_+$. It can also be proved that $\eta = \{x \ \in \ \mathfrak{a} \mid \theta x = x\ \}$.

Here $K,\ p_i = t^i \otimes E_{i'},\ q_i = i^{-1}t^{-i} \otimes E_{4-i'}$, where $i \in \underline{E}_+$ such that $i' = i$ mod4 with $i' \in \{1, 3\}$ are the generators of η. They satisfy the relation $[p_i, q_j] = K\delta_{ij}$.

Let us consider the space $\mathbb{C}[x_i\ ;\ i \in \mathbb{Z}_{+\text{odd}}]$ and consider it as an η module on which p_i acts as $\frac{\partial}{\partial x_i}$, q_i acts as multiplication by ix_i and K acts as Id.

Let us take $\bar{\gamma}_1 \in \Delta_{\underline{\eta}}$. Then

$[p_i, X_{\bar{\gamma}_1}(z)] = [t^i \otimes E_{i'}, \sum_{j\in\mathbb{Z}_{+\text{odd}}} z^{-j}(t^j \otimes X_{\bar{\gamma}_1,j\text{mod}4})] = z^i(1 + I^{i'})X_{\bar{\gamma}_1}(z)$.

Here $i' \ \in \{1, 3\}$ such that $i' \equiv i$ mod4 with $i \in \underline{E}_+$. Similarly, $[q_i, X_{\bar{\gamma}_1}(z)] = z^{-i}(1 + I^{4-i'})X_{\bar{\gamma}_1}(z)$.

Using Kac et al [8], Kac [9], the principal vertex operator corresponding to the root $\bar{\gamma}_1$ is

$$\Gamma^{\bar{\gamma}_1}(z) = \exp\left(\sum_{j\in\mathbb{Z}_{+\text{odd}}} (1 + I^{j'})z^j x_j\right) \exp\left(-\sum_{j\in\mathbb{Z}_{+\text{odd}}} (1 + I^{4-j'})z^{-j}\frac{\partial}{\partial x_j}\right)$$

(where $j' \in \{1, 3\}$ with $j' \equiv$ mod4).

Similarly, we can have the vertex operator expressions for all the roots $\bar{\gamma} \in (\Delta_{\underline{\eta}})$.

Let us consider the fermionic oscillators ψ_n satisfying the following anticommutation relation: $[\psi_m, \psi_n]_+ = \psi_m\psi_n + \psi_n\psi_m = \delta_{m,-n}\ \ (m, n \in \delta + \mathbb{Z})$, where either $\delta = 0$ (or) $\delta = \frac{1}{2}$ so $\psi_0^2 = \frac{1}{2}$.

Denote by A_δ the algebra generated by $\{\psi_n | n \in \delta + \mathbb{Z}\}$. Consider the space $V_\delta = \wedge[\xi_i \mid i \geq 0,\ i \in \delta + \mathbb{Z}]$, where $\wedge$ means the exterior algebra generated by ξ_j.

The algebra A_δ can be represented in V_δ by

$$\psi_n \mapsto \frac{\partial}{\partial \xi_n}; \psi_{-n} \mapsto \xi_n \text{for } \ n > 0; \psi_0 \mapsto \frac{1}{\sqrt{2}}(\xi_0 + \frac{\partial}{\partial \xi_0}), \quad \text{for } \ n = 0.$$

The proof of the following proposition is in Kac and Raina [10].

