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PERSISTENCE OF A TWO-SPECIES COMPETITION SYSTEM IN A POLLUTED ENVIRONMENT WITH PULSE TOXICANT INPUT

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Abstract: The aim of this paper is to investigate persistence of a Lotka-Volterra competition system in a polluted environment with impulsive toxicant input at fixed moments, in which two species have different concentration of the toxicant. The effects of pulse toxicant input with constant rate on two-species Lotka-Volterra competition system is considered, and the thresholds between persistence and extinction of each population are obtained. Finally, numerical simulations are presented to illustrate our main results.

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1. Introduction

With the rapid development of modern industry and agriculture, a great quantity of toxicant and contaminant enter into ecosystems one after another. These pollutants seriously threaten the survival of the exposed species. In order to use and regulate toxic substances wisely, we must assess the risk of the populations exposed to toxicant. Therefore, it is important to study the effects of toxicant on populations and to find a theoretical threshold value, which determines the permanence or extinction of a population or species.

In recent years, using the mathematical models many investigations [2, 3] have been performed to study the effect of toxicant on biological species. In these models it is assumed invariably that the exogenous input of toxicant is continuous. However, the practical situation is often that toxicant are emitted in regular pulses. For

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example, pesticides can be sprayed instantaneously and regularly. Chemical plant and other artificial industry often termly let sewage or other pollutant into rivers, soil and air. Therefore, the assumption of the continuous input of toxicant should be replaced with an impulsive perturbation, see [5]. In [4], Liu and Zhang investigated the the Lotka-Volterra system in a polluted environment with pulse toxicant input and concluded that the population is extinct when the pulse period is less than some critical value, otherwise the population is persistent and the persistent condition also assures that there exists a unique positive periodic solution which is globally asymptotically stable. But, they assumed that two-species have identical toxicant concentration. In the real world, different species take up different toxicant in same polluted environment, and the toxicant concentrations of two species are different at identical time.

Motivated by literatures [4, 7], we considered the following two-species Lotka-Volterra competition model in a polluted environment:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{dx(t)}{dt} = x(t)\left[r_{10} - r_{11}c_1(t) - a_{11}x(t) - a_{12}y(t)\right], \\ \dfrac{dy(t)}{dt} = y(t)\left[r_{20} - r_{21}c_2(t) - a_{21}x(t) - a_{22}y(t)\right], \\ \dfrac{dc_1(t)}{dt} = k_1c_e(t) - g_1c_1(t) - m_1c_1(t), \\ \dfrac{dc_2(t)}{dt} = k_2c_e(t) - g_2c_2(t) - m_2c_2(t), \\ \dfrac{dc_e(t)}{dt} = -hc_e(t), \end{array}\right\} t \neq n\tau, \\ \left.\begin{array}{l} \Delta x(t) = 0, \Delta y(t) = 0, \\ \Delta c_i(t) = 0, (i = 1, 2), \Delta c_e(t) = b, \end{array}\right\} t = n\tau, \end{array}\right. \tag{1.1}$$

where $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$, $\Delta c_i(t) = c_i(t^+) - c_i(t)$ $(i = 1, 2)$, $\Delta c_e(t) = c_e(t^+) - c_e(t)$, $0 \leq c_i \leq 1$ $(i = 1, 2)$, $0 \leq c_e \leq 1$. The initial conditions are $x(0) > 0$, $y(0) > 0$, $c_i(0) \geq 0$ $(i = 1, 2)$, $c_e(0) \geq 0$. $x(t)$ and $y(t)$ represent the density of the two-species at time t, respectively; $c_i(t)$ $(i = 1, 2)$ is the concentration of the toxicant in the organism at time t; $c_e(t)$ is the concentration of the toxicant in the environment time t; τ is the period of the pulse effect about the exogenous input of toxicant and b is the toxicant input amount at every time; r_{i0} $(i = 1, 2)$ is the intrinsic growth rate of the ith population in the environment without toxicant concentration; r_{i1} $(i = 1, 2)$ is the dose response parameter of species i to the organismal toxicant concentration; a_{ij} measures the action of species j upon the growth rate of species i (in particular, a_{ii} represents the intraspecific competition coefficient of species i; k_i $(i = 1, 2)$ denotes environmental toxicant take up rate per unit mass organism; g_i $(i = 1, 2)$ and m_i $(i = 1, 2)$ are organismal net ingestion and depuration rates of toxicant, respectively; h represents the loss rate of toxicant from the environment itself by volatilization and all the coefficients are positive; $n \in Z^+$. By applying the theory of impulsive differential equations [7], comparison theorem and some basic properties of Lotka-Volterra competition system, we derive some thresholds between extinction and weak average persistence for the two species respectively.

