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THE EXTENSION CONSTRUCTION OF A GAME PROBLEM IN THE CLASS OF FINITELY ADDITIVE MEASURES

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Abstract: A maximin problem under constraints of asymptotic character is considered. For the investigation of the "usual" maximin asymptotics, the extension construction in the class of finitely additive measures is used. The generalized problem for which the payoff maximin defines the above-mentioned asymptotics is constructed. The sufficient conditions for the maximin stability are established.

Key Words: linear control system, maximin, attraction set, finitely additive measure

1. Introduction

It is known that in many control problems, the stability under a weakening of constraints is lacking. Namely, the small weakening of constraints can be lead to a spasmodic change of the attainable result. But, in this case, we obtain the improvement of our result. If we consider a game problem and the above-mentioned possibility can be realized for both participants called players, then the more complicated effect connected with the result change arises.

For investigation of the above-mentioned problems of control theory, traditionally extension constructions are used. Such constructions are used in game control problems too. In this connection, we note investigations of J. Warga, R.V. Gamkrelidze, N.N. Krasovskii, A.I. Subbotin and other mathematicians; see [17]-[16]. In

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this approach, generalized control problems are constructed. The results realized in the class of generalized controls correspond to results attainable in asymptotic regime. This fact is known for classical control problems with geometric constraints; the systematic investigation of control problems with such constraints was begun by L.S. Pontryagin.

In the case of impulse constraints, difficulties connected with effects of type of product of discontinuous and generalized functions arise. We note the original approach of N.N. Krasovskii connected with the employment of distributions; see [12]. This approach served by foundation for many investigations in theory of impulse control.

In this investigations, we follow (see [7]) the approach connected with the employment of finitely additive measures (FAM) for extension constructing. This approach is stated in [1]-[6]. Now we use this general approach for investigation of abstract game control problem with instability under the weakening of constraints.

The following maximin problem is considered:

$$f_0\left(\left(\int_{I_1} \alpha_i u\, d\eta_1\right)_{i\in\overline{1,k}}, \left(\int_{I_2} \beta_j v\, d\eta_2\right)_{j\in\overline{1,l}}\right) \to \sup_{v\in\mathcal{V}} \inf_{u\in\mathcal{U}}. \qquad (1.1)$$

Here $f_0 : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$, I_1 and I_2 are nonempty sets equipped with semialgebras $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively,

$$\eta_1 : \mathcal{L}_1 \to [0, \infty[, \quad \eta_2 : \mathcal{L}_2 \to [0, \infty[$$

are finitely additive measures. In (1.1), α_i and β_j are the uniform limits of real-valued step-functions defined on $(I_1, \mathcal{L}_1)$ and $(I_2, \mathcal{L}_2)$ respectively. Moreover, suppose that (see (1.1))

$$u : I_1 \to \mathbb{R}; \quad v : I_2 \to \mathbb{R}$$

are the uniform limits of step-functions of the above-mentioned type. We can choose $u \in U$ and $v \in V$, where $U \neq \varnothing$ and $V \neq \varnothing$, with the validity of the constraints

$$\left(\left(\int_{I_1} \gamma_i u\, d\eta_1\right)_{i\in\overline{1,p}} \in Y\right) \;\&\; \left(\left(\int_{I_2} \omega_j v\, d\eta_2\right)_{j\in\overline{1,q}} \in Z\right), \qquad (1.2)$$

where Y and Z are nonempty closed sets in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. It is known that extremal problems with the constraints (1.2) are (generally speaking) unstable under the weakening of these constraints. Namely, if the replacements

$$(Y \to Y_\varepsilon) \;\&\; (Z \to Z_\delta),$$

where Y_ε is the ε-neighborhood of Y and Z_δ is the δ-neighborhood Z, are realized ($\varepsilon > 0$, $\delta > 0$), then the maximin of the game problem with the (ε, δ)-weakened constraints can be very different from extremum of (1.1). In this connection the question about the maximin asymptotics representation of above-mentioned game problem with weakened constraints arises. For the obtaining of this representation we use special construction of extension in the class of finitely additive measures (FAM). Namely, the above-mentioned representation is connected with a generalized problem on the space of FAM. Another question is connected with stability conditions by results. The corresponding sufficient conditions are obtained too. But now we consider some example for which the corresponding stability is lacking.