Proposition 3.1. *Let l_k $(k \in \mathbb{Z})$ be the operators in V_δ defined by*

$$l_k = \delta_{k,0}\frac{1 - 2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j : \psi_{-j}\psi_{j+k} : \qquad \text{where}$$

$$: \psi_m\psi_n : = \begin{cases} \psi_m\psi_n, & \text{if } n \geq m \\ -\psi_n\psi_m, & \text{if } n < m \ \ (m, n \in \delta + \mathbb{Z}). \end{cases}$$

Then $[l_m, l_n] = (m - n)l_{m+n} + \delta_{m,-n}\left(\frac{m^3-m}{24}\right)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Here $l_0 = \frac{1-2\delta}{16} + \sum_{j>0} j\psi_{-j}\psi_j$ the summation is over $j \in \delta + \mathbb{Z}$ and $[\psi_m, l_k] = (m + \frac{k}{2})\psi_{m+k}$, for $m \in \mathbb{Z}$. The operators $l_n, n \in \mathbb{Z}$ along with the identity operator form a Virasoro algebra Vir with central charge $\frac{1}{2}$ in Goddard and Olive [7], Wakimoto [20], and Kac et al [8].

The subspaces V_δ^+ and V_δ^- of even and odd elements respectively are sub representations of V_δ. If c is the central charge and k is the eigen value of the energy operator in Kac [9], Kac and Raina [10], then the corresponding Virasoro module can be denoted by $V(c, k)$. Using this notation we can write: $V_{\frac{1}{2}}^+ = V(\frac{1}{2}, 0)$ and $V_{\frac{1}{2}}^- = V(\frac{1}{2}, \frac{1}{2})$, with $V_{\frac{1}{2}} = V_{\frac{1}{2}}^+ \oplus V_{\frac{1}{2}}^-$ and $V_0^+ = V_0^- = V(\frac{1}{2}, \frac{1}{16})$.

Let $a_j = \frac{\partial}{\partial x_j}, a_{-j} = jx_j$ for $j \in \mathbb{N}_{odd}$ be the operators defined on $\mathbb{C}[x]$ and $\dot{L}_n^{(h)} = \frac{1}{2h}\Sigma_{j\in\mathbb{N}_{odd}} : a_{nh-j}a_j :, n \in \mathbb{Z}$. Here h is the coxeter number of $C_2^{(1)}$.

Using the construction given in Goddard and Olive [7], and also using Kac [9] and Kac and Raina [10] one can directly prove the following theorem (we omit the proof) which gives the coset Virasaro algebra representation of $C_2^{(1)}$:

Theorem 3.2. *The representation of* $\underline{\hat{\mathfrak{g}}}'_{[0]} + Vir = C_2^{(1)} \oplus Vir$ *on the space* $\mathbb{C}[x_i,\ i \in \mathbb{N}_{\text{odd}}] \otimes \wedge[\ \xi_i \mid i \in \frac{1}{2} + \mathbb{Z}_+]$ *can be given by:*

$$\begin{array}{lll} p_j & \mapsto & \dfrac{\partial}{\partial x_j};\ \ q_j \mapsto jx_j;\ \ K \mapsto Id;\ \ d_j \mapsto \dot{L}_j^{(h)} + l_j; \\ X_{\bar{\gamma}_i}(z) & \mapsto & \Gamma^{\bar{\gamma}_i}(z)\psi(z), \qquad \text{if} \qquad (\bar{\gamma}_i|\bar{\gamma}_i) = 1 \text{ for } i = 1,2 \text{ and} \\ X_{\bar{\gamma}_i}(z) & \mapsto & \Gamma^{\bar{\gamma}_i}(z), \qquad\qquad \text{if} \qquad (\bar{\gamma}_i|\bar{\gamma}_i) = 2 \text{ for } i = 1,2. \end{array}$$

4. Super Hirota Bilinear Equations for $C_2^{(1)}$

Now we use the method followed in Kac and Wakimoto [11], to write down the Hirota bilinear equations for $C_2^{(1)}$.