For sake of convenience, we will employe the following notation:

$$K_i = \frac{k_i b}{h\left(g_i + m_i\right)} \ \left(i = 1, 2\right), \ \Delta = a_{11}a_{22} - a_{12}a_{21}, \ \Delta 1 = r_{10}a_{22} - a_{12}r_{20},$$

$$\Delta_2 = r_{20}a_{11} - a_{21}r_{10}, \ \widetilde{\Delta_1} = r_{11}a_{22} - a_{12}r_{21}, \ \widetilde{\Delta_2} = r_{21}a_{11} - a_{21}r_{11},$$

$$\gamma = r_{11}r_{20} - r_{10}r_{21}, \ \overline{\gamma} = r_{10}r_{21} - r_{20}r_{11}, \ \overline{x\left(t\right)} = \frac{1}{t}\int_0^t x\left(s\right) ds,$$

$$\overline{x\left(t\right)^*} = \lim_{t \to +\infty} \sup \frac{1}{t}\int_0^t x\left(s\right) ds, \ \overline{x\left(t\right)_*} = \lim_{t \to +\infty} \inf \frac{1}{t}\int_0^t x\left(s\right) ds.$$

2. The Basic Model and Auxiliary Lemmas

Definition 1. The species $x(t)$ (or $y(t)$) is said to be extinct if $\lim_{t\to+\infty} x(t) = 0$ (or $\lim_{t\to+\infty} y(t) = 0$).

Definition 2. It is said that the species $x(t)$ (or $y(t)$) is weak average persistent if $\overline{x(t)^*} > 0$ (or $\overline{y(t)^*} > 0$).

Definition 3. It is said that the species $x(t)$ (or $y(t)$) is nonpersistent if $\overline{x(t)^*} = 0$ (or $\overline{y(t)^*} = 0$).

Now we give some basic properties of the following subsystem of model (1.1),

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{dc_1\left(t\right)}{dt} = k_1c_e\left(t\right) - g_1c_1\left(t\right) - m_1c_1\left(t\right), \\ \dfrac{dc_2\left(t\right)}{dt} = k_2c_e\left(t\right) - g_2c_2\left(t\right) - m_2c_2\left(t\right), \\ \dfrac{dc_e\left(t\right)}{dt} = -hc_e\left(t\right), \end{array}\right\} t \neq n\tau, \\ \Delta c_i\left(t\right) = 0 \ \left(i = 1, 2\right), \ \Delta c_e\left(t\right) = b, \} \ t = n\tau. \end{array}\right. \tag{2.1}$$

Lemma 1. (see [1]) *System (2.1) has a unique positive τ-periodic solution $(\widetilde{c_1}(t), \widetilde{c_2}(t), \widetilde{c_e}(t))^T$; and for each solution $(c_1(t), c_2(t), c_e(t))^T$ of (2.1), $c_i(t) \to \widetilde{c_i}(t)$ $(i = 1, 2)$ and $c_e(t) \to \widetilde{c_e}(t)$ as $t \to +\infty$. Moreover $c_i(t) > \widetilde{c_i}(t)$ $(i = 1, 2)$ and*

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$c_e(t) > \widetilde{c_e}(t)$ *for all* $t \geq 0$ *if* $c_i(0) > \widetilde{c_i}(0)$ $(i = 1, 2)$ *and* $c_e(0) > \widetilde{c_e}(0)$, *where*

$$\begin{cases} \widetilde{c_1}(t) = \widetilde{c_1}(0)\, e^{-(g_1+m_1)(t-n\tau)} + \dfrac{kb(e^{-(g_1+m_1)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g_1-m_1)(1-e^{-h\tau})}, \\ \widetilde{c_2}(t) = \widetilde{c_2}(0)\, e^{-(g_2+m_2)(t-n\tau)} + \dfrac{kb(e^{-(g_2+m_2)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g_2-m_2)(1-e^{-h\tau})}, \\ \widetilde{c_e}(t) = \dfrac{be^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ \widetilde{c_1}(0) = \dfrac{kb(e^{-(g_1+m_1)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g_1-m_1)(1-e^{-(g_1+m_1)\tau})(1-e^{-h\tau})}, \\ \widetilde{c_2}(0) = \dfrac{kb(e^{-(g_2+m_2)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g_2-m_2)(1-e^{-(g_2+m_2)\tau})(1-e^{-h\tau})}, \\ \widetilde{c_e}(0) = \dfrac{b}{1-e^{-h\tau}}, \end{cases}$$

for $t \in (n\tau, (n+1)\tau)$.