Example. Consider the case $I_1 = I_2 = I = [0, 1[$; suppose that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, where $\mathcal{L}$ is the semialgebra of all half-intervals $[a, b[$, where $a \in [0, 1]$ and $b = [0, 1]$. Then $(I_1, \mathcal{L}_1) = (I_2, \mathcal{L}_2) = (I, \mathcal{L})$ is the known pointer-space. Let $\eta_1 = \eta_2 = \eta$, where η is the length function: $\eta([a, b[) = b - a$ for $0 \leqslant a < b \leqslant 1$ and $\eta(\varnothing) \triangleq 0$ (here and below $\triangleq$ is the equality by definition). Consider the following conditions

$$\int_I tu(t)\eta(dt) = æ_1 \qquad \text{and} \qquad \int_I tv(t)\eta(dt) = æ_2 \tag{1.3}$$

on the choice of nonnegative piece-wise constant and continuous from the right real-valued functions on I for which

$$\int_I u(t)\eta(dt) \leqslant c_1 \qquad \text{and} \qquad \int_I v(t)\eta(dt) \leqslant c_2. \tag{1.4}$$

In (1.3) and (1.4), $æ_1, æ_2, c_1$ and c_2 are nonnegative constants. For our goals the cases $æ_1 \in \{0; c_1\}$ and $æ_2 \in \{0; c_2\}$ are important. In this connection, we note that, under $æ_1 = c_1$ or $æ_2 = c_2$, the incompatible variant of our problem is realized (see (1.3) and (1.4)). Namely, we have the incompatible constraints for one from participants.

Suppose that in our example $f_0 : \mathbb{R} \times \mathbb{R} \to [0, \infty[$ is defined by the rule:

$$f_0(x, y) \triangleq |x - y|.$$

Moreover, we suppose that

$$\alpha : I \to [0, \infty[\qquad \text{and} \qquad \beta : I \to [0, \infty[$$

are defined by the simplest rule $\alpha(t) \equiv 1$ and $\beta(t) \equiv 1$. Then in the capacity of (1.1) we obtain the problem

$$\left| \int_I u(t)\eta(dt) - \int_I v(t)\eta(dt) \right| \to \sup_v \inf_u \tag{1.5}$$

where u and v satisfy (along with (1.4)) to precise or approximate variant of (1.3). For the case $æ = c_1$ and $æ = c_2$, under the weakened constrains

$$\left| \int_I tu(t)\eta(dt) - æ_1 \right| < \varepsilon, \quad \left| \int_I tv(t)\eta(dt) - æ_2 \right| < \delta, \tag{1.6}$$

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where $\varepsilon > 0$ and $\delta > 0$, the extremum of (1.5) (maximin) coincides with the member

$$\sup(\{0; c_2 - c_1\}). \tag{1.7}$$

For any variants of c_1 and c_2 we have the problem differing from the nonperturbed (incompatible) problem with precise constraints (1.3). In addition, the basic effects connected with the obtaining of (1.7) are realized by the employment of "large" rectangle impulses localized next to 1.

The analogous situation is realized under the weakening of (1.3) in the case $æ_1 = æ_2 = 0$. The number (1.7) coincides with maximin in (1.5) for the case of the constraints (1.6). But in this case the required (for the realization of (1.7)) effects are greated by "large" impulses localized next to 0. We note that in this case $æ_1 = æ_2 = 0$ the nonperturbed problem with the constraints (1.3) is compatible. The corresponding admissible usual controls are defined as the functions on I identical equal zero. As a corollary, the corresponding "precise" maximin in (1.5) coincidences with 0.