Theorem 4.1. (see Peterson and Kac [17]) *Let $\mathfrak{g}$ be a Kac-Moody algebra with a symmetrizable Cartan matrix and let G be the associated group. Let $\{u_j\}$ and $\{u^j\}$ be bases of $\mathfrak{g}$ dual with respect to a nondegenerate invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ and consistent with a triangular decomposition of $\mathfrak{g}$. Let $L(\Lambda)$ be an integrable representation of $\mathfrak{g}$ with highest weight Λ and let v_Λ be its highest weight vector. Then:*

a) A nonzero vector v of $L(\Lambda)$ lies in the orbit $G.v_\Lambda$ if and only if $\sum u_j(v) \otimes u^j(v) = (\Lambda|\Lambda)(v \otimes v)$ in $L(\Lambda) \otimes L(\Lambda)$.

b) A vector v of $L(\Lambda)$ satisfies the equation $(*)$ if and only if $v \otimes v$ lies in the highest component of $L(\Lambda) \otimes L(\Lambda)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Thus we conclude, using the above theorem and the Taylor's expansion for the $\mathfrak{g}$ module $L(\Lambda)$ that the element τ of $L(\Lambda)$ lies in the orbit $G.1$ if and only if, $S(\tau\otimes\tau)=0$, where:

$$S=\sum u_j\otimes u^j,$$

see Kac and Wakimoto [11].

Let us find $S(\tau\otimes\tau)$ for $C_2^{(1)}$. We choose dual bases for $C_2^{(1)}$ as follows:

$$u_j:\{\frac{1}{\sqrt{8}}p_j,\quad \frac{1}{\sqrt{8}}q_j,\quad t^j\otimes X_{\bar{\gamma}_1,j\mathrm{mod}4},\quad t^j\otimes X_{\bar{\gamma}_2,j\mathrm{mod}4},\qquad K,\quad d\}$$

$$u^j:\{\frac{1}{\sqrt{8}}q_j,\quad \frac{1}{\sqrt{8}}p_j,\quad t^{-j}\otimes X_{-\bar{\gamma}_1,-j\mathrm{mod}4},\quad t^{-j}\otimes X_{-\bar{\gamma}_2,-j\mathrm{mod}4},d,\ K\}$$

The operator S acts on the module

$$\mathbb{C}[x'_i,\ i\in\mathbb{Z}_{+\mathrm{odd}}]\otimes\wedge[\xi'_i\mid i\in\frac{1}{2}+\mathbb{Z}_+]\bigotimes\mathbb{C}[x''_i,\ i\in\mathbb{Z}_{+\mathrm{odd}}]\otimes\wedge[\xi''_i\mid i\in\frac{1}{2}+\mathbb{Z}_+]$$
$$=\mathbb{C}[x'_i,\ x''_i,\ i\in\mathbb{Z}_{+\mathrm{odd}}]\bigotimes\wedge[\xi'_i,\ \xi''_i\ \mid\ i\in\frac{1}{2}+\mathbb{Z}_+].$$

Here $'$ and $''$ denote the operators acting on the first and second factors of the tensor product respectively. Hence,

$$S=\frac{1}{8}\sum_{j\in\underline{E}_+}\left(\frac{\partial}{\partial x'_j}\otimes jx''_j+jx'_j\frac{\partial}{\partial x''_j}\right)+\text{Coeff. of } z^0 \text{ in}$$
$$\left(\sum_{i=1}^{2}X_{\bar{\gamma}_i}(z)\otimes X_{-\bar{\gamma}_i}(z)\right)-1\otimes(\dot{L}_0^{(\mathrm{h})''}+l''_0)-(\dot{L}_0^{(\mathrm{h})'}+l'_0)\otimes 1$$
$$=\sum_{j\in\underline{E}_+}\frac{j}{8}\left(\frac{\partial}{\partial x'_j}\otimes x''_j+x'_j\otimes\frac{\partial}{\partial x''_j}\right)+\text{Coeff. of } z^0 \text{ in}\sum_{i=1}^{2}X_{\bar{\gamma}_i}(z)\otimes X_{-\bar{\gamma}_i}(z)-$$
$$\frac{1}{8}\sum_{j\in\underline{E}_+}\frac{\partial}{\partial x'_j}jx'_j\otimes 1-1\otimes\frac{1}{8}\sum_{j\in\underline{E}_+}\frac{\partial}{\partial x''_j}jx''_j-\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}j\psi''_{-j}\psi''_j-\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}j\psi'_{-j}\psi'_j$$
$$=\frac{-1}{8}\sum_{j\in\underline{E}_+}j(x'_j-x''_j)\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)-\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}j(\psi'_{-j}\psi'_j+\psi''_{-j}\psi''_j)$$
$$+\text{Coeff. of }\ z^0 \text{ in }\sum_{i=1}^{2}(X_{\bar{\gamma}_i}(z)\otimes X_{-\bar{\gamma}_i}(z)).$$