By simple computation, we can obtain

$$\int_{n\tau}^{(n+1)\tau} \widetilde{c_i}(t)dt = \frac{k_i b}{h(g_i+m_i)} = K_i \quad (i = 1, 2). \tag{2.2}$$

From Lemma 1, we conclude that $\forall \varepsilon > 0$, the following inequalities

$$\widetilde{c_i}(t) - \varepsilon < c_i(t) < \widetilde{c_i}(t) + \varepsilon \quad (i = 1, 2) \tag{2.3}$$

hold for all t large enough.

For the convenience of description, we assume that (2.3) holds for all $t > 0$. Note that $\widetilde{c_i}(t)$ $(i = 1, 2)$ are periodic function, it indicates

$$\overline{\widetilde{c_i}(t)^*} = \overline{\widetilde{c_i}(t)_*} = \lim_{t\to+\infty} \overline{\widetilde{c_i}(t)} = \frac{K_i}{\tau} \quad (i = 1, 2). \tag{2.4}$$

Lemma 2. (see [6]) *Assume that* $f(t) \in C[R_+, R_+ - 0]$, λ *and* λ_0 *are positive constants. If there exists a positive constant* T *such that* $f(t)$ *satisfies inequality* $\ln f(t) < \lambda t - \lambda_0 \int_0^t f(s)ds$ *for all* $t > T$, *then* $\overline{f^*} \leq \frac{\lambda}{\lambda_0}$, *where*

$$\overline{f^*} = \lim_{t\to+\infty} \sup \frac{1}{t} \int_0^t f(s)\, ds.$$

The model (1.1) discussed here is a five-dimensional system. Since $c_1(t)$, $c_2(t)$ and $c_e(t)$ can be solved successfully, the model (1.1) is equivalent to the following two-dimensional subsystem of (1.1), that is

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)\left[r_{10} - r_{11}c_1(t) - a_{11}x(t) - a_{12}y(t)\right], \\ \dfrac{dy(t)}{dt} = y(t)\left[r_{20} - r_{21}c_2(t) - a_{21}x(t) - a_{22}y(t)\right], \\ x(0) > 0\ ,\ y(0) > 0, \end{cases} \tag{2.5}$$

where $c_i(t)$ $(i = 1, 2)$ satisfies the properties of system (2.1).

From system (2.5), we can obtain the following equations:

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} = r_{10} - r_{11}\overline{c_1(t)} - a_{11}\overline{x(t)} - a_{12}\overline{y(t)}, \tag{2.6}$$

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} = r_{20} - r_{21}\overline{c_2(t)} - a_{21}\overline{x(t)} - a_{22}\overline{y(t)}. \tag{2.7}$$

Here we have two assumptions:

H_1) The capacity of the environment is so large that the change of toxicant in the environment that comes from uptake and assimilation by the organisms can be neglected.

H_2) The individuals in the two-species have the different organismal toxicant concentration at time t.

Remark 1. $c_i(t)$ $(i = 1, 2)$ and $c_e(t)$ are the concentration of toxicant. To assure $0 \le c_i(t) \le 1$ $(i = 1, 2)$ and $0 \le c_e(t) \le 1$, it is necessary that $g_i \le k_i \le g_i + m_i$, $b \le 1 - e^{-h\tau}$.

Remark 2. For each solution $(x(t), y(t))^T$ of (2.5), there exist two constants $M_1 > 0$ and $M_2 > 0$ such that $0 < x(t) < M_1$ and $0 < y(t) < M_2$ for all t large enough.

Remark 3. Solutions of model (2.5) exist for all time $t \in R_+ = [0, +\infty]$, so there is no extinction of the population x and y in a finite time and any solution of (2.5) remains positive for all $t \in R_+$.

Remark 4. The necessary and sufficient condition for persistence of system (1.1) in the absence of toxicant is $\Delta > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$, that is $\frac{a_{11}}{a_{21}} > \frac{r_{10}}{r_{20}} > \frac{a_{12}}{a_{22}}$.

Remark 5. $\gamma > 0$, $0 < \frac{\Delta_1}{\widetilde{\Delta}_1} < \frac{r_{10}}{r_{11}}$; $\gamma = 0$, $\frac{\Delta_1}{\widetilde{\Delta}_1} = \frac{r_{10}}{r_{11}} = \frac{r_{20}}{r_{21}} = \frac{\Delta_2}{\widetilde{\Delta}_2}$; $\gamma < 0$, $0 < \frac{\Delta_2}{\widetilde{\Delta}_2} < \frac{r_{20}}{r_{21}}$.