2. General Definitions and Designations

We use the standard set-theoretical symbolics (quantors, connectives and other special symbols). By $\mathcal{P}(E)$ (by $\mathcal{P}'(E)$) we denote the family of all (of all nonempty) subsets of a set E. If A and B are sets, $C \in \mathcal{P}'(A)$ and f is mapping from A into B, then $f^1(C) \triangleq \{f(x) : x \in C\}$ and

$$(f \mid C) \triangleq (f(x))_{x \in C}$$

is the usual restriction of this mapping f to the set C; of course, $(f \mid C)$ operates from C into B by the rule $(f \mid C)(x) = f(x)$. In the following, $\mathbb{N} \triangleq \{1; 2; 3; \ldots\}$ and $\mathbb{R}$ is the real line. If $m \in \mathbb{N}$, then $\overline{1,m} \triangleq \{i \in \mathbb{N} \mid i \leqslant m\}$ and for any set T by T^m we denote the set of all processions $(t_i)_{i \in \overline{1,m}} : \overline{1,m} \to T$; so, T^m is the set of all mappings from $\overline{1,m}$ into T. In particular, we consider elements of $\mathbb{R}^m$ as processions $(\xi_i)_{i \in \overline{1,m}}$, where $\xi_j \in \mathbb{R}\ \forall j \in \overline{1,m}$.

In the following, linear operations, product and order in spaces of real-valued functions are defined pointwisely.

Topological Constructions. For a topological space (TS) (X, τ) and a set $A \in \mathcal{P}(X)$, by $\mathrm{cl}(A, \tau)$ and $\tau|_A$ we denote respectively the closure of the set A and the topology of A induced from (X, τ); moreover, we suppose that $\mathcal{N}^0_\tau[A] \triangleq \{G \in \tau \mid A \subset G\}$ and

$$\mathcal{N}_\tau[A] \triangleq \{H \in \mathcal{P}(X) \mid \exists\, G \in \mathcal{N}^0_\tau[A] : G \subset H\}$$

(the family of all neighborhoods of A). If (X, τ) is a TS and $x \in X$, then

$$\left(\mathcal{N}^0_\tau(x) \triangleq \mathcal{N}^0_\tau[\{x\}]\right)\ \&\ \left(\mathcal{N}_\tau(x) \triangleq \mathcal{N}_\tau[\{x\}]\right), \tag{2.1}$$

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where $\{x\}$ is singleton (one-element set) containing the point x. Last family in (2.1) is the filter of neighborhoods of x in TS (X, τ).

If $(X, \tau_1),\ X \neq \varnothing$, and $(Y, \tau_2),\ Y \neq \varnothing$, are TS, then $C(X, \tau_1, Y, \tau_2)$ is the set of all (τ_1, τ_2)-continuous functions from X into Y. For any TS (T, τ) by $(\tau - \mathrm{comp})[T]$, we denote the family of all nonempty compact sets in the sense of (T, τ).

Nets. If $(D, \preceq)$ is a nonempty directed set and h is a mapping from D into X, then we call the triplet $(D, \preceq, h)$ by a net in a set X. If $(D, \preceq, h)$ is a net in a set X, then

$$(X - \mathrm{ass})[D, \preceq, h] \triangleq \{T \in \mathcal{P}(X) \mid \exists\, d \in D\, \forall\, \delta \in D \quad (d \preceq \delta) \Rightarrow (h(\delta) \in T)\}$$

is the filter of X associated with the net $(D, \preceq, h)$. Now, we introduce the usual Moore-Smith convergence: if $(D, \preceq, h)$ is a net in a TS (X, τ) and $x \in X$, then

$$\Big((D, \preceq, h) \xrightarrow{\tau} x\Big) \overset{\mathrm{def}}{\Leftrightarrow} \left(\mathcal{N}_\tau(x) \subset (X - \mathrm{ass})[D; \preceq; h]\right).$$