Here $\bar{\gamma}_1$ is a short root and $\bar{\gamma}_2$ is a long root. Therefore:

$$\text{Coeff. of } z^0 \text{ in } X_{\bar{\gamma}_1}(z)\otimes X_{-\bar{\gamma}_1}(z)$$
$$=\text{Coeff. of } z^0 \text{ in } \Gamma^{\bar{\gamma}_1}(z)\psi'_{1/2}(z)\otimes\Gamma^{-\bar{\gamma}_1}(z)\psi''_{1/2}(-z)$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

Coeff. of z^0 in $\Gamma^{\bar{\gamma}_1}(z)\psi'_{1/2}(z)\otimes\Gamma^{-\bar{\gamma}_1}(z)\psi''_{1/2}(-z)$

$$= \text{Coeff. of } \mathrm{z}^0 \text{ in } \exp\left(\sum_{j\in\underline{E}_+}(1+I^{j'})z^j x'_j\right)$$

$$\exp\left(-\sum(1+I^{4-j'})z^{-j}\frac{\partial}{\partial x'_j}\right)\psi'_{1/2}(z)\otimes$$

$$\exp\left(-\sum_{j\in\underline{E}_+}(1+I^{j'})z^j x''_j\right)\exp\left(\sum(1+I^{4-j'})z^{-j}\frac{\partial}{\partial x''_j}\right)\psi''_{1/2}(-z)$$

$$= \text{Coeff. of } \mathrm{z}^0 \text{ in } \exp\left(\sum_{j\in\underline{E}_+}(1+I^{j'})(x'_j-x''_j)z^j\right)$$

$$\exp\left(-\sum_{j\in\underline{E}_+}(1+I^{4-j'})\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)z^{-j}\right)\otimes\psi'_{1/2}(z)\psi''_{1/2}(-z).$$

Similarly Coeff. of z^0 in $X_{\bar{\gamma}_2}(z)\otimes X_{-\bar{\gamma}_2}(z)=$ Coeff. of z^0 in $\Gamma^{\bar{\gamma}_2}(z)\otimes\Gamma^{-\bar{\gamma}_2}(z)$. Hence, coeff. of z^0 in $\Gamma^{\bar{\gamma}_2}(z)\otimes\Gamma^{-\bar{\gamma}_2}(z)$

$$= \text{Coeff. of } \mathrm{z}^0 \text{ in } \left\{\exp\left(\sum_{j\in\underline{E}_+}(-2)I^{j'}(x'_j-x''_j)z^j\right)\right.$$

$$\left.\exp\left(\sum_{j\in\underline{E}_+}2I^{4-j'}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)z^{-j}\right)\right\}.$$

Assume

$$x_j=\frac{1}{2}(x'_j+x''_j);\quad y_j=\frac{1}{2}(x'_j-x''_j);$$

$$\frac{\partial}{\partial x_j}=\frac{\partial}{\partial x'_j}+\frac{\partial}{\partial x''_j} \text{ and } \frac{\partial}{\partial y_j}=\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}.$$