3. Existence and Persistence of Competition System

Theorem 1. *Assume that $\gamma < 0$ holds, then x is weak average persistent if $\tau > \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$, nonpersistent if $\tau = K_1\frac{r_{11}}{r_{10}}$, and extinct if $\tau < K_1\frac{r_{11}}{r_{10}}$.*

Proof. (1) If $\tau > \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$, then we can choose a $\varepsilon > 0$ such that $\delta = \Delta_1 - \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\tau}$. From Remark 2, we know that $x(t)$ and $y(t)$ are bounded. Thus

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{x(t)}{x(0)} \le 0, \tag{3.1}$$

and

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{y(t)}{y(0)}\le 0, \tag{3.2}$$

hold. From (2.3), (2.4), (2.6) and (3.1), it follows that

$$\begin{aligned} a_{11}\overline{x(t)^*}+a_{12}\overline{y(t)^*} &\ge r_{10}-r_{11}\overline{c_1(t)^*}\ge r_{10}-r_{11}(\overline{\widetilde{c_1}(t)^*}+\varepsilon)\\ &= r_{10}-r_{11}\frac{K_1}{\tau}-r_{11}\varepsilon>0. \end{aligned} \tag{3.3}$$

So

$$a_{11}\overline{x(t)^*}+a_{12}\overline{y(t)^*}>0. \tag{3.4}$$

We can conclude that $\overline{x(t)^*}>0$. We assume for the purpose of contradiction that $\overline{x(t)^*}=0$ and $\overline{y(t)^*}>0$. Note that $\gamma<0$ implies that $\frac{r_{11}}{r_{21}}<\frac{r_{10}}{r_{20}}$, and $\Delta_1>0$ and $\Delta_2>0$ imply that $\frac{a_{11}}{a_{21}}>\frac{r_{10}}{r_{20}}>\frac{a_{12}}{a_{22}}$. Thus $\frac{r_{11}}{r_{21}}\in(0,\frac{r_{10}}{r_{20}})$.

(I) If $\frac{r_{11}}{r_{21}}\in(0,\frac{a_{12}}{a_{22}})$, then $\widetilde{\Delta_1}\le 0$.
From (2.6)$\times a_{22}-$(2.7)$\times a_{12}$, we obtain

$$a_{22}\frac{1}{t}\ln\frac{x(t)}{x(0)}-a_{12}\frac{1}{t}\ln\frac{y(t)}{y(0)}=\Delta_1-\Big(r_{11}a_{22}\overline{c_1(t)}-r_{21}a_{12}\overline{c_2(t)}\Big)-\Delta\overline{x(t)}. \tag{3.5}$$

It is inferred from (3.1) and (3.5) that

$$\begin{aligned} -a_{12}\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{y(t)}{y(0)} &\ge \Delta_1-[r_{11}a_{22}\overline{\widetilde{c_1}(t)^*}-r_{21}a_{12}\overline{\widetilde{c_2}(t)^*}]\\ &\ge \Delta_1-[r_{11}a_{22}(\overline{\widetilde{c_1}(t)^*}+\varepsilon)-r_{21}a_{12}(\overline{\widetilde{c_2}(t)^*}+\varepsilon)]\\ &= \Delta_1-\frac{a_{22}r_{11}K_1-a_{12}r_{21}K_2}{\tau}-\widetilde{\Delta_1}\varepsilon>0, \end{aligned}$$

that is, $\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{y(t)}{y(0)}<0$. So $\lim_{t\to+\infty}y(t)=0$, which implies that $\overline{y(t)^*}=0$ contradicts $\overline{y(t)^*}>0$.

(II) If $\frac{r_{11}}{r_{21}}\in(\frac{a_{12}}{a_{22}},\frac{r_{10}}{r_{20}})$, then $\widetilde{\Delta_1}>0$.
From(2.7)$\times r_{11}-$(2.6)$\times r_{21}$, it follows that

$$\begin{aligned} &r_{11}\frac{1}{t}\ln\frac{y(t)}{y(0)}-r_{21}\frac{1}{t}\ln\frac{x(t)}{x(0)}\\ &\qquad=\gamma+\widetilde{\Delta_2}\overline{x(t)}-\widetilde{\Delta_1}\overline{y(t)}-(r_{11}r_{21}\overline{c_2(t)}+r_{21}r_{11}\overline{c_1(t)}). \end{aligned} \tag{3.6}$$

It is inferred from (3.1) and (3.6) that $\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{y(t)}{y(0)}\le r_{11}^{-1}[\gamma-\widetilde{\Delta_1}\overline{y^*(t)}-r_{11}r_{21}(\overline{\widetilde{c_1}(t)^*}+\overline{\widetilde{c_2}(t)^*}+2\varepsilon)]$. From $\gamma<0$, we know $\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{y(t)}{y(0)}<0$. So $\overline{y(t)^*}=0$ which contradicting $\overline{y(t)^*}>0$. This completes the proof of (1).

(2) According to the specific property of the inferior limit, we know that for an arbitrarily given and sufficiently small positive ε, there is a $T > 0$ such that $\overline{c_1(t)} \geq \overline{\widetilde{c_1}(t)_*} - \varepsilon$, for all $t \geq T$.

From (2.6), it follows that

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} \leq r_{10} - r_{11}\overline{\widetilde{c_1}(t)_*} + r_{11}\varepsilon - a_{11}\overline{x(t)},$$

this is

$$\ln\frac{x(t)}{x(0)} \leq \lambda t - a_{11}\int_0^t x(s)ds,$$

where $\lambda = r_{10} - r_{11}\overline{\widetilde{c_1}(t)_*} + r_{11}\varepsilon = r_{10} - r_{11}\frac{K_1}{\tau} + r_{11}\varepsilon$. If $\tau = K_1\frac{r_{11}}{r_{10}}$, $\lambda = r_{11}\varepsilon$, in term of Lemma 2, the following inequality can be obtained,

$$\overline{x(t)^*} \leq \frac{\lambda}{a_{11}} = \frac{r_{11}\varepsilon}{a_{11}}. \tag{3.7}$$

Because ε is arbitrarily small, $\overline{x(t)^*} \leq 0$. But $\overline{x(t)^*} \geq 0$, so $\overline{x(t)^*} = 0$ is nonpersistent.

(3) If $\tau < K_1\frac{r_{11}}{r_{10}}$, then we can choose a $\varepsilon > 0$ such that $\delta = -r_{10}\tau + r_{11}K_1 - r_{11}\varepsilon\tau > 0$. From (2.6), we obtain

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t}\ln\frac{x(t)}{x(0)} &\leq r_{10} - r_{11}(\overline{\widetilde{c_1}(t)_*} - \varepsilon) - a_{11}\overline{x(t)_*} - a_{12}\overline{y(t)_*} \\ &= -\frac{\delta}{\tau} - a_{11}\overline{x(t)_*} - a_{12}\overline{y(t)_*} < 0.\end{aligned}$$

So $\lim_{t\to+\infty} x(t) = 0$, and $x(t)$ is extinct. □

Theorem 2. *Assume that $\gamma = 0$ holds, then x is weak average persistent if $\tau > \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$, nonpersistent if $\tau = K_1\frac{r_{11}}{r_{10}}$, and extinct if $\tau < K_1\frac{r_{11}}{r_{10}}$.*

Proof. (1) We can deduce from $\Delta_1 > 0$ and $\Delta_2 > 0$ that $\widetilde{\Delta_1} > 0$ and $\widetilde{\Delta_2} > 0$. It is easily verified that $\frac{\Delta_1}{\widetilde{\Delta_1}} = \frac{r_{10}}{r_{11}} = \frac{r_{20}}{r_{21}} = \frac{\Delta_2}{\widetilde{\Delta_2}}$, From (3.4), we can conclude that $\overline{x(t)^*} > 0$. If $\overline{x(t)^*} = 0$, and $\overline{y(t)^*} \leq 0$ hold, $\lim_{t\to+\infty}\frac{1}{t}\ln\frac{x(t)}{x(0)} \leq 0$ since $\lim_{t\to+\infty}\overline{x(t)} = 0$. For an arbitrarily given positive ε, there exists a $T > 0$ such that

$$\left|\widetilde{\Delta_2}\overline{x(t)}\right| < \frac{\varepsilon}{2} \quad \text{and} \quad r_{21}\frac{1}{t}\ln\frac{x(t)}{x(0)} < \frac{\varepsilon}{2},$$

for $t \geq T$.

From (3.6), we have

$$\ln\frac{y(t)}{y(0)} < (\gamma + \varepsilon)r_{11}^{-1}t - \widetilde{\Delta_1}r_{11}^{-1}\int_0^t y(s)ds.$$

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According to the Lemma 2, it follows that

$$\overline{y(t)^*} \leq \frac{\gamma+\varepsilon}{\widetilde{\Delta_1}} < 0. \tag{3.8}$$

Because of the arbitrary property of ε, $\overline{y(t)^*} \leq \frac{\gamma}{\widetilde{\Delta_1}} = 0$, which contradicts $\overline{y(t)^*} > 0$. So $x(t)$ is weak average persistent.