Therefore Coeff. of z^0 in $X_{\bar{\gamma}_1}(z)\otimes X_{-\bar{\gamma}_1}(z)$

$$= \text{Coeff. of } z^0 \text{ in } \exp\left(\sum_{j\in\underline{E}_+}2(1+I^{j'})y_j z^j\right)\exp\left(-\sum_{j\in\underline{E}_+}(1+I^{4-j'})\frac{\partial}{\partial y_j}z^{-j}\right)$$

$$\otimes\psi'_{1/2}(z)\psi''_{1/2}(-z).$$

Coeff. ofz^0 in $X_{\bar{\gamma}_2}(z)\otimes X_{-\bar{\gamma}_2}(z)$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

$$= \text{Coeff. of } z^0 \text{ in } \exp\left(\sum_{j\in\underline{E}_+}(-2)I^{j'}2y_jz^j\right)\exp\left(\sum_{j\in\underline{E}_+}2I^{4-j'}\frac{\partial}{\partial y_j}z^{-j}\right).$$

We note that $\psi_{1/2}(-z) = -\ \psi_{1/2}(z)$. Therefore we have

$$S = -\sum_{j\in\underline{E}_+}\frac{j}{4}y_j\frac{\partial}{\partial y_j} - \sum_{j\in\frac{1}{2}+\mathbb{Z}_+} j(\psi'_{-j}\psi'_j + \psi''_{-j}\psi''_j) - \text{ Coeff. of z}^0 \text{ in}$$

$$\left\{\exp\left(\sum_{j\in\underline{E}_+}2(1+I^{j'})y_jz^j)\right)\exp\left(-\sum_{j\in\underline{E}_+}(1+I^{4-j'})\frac{\partial}{\partial y_j}z^{-j}\right)\right.$$
$$\left.\otimes\psi'_{1/2}(z)\psi''_{1/2}(z)\right\}$$
$$+\text{ Coeff. of z}^0 \text{ in}\left\{\exp\left(\sum_{j\in\underline{E}_+}-4I^{j'}y_jz^j\right)\exp\left(\sum_{j\in\underline{E}_+}2I^{4-j'}\frac{\partial}{\partial y_j}z^{-j}\right)\right\}.$$

Let $P_k^{\underline{E}_+}(x),\ k\in E_+$ be the Schur polynomials defined as follows:

$$\sum_{k\geq 0}P_k^{\underline{E}_+}(x)z^k = \exp\left(\sum_{k\in\underline{E}_+}x_kz^k\right),\quad \text{where } P_k^{\underline{E}_+}(x) = P_k(x_k),\quad k\in\underline{E}_+.$$

Hence

$$S(\tau\otimes\tau) = \left\{-\sum_{j\in\underline{E}_+}\frac{j}{4}y_j\frac{\partial}{\partial y_j} - \sum_{j\in\frac{1}{2}+\mathbb{Z}_+}j(\psi'_{-j}\psi'_j + \psi''_{-j}\psi''_j) - \text{Coeff. of } z^0\text{in}\right.$$

$$\left\{\left(\sum_{m\in\mathbb{Z}_+}P_m^{\underline{E}_+}(2(1+I^{m'})y_m)z^m\right)\left(\sum_{n\in\mathbb{Z}_+}P_n^{\underline{E}_+}(-(1+I^{4-n'})\frac{\partial}{\partial y_n})z^{-n}\right)\right.$$

$$\left.\otimes\sum_{j,k\in 1/2+\mathbb{Z}}\psi'_j\psi''_kz^{-(j+k)}\right\} + \text{ Coeff. of } z^0 \text{ in}$$

$$\left.\left(\sum_{m\in\mathbb{Z}_+}P_m^{\underline{E}_+}((-4I^{m'})y_m)z^m\right)\left(\sum_{n\in\mathbb{Z}_+}P_n^{\underline{E}_+}((2I^{4-n'})\frac{\partial}{\partial y_n})z^{-n}\right)\right\}(\tau\otimes\tau)$$

$$= \left\{-\sum_{j\in\underline{E}_+}\frac{j}{4}y_j\frac{\partial}{\partial y_j} - \sum_{j\in\frac{1}{2}+\mathbb{Z}_+}j(\psi'_{-j}\psi'_j + \psi''_{-j}\psi''_j)\right.$$