The proofs of nonpersistence and extinction are omitted, since they are the same as those of nonpersistence and extinction in Theorem 1. □

Theorem 3. *Assume that $\gamma > 0$ holds, then x is weak average persistent if $\tau > K_1\frac{\widetilde{\Delta_1}}{\Delta_1}$, nonpersistent if $\tau = \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$, and extinct if $\tau < \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$.*

Proof. (1) $\gamma > 0$ implies that $\frac{r_{10}}{r_{20}} < \frac{r_{11}}{r_{21}}$, $\Delta_1 > 0$. So $K_1\frac{\widetilde{\Delta_1}}{\Delta_1}$ is positive when $\gamma > 0$. If $\tau > K_1\frac{r_{11}}{r_{10}}$, from (3.3), it follows that

$$a_{11}\overline{x(t)^*} + a_{12}\overline{y(t)^*} \geq r_{10} - r_{11}\overline{c_1(t)^*} > r_{10} - r_{11}\frac{K_1}{\tau} = a_{12}\frac{\gamma}{\widetilde{\Delta_1}}. \tag{3.9}$$

So $a_{11}\overline{x(t)^*} + a_{12}\overline{y(t)^*} > 0$.

If $\overline{x(t)^*} = 0$, then $\overline{y(t)^*} > 0$. It follows from (3.9) that $\overline{y(t)^*} > \frac{\gamma}{\widetilde{\Delta_1}}$. On the other hand, from (3.8), taking $\varepsilon = \frac{(\widetilde{\Delta_1}\overline{y(t)^*} - \gamma)}{2}$, we can obtain $\overline{y(t)^*} < \frac{\gamma}{\widetilde{\Delta_1}}$. This contradiction shows that $x(t)$ is weak average persistent.

(2) If $\tau = \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$, similar to those used in the proofs of (3.7) and (3.8), it follows from (3.5) that

$$\begin{aligned}\overline{x(t)^*} &\leq \frac{\Delta_1 - [r_{11}a_{22}\overline{\widetilde{c_1}(t)^*} - r_{21}a_{12}\overline{\widetilde{c_2}(t)^*}]}{\Delta} \\ &< \frac{\Delta_1 - [r_{11}a_{22}(\overline{\widetilde{c_1}(t)^*} - \varepsilon) - r_{21}a_{12}(\overline{\widetilde{c_2}(t)^*} - \varepsilon)]}{\Delta} = \frac{\widetilde{\Delta_1}\varepsilon}{\Delta}.\end{aligned}$$

Because of the arbitrary property of ε, $\overline{x(t)^*} \geq 0$. But since $\overline{x(t)^*} \geq 0$, we have $\overline{x(t)^*} = 0$, which shows that $x(t)$ is nonpersistent.

(3) If $\tau < \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}$, we can choose a $\varepsilon > 0$ such that $\delta = -\Delta_1\tau + (r_{11}a_{22}K_1 - r_{21}a_{12}K_2) - \widetilde{\Delta_1}\varepsilon\tau > 0$. From (3.2) and (3.5), it follows that

$$\begin{aligned}\lim_{t\to+\infty}\sup\frac{1}{t}\ln\frac{x(t)}{x(0)} &\leq a_{22}^{-1}[\Delta_1 - (r_{11}a_{22}\overline{\widetilde{c_1}(t)^*} - r_{21}a_{12}\overline{\widetilde{c_2}(t)^*}) - a_{22}^{-1}\Delta\overline{x(t)_*} \\ &= -a_{22}^{-1}\frac{\delta}{\tau} - a_{22}^{-1}\Delta\overline{x(t)_*} < 0.\end{aligned}$$

So $\lim_{t\to+\infty}\overline{x(t)} = 0$ which shows that $x(t)$ is extinct. □

Now we turn to the population $y(t)$. By the transformation:

$$y_1 = y, \overline{r_{10}} = r_{20}, \overline{r_{11}} = r_{21}, b_{11} = a_{22}, b_{12} = a_{21}$$

$$y_2 = x, \overline{r_{20}} = r_{10}, \overline{r_{21}} = r_{11}, b_{22} = a_{11}, b_{21} = a_{12}.$$

We have $\overline{\gamma} = -\gamma$ and system (2.5) becomes

$$\begin{cases} \dfrac{dy_1(t)}{dt} = y_1(t)\left[\overline{r_{10}} - \overline{r_{11}}c_1(t) - b_{11}y_1(t) - b_{12}y_2(t)\right], \\ \dfrac{dy_2(t)}{dt} = y_2(t)\left[\overline{r_{20}} - \overline{r_{21}}c_2(t) - b_{21}y_1(t) - b_{22}y_2(t)\right]. \end{cases}$$

Applying the results of $x(t)$ to $y_1(t)$ gives the conclusions for the population $y(t)$ immediately.

By summing up the conclusions of Theorems 1-3, we can also obtain the following threshold theorem which generializes the threshold value for persistence and extinction of each population in system (2.5).

Theorem 4. *For the population x and y, there exist the threshold value $\mu_i (i = 1, 2)$ and $\nu_i (i = 1, 2)$ respectively such that:*

(1) If $\tau > \mu_1$, $x(t)$ is weak average persistent, if $\tau > \nu_1$, $y(t)$ is weak average persistent.