$$-\sum_{n\in\mathbb{Z}_+,\ r\in\mathbb{Z}}P_n^{\underline{E}_+}(2(1+I^{n'})y_n)P_{n-r}^{\underline{E}_+}\left(-(1+I^{4-n'})\frac{\partial}{\partial y_n}\right)\sum_{\substack{j+k=r\\ j,k\in\frac{1}{2}+\mathbb{Z}}}\psi'_j\psi''_k$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

$$+ \sum_{n\in\mathbb{Z}_+} P_n^{E_+}(-4I^{n'}y_n)P_n^{E_+}\left(2I^{4-n'}\frac{\partial}{\partial y_n}\right)\Bigg\}\tau(x+y,\xi')\cdot\tau(x-y,\xi'').$$

Now the module is $\mathbb{C}[x',x'']\otimes V_{\frac{1}{2}}(\xi',\xi'')$.

As in Kac [9], let us change the variables ξ',ξ'' into ξ,η by using

$$\xi_j = \frac{1}{2}(\xi'_j+\xi''_j), \quad \eta_j = \frac{1}{2}(\xi'-\xi'').$$

Therefore $f(x+y,\xi')\cdot g(x-y,\xi'') = f(x+y,\xi+\eta)\cdot g(x-y,\xi-\eta)$ and the operators ψ'_j,ψ''_j act on $V_{\frac{1}{2}}(\xi,\eta)$. Let $\psi_j = \frac{1}{2}(\psi'_j+\psi''_j)$ and $\theta_j = \frac{1}{2}(\psi'_j-\psi''_j)$, $j\in\frac{1}{2}+\mathbb{Z}_+$. The action of the above elements is given by $\psi_j \mapsto \frac{1}{2}\frac{\partial}{\partial\xi_j}$ and $\theta_j \mapsto \frac{1}{2}\frac{\partial}{\partial\eta_j}$ $\psi_{-j}\mapsto\xi_j$ and $\theta_{-j}\mapsto\eta_j$ for $j\in\frac{1}{2}+\mathbb{Z}_+$. They satisfy the following relations:

$$[\psi_j,\psi_k]_+ = \frac{1}{2}\delta_{j,-k}\ ;\ [\theta_j,\theta_k]_+ = \frac{1}{2}\delta_{j,-k} \quad \text{and} \quad [\psi_j,\theta_k]_+ = 0.$$

Therefore

$$\begin{aligned}\sum_{j\in\frac{1}{2}+\mathbb{Z}_+} j(\psi'_{-j}\psi'_j+\psi''_{-j}\psi''_j) &= 2\sum_{j\in\frac{1}{2}+\mathbb{Z}_+} j(\psi_{-j}\psi_j+\theta_{-j}\theta_j) \ \text{ and}\\ \sum_{\substack{i,j\in\frac{1}{2}+\mathbb{Z}\\ i+j=s}}\psi'_i\psi''_j &= 2\sum_{\substack{j,k\in\frac{1}{2}+\mathbb{Z}_+\\ j+k=s}}\psi_j\theta_k + 2\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}(\psi_{-j}\psi_{j+s}-\theta_{-j}\theta_{j+s}).\end{aligned}$$

Let us use here Taylor's expansion:

$$e^{\sum y_j\frac{\partial}{\partial u_j}}f(x+u,\xi)\cdot g(x-u,\eta)|_{u=0} = f(x+y,\xi)\cdot g(x-y,\eta)$$

$$e^{\sum\xi_j\frac{\partial}{\partial\alpha_j}}f(\alpha)|_{\alpha=0} = f(\xi) \ \text{ and } \ e^{\sum\eta_j\frac{\partial}{\partial\beta_j}}f(\beta)|_{\beta=0} = f(\eta).$$