(2) If $\tau = \mu_2$, $x(t)$ is nonpersistent, if $\tau = \nu_2$, $y(t)$ is nonpersistent.

(3) If $\tau < \mu_2$, $x(t)$ is extinct, if $\tau < \nu_2$, $y(t)$ is extinct.

Here:

$$\mu_1 = \begin{cases} \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}, & \gamma \le 0, \\ K_1\frac{\widetilde{\Delta_1}}{\Delta_1}, & \gamma > 0, \end{cases} \quad \mu_2 = \begin{cases} K_1\frac{r_{11}}{r_{10}}, & \gamma \le 0, \\ \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1}, & \gamma > 0, \end{cases}$$

and $$\nu_1 = \begin{cases} \frac{a_{11}r_{21}K_2 - a_{21}r_{11}K_1}{\Delta_2}, & \overline{\gamma} \le 0, \\ K_2\frac{\widetilde{\Delta_2}}{\Delta_2}, & \overline{\gamma} > 0, \end{cases} \quad \nu_2 = \begin{cases} K_2\frac{r_{21}}{r_{20}}, & \overline{\gamma} \le 0, \\ \frac{a_{11}r_{21}K_2 - a_{21}r_{11}K_1}{\Delta_2}, & \overline{\gamma} > 0. \end{cases}$$

4. Numeric Simulation and Discussion

In this paper, we modeled the process of period input of toxicant at fixed time by the help of impulsive differential equation and studied the effect of toxicant on the two populations. We found that the pulse period τ and the release amount of toxicant b will affect the fates of each population.

Theorem 4 shows that both of species will be extinct if the pulse period τ is too short (see Figure 1). When $\tau = K_1\frac{r_{11}}{r_{10}}$, we know the species x is nonpersistent and species y is weak average persistent (see Figure 2). Both of species will be weak average persistent if this value is large enough (see Figure 3).

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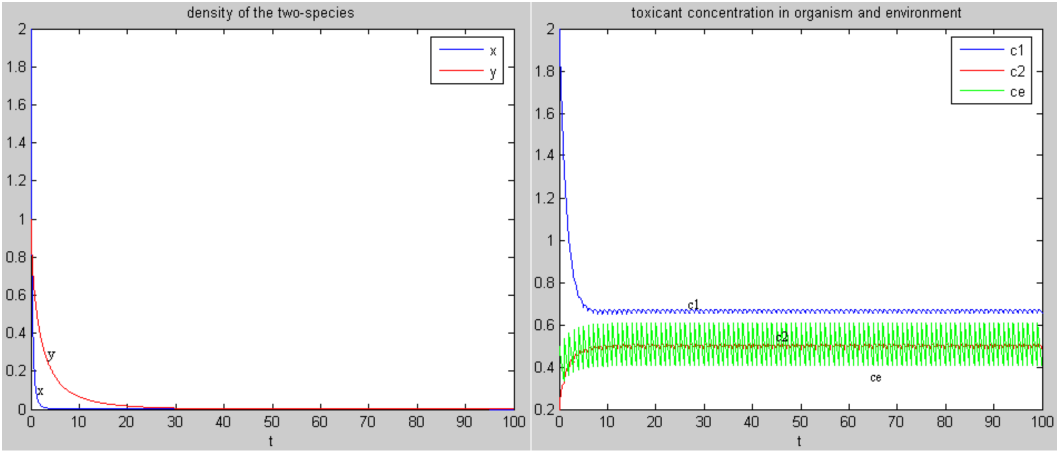

Figure 1: Dynamical behavior of the competition system (2.5) with $r_{10} = 1$, $r_{11} = 2$, $r_{20} = 0.8$, $r_{21} = 1.8$, $a_{11} = 0.6$, $a_{12} = 0.4$, $a_{21} = 0.3$, $a_{22} = 1$, $k_1 = 0.8$, $g_1 = 0.2$, $m_1 = 0.4$, $k_2 = 1$, $g_2 = 0.4$, $m_2 = 0.6$, $h = 0.4$, $b = 0.2$, $((x, y) = (2, 1))$, $\gamma = -0.2$, $T_1 = K_1 \frac{r_{11}}{r_{10}} = 1.334$, $T_2 = \frac{a_{22} r_{11} K_1 - a_{12} r_{21} K_2}{\Delta_1} = 2.029$, $\tau = 1$; $\tau < T_1$.

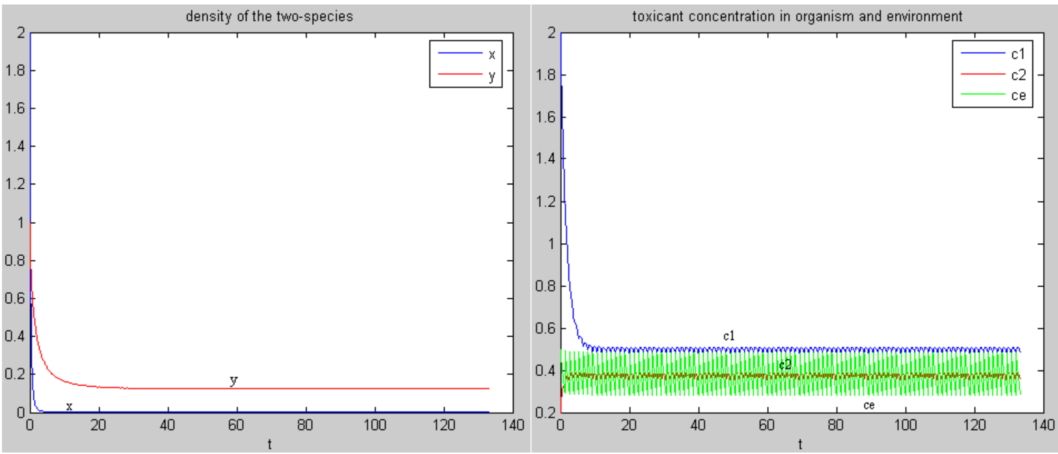

Figure 2: Dynamical behavior of the competition system (2.5) with $r_{10} = 1$, $r_{11} = 2$, $r_{20} = 0.8$, $r_{21} = 1.8$, $a_{11} = 0.6$, $a_{12} = 0.4$, $a_{21} = 0.3$, $a_{22} = 1$, $k_1 = 0.8$, $g_1 = 0.2$, $m_1 = 0.4$, $k_2 = 1$, $g_2 = 0.4$, $m_2 = 0.6$, $h = 0.4$, $b = 0.2$, $((x, y) = (2, 1))$, $\gamma = -0.2$, $T_1 = K_1 \frac{r_{11}}{r_{10}} = 1.334$, $T_2 = \frac{a_{22} r_{11} K_1 - a_{12} r_{21} K_2}{\Delta_1} = 2.029$, $\tau = 1.334$; $\tau = T_1$.

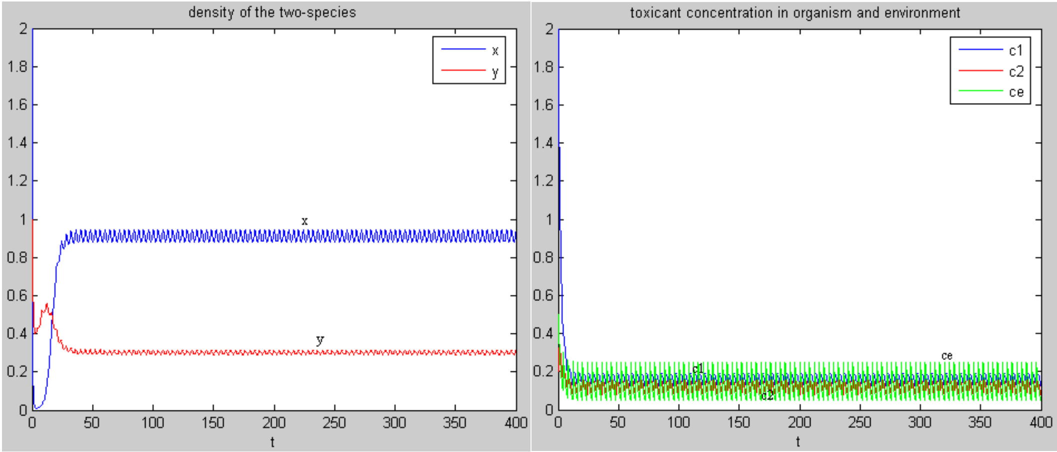

Figure 3: Dynamical behavior of the competition system (2.5) with $r_{10} = 1$, $r_{11} = 2$, $r_{20} = 0.8$, $r_{21} = 1.8$, $a_{11} = 0.6$, $a_{12} = 0.4$, $a_{21} = 0.3$, $a_{22} = 1$, $k_1 = 0.8$, $g_1 = 0.2$, $m_1 = 0.4$, $k_2 = 1$, $g_2 = 0.4$, $m_2 = 0.6$, $h = 0.4$, $b = 0.2$, $((x, y) = (2, 1))$, $\gamma = -0.2$, $T_1 = K_1 \frac{r_{11}}{r_{10}} = 1.334$, $T_2 = \frac{a_{22}r_{11}K_1 - a_{12}r_{21}K_2}{\Delta_1} = 2.029$, $\tau = 4$; $\tau > T_2$.

Acknowledgments

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