Let us use Theorem 4.1. Hence τ satisfies the following hierarchy of Hirota's bilinear equations corresponding to $C_2^{(1)}$:

$$\begin{aligned}S(\tau\otimes\tau) = \Bigg\{&\frac{1}{2}-\sum_{j\geq 0}\frac{j}{4}y_j\frac{\partial}{\partial u_j}-2K\\ &-\sum_{n\geq 0,\ r\in\mathbb{Z}} P_n^{E_+}(2(1+I^{n'})y_n)P_{n-r}^{E_+}(-(1+I^{4-n'})\frac{\partial}{\partial u_n})K_r\\ &+\sum_{n\geq 0} P_n^{E_+}(-4I^{n'}y_n)P_n^{E_+}(2I^{4-n'}\frac{\partial}{\partial u_n})\Bigg\}e^{\sum_{j\geq 0}y_j\frac{\partial}{\partial u_j}}\cdot e^{\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}\xi_j\frac{\partial}{\partial\alpha_j}}\cdot\\ &e^{\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}\eta_j\frac{\partial}{\partial\beta_j}}\tau(x+u,\alpha+\beta)\cdot\tau(x-u,\alpha-\beta)\big|_{u=\alpha=\beta=0} = 0,\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

where

$$\begin{aligned} K &= \sum_{j\in\frac{1}{2}+\mathbb{Z}_+} j\left(\xi_j\frac{\partial}{\partial\alpha_j}+\eta_j\frac{\partial}{\partial\beta_j}\right) \\ K_r &= \frac{1}{2}\sum_{\substack{j,k\in\frac{1}{2}+\mathbb{Z}_+\\ j+k=r}}\left(\frac{\partial}{\partial\alpha_j}\frac{\partial}{\partial\beta_k}\right)+\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}\left(\xi_j\frac{\partial}{\partial\alpha_{j+r}}-\eta_j\frac{\partial}{\partial\beta_{j+r}}\right) \quad \text{and} \\ K_{-r} &= 2\sum_{\substack{j,k\in\frac{1}{2}+\mathbb{Z}_+\\ j+k=r}}\eta_j\xi_k+\sum_{j\in\frac{1}{2}+\mathbb{Z}_+}\left(\xi_{j+r}\frac{\partial}{\partial\alpha_j}-\eta_{j+r}\frac{\partial}{\partial\beta_j}\right). \end{aligned}$$

Let us consider some particular coefficients from the above hierarchy:

Coeff. of $\xi_j\eta_k$:

$$\Bigg\{\frac{1}{2}-2(j+k)\frac{\partial}{\partial\alpha_j}\frac{\partial}{\partial\beta_k} - \Bigg(2P^{E_+}_{j+k}\left(-(1+I^{4-n'})\frac{\partial}{\partial u_n}\right) + \sum_{0<s\le j}P^{E_+}_{j-s}\left(-(1+I^{4-n'})\frac{\partial}{\partial u_n}\right)\frac{\partial}{\partial\alpha_s}\frac{\partial}{\partial\beta_k}\Bigg\} - \sum_{0<s\le k}P^{E_+}_{k-s}\left(-(1+I^{4-n'})\frac{\partial}{\partial u_n}\right)\frac{\partial}{\partial\beta_s}\frac{\partial}{\partial\alpha_j}\Bigg)\Bigg\}\tau\cdot\tau\Big|_{u=\alpha=\beta=0}=0.$$

After making suitable proportionate transformation to the Hirota operators in the above hierarchy we get the following:

Coeff. of $\xi_{\frac{3}{2}}\eta_{\frac{3}{2}}$: $\left\{\frac{\partial}{\partial\alpha_{\frac{3}{2}}}\cdot\frac{\partial}{\partial\beta_{\frac{3}{2}}}\right\}\tau\cdot\tau=0$

Coeff. of $\xi_{\frac{3}{2}}\eta_{\frac{5}{2}}$: $\left\{D_1^4-12D_1D_3+\frac{3}{2}D_1^2\left(\frac{\partial}{\partial\beta_{\frac{1}{2}}}\frac{\partial}{\partial\alpha_{\frac{3}{2}}}\right)-\frac{39}{2}\frac{\partial}{\partial\alpha_{\frac{3}{2}}}\frac{\partial}{\partial\beta_{\frac{5}{2}}}\right\}\tau\cdot\tau=0.$

Coeff. of $\xi_{\frac{7}{2}}\eta_{\frac{1}{2}}$: $\left\{D_1^4-4D_1D_3-D_1^2\left(\frac{\partial}{\partial\alpha_{\frac{3}{2}}}\cdot\frac{\partial}{\partial\beta_{\frac{1}{2}}}\right)-\frac{3}{2}\frac{\partial}{\partial\alpha_{\frac{7}{2}}}\cdot\frac{\partial}{\partial\beta_{\frac{1}{2}}}\right\}\tau\cdot\tau=0$

Coeff. of $\xi_{\frac{7}{2}}\eta_{\frac{5}{2}}$:

$$\Bigg\{D_1^6-20D_1^3D_3-8D_3^2+18D_1D_5-\frac{D_1^2}{2}\left(\frac{\partial}{\partial\alpha_{\frac{3}{2}}}\cdot\frac{\partial}{\partial\beta_{\frac{5}{2}}}-\frac{\partial}{\partial\beta_{\frac{1}{2}}}\cdot\frac{\partial}{\partial\alpha_{\frac{7}{2}}}\right) -\frac{9}{2}\frac{\partial}{\partial\alpha_{\frac{7}{2}}}\cdot\frac{\partial}{\partial\beta_{\frac{5}{2}}}\Bigg\}\tau\cdot\tau=0.$$

Here we recall the definition of a superbilinear equation from Kac and Wakimoto [11] which is defined as follows:

$$P(D_x,D^{\alpha}_{\frac{1}{2}},D^{\beta}_{\frac{1}{2}})\tau\cdot\tau=0.$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 4 (2010)

That is, $P\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \alpha_{\frac{1}{2}}}, \frac{\partial}{\partial \beta_{\frac{1}{2}}}\right) \tau(x+u, \alpha+\beta) \cdot \tau(x-u, \alpha-\beta)|_{u=\alpha=\beta=0} = 0$. Hence the above equations can be rewritten as follows:

Coeff. of $\xi_{\frac{3}{2}}\eta_{\frac{3}{2}}$: $\left\{D^{\alpha}_{\frac{3}{2}} D^{\beta}_{\frac{3}{2}}\right\} \tau \cdot \tau = 0.$

Coeff. of $\xi_{\frac{3}{2}}\eta_{\frac{5}{2}}$: $\left\{D_1^4 - 12D_1D_3 + \frac{3}{2}D_1^2\left(D^{\alpha}_{\frac{3}{2}} D^{\beta}_{\frac{1}{2}}\right) - \frac{39}{2}D^{\alpha}_{\frac{3}{2}} D^{\beta}_{\frac{5}{2}}\right\} \tau \cdot \tau = 0.$

Coeff. of $\xi_{\frac{7}{2}}\eta_{\frac{1}{2}}$: $\left\{D_1^4 - 4D_1D_3 - D_1^2\left(D^{\alpha}_{\frac{3}{2}} D^{\beta}_{\frac{1}{2}}\right) - \frac{3}{2}D^{\alpha}_{\frac{7}{2}} D^{\beta}_{\frac{1}{2}}\right\} \tau \cdot \tau = 0.$

Coeff.of $\xi_{\frac{7}{2}}\eta_{\frac{5}{2}}$:

$$\Big\{D_1^6 - 20D_1^3D_3 - 8D_3^2 + 18D_1D_5 - \frac{D_1^2}{2}\left(D^{\alpha}_{\frac{3}{2}} \cdot D^{\beta}_{\frac{5}{2}} - D^{\beta}_{\frac{1}{2}} \cdot D^{\alpha}_{\frac{7}{2}}\right) - \frac{9}{2}D^{\alpha}_{\frac{7}{2}} \cdot D^{\beta}_{\frac{5}{2}}\Big\} \tau \cdot \tau = 0.$$

Remark. So these equations may be viewed as a subclass of super KP equations, a particular case of super Hirota bilinear equations.

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