Of course, in these terms, closure of a set in TS is realized in the following known [11, 9] form: for any TS (X, τ) and a set $A \in \mathcal{P}(X)$, $\mathrm{cl}(A, \tau)$ is the set of all $x \in X$, for which there exists a net $(D, \preceq, h)$ in the set A with the following convergence property: $(D, \preceq, h) \xrightarrow{\tau} x$. For the definiteness, under $m \in \mathbb{N}$, we equip the space $\mathbb{R}^m$ with the norm $\|\cdot\|^{(m)} = \left(\|x\|^{(m)}\right)_{x \in \mathbb{R}^m}$: if $y = (y_i)_{i \in \overline{1,m}} \in \mathbb{R}^m$ then

$$\|y\|^{(m)} \triangleq \max_{i \in \overline{1,m}} |\, y_i| \in [0, \infty[;$$

then the topology $\tau_{\mathbb{R}}^{(m)}$ of the coordinate-wise convergence of $\mathbb{R}^m$ is generated by the norm $\|\cdot\|^{(m)}$. If $m \in \mathbb{N}$, $S \in \mathcal{P}(\mathbb{R}^m)$ and æ $\in]0, \infty[$, then

$$O_{æ}^{(m)}[S] \triangleq \Big\{x \in \mathbb{R}^m \mid \exists\, s \in S : \|x - s\|^{(m)} < æ\Big\} \in \tau_{\mathbb{R}}^{(m)} \tag{2.2}$$

is the open æ-neighborhood of the set S.

3. Finitely Additive Measures on a Semialgebra of Sets

In the following, we consider an abstract setting of the game problem. Namely, in the capacity of players strategies, we use step and stratum functions on a measurable spaces with semialgebras of sets. We suppose that, on the above-mentioned spaces, some FAM are fixed. In terms of integral representations realized under these fixed FAM, the game criterion is defined. As a result, we obtain the game problem (1.1).

Now, we consider the basic elements of these constructions under arbitrary fixed (in this section) space with a nonnegative FAM. For the case of a control problem, this measure is identified with the usual (countably additive) Lebesgue measure.

But, we consider an abstract variant of the game problem (1.1). Therefore, we should to introduce the corresponding spaces of players strategies. These spaces are realized as variants of the common construction. This construction is considered now.

In this section, we fix a nonempty set I and a semialgebra $\mathcal{L}$ of subsets of I. So, $(I, \mathcal{L})$ is the measurable space with a semialgebra of sets. We denote by $(\mathrm{add})_+[\mathcal{L}]$ the cone of all real-valued FAM on $\mathcal{L}$. By $\mathbb{A}(\mathcal{L})$ we denote the linear space generated by the cone $(\mathrm{add})_+[\mathcal{L}]$; elements of $\mathbb{A}(\mathcal{L})$ are FAM on $\mathcal{L}$ with the bounded total variation.

In this section, we fix $\eta \in (\mathrm{add})_+[\mathcal{L}]$; then

$$(\mathrm{add})^+[\mathcal{L};\eta] \triangleq \{\mu \in (\mathrm{add})_+[\mathcal{L}] \mid \forall\, L \in \mathcal{L}\ ((\eta(L) = 0) \Rightarrow (\mu(L) = 0))\}$$

is the cone of all weakly absolutely continuous (with respect to η) nonnegative FAM on $\mathcal{L}$. Then, the linear space $\mathbb{A}_\eta[\mathcal{L}]$ generated by the cone $(\mathrm{add})^+[\mathcal{L};\eta]$ is the set

$$\nu_1 - \nu_2,\ \nu_1 \in (\mathrm{add})^+[\mathcal{L};\eta],\ \nu_2 \in (\mathrm{add})^+[\mathcal{L};\eta];$$

$\mathbb{A}_\eta[\mathcal{L}]$ plays the important role in questions of approximate realization of FAM in the class of step-functions; see [1]-[3].

In addition,

$$(\mathrm{add})^+[\mathcal{L};\eta] \subset \mathbb{A}_\eta[\mathcal{L}] \subset \mathbb{A}(\mathcal{L}). \tag{3.1}$$

Using (3.1), we consider FAM of $(\mathrm{add})^+[\mathcal{L};\eta]$ and $\mathbb{A}_\eta[\mathcal{L}]$ as generalized elements. In connection with this, we note known statements of functional analysis (see [8, Chapter III, IV]) connected with $*$-weak compactness. Now, we consider the linear span $B_0(I, \mathcal{L})$ of all indicator functions [15, Chapter II] of sets of $\mathcal{L}$. So, $B_0(I, \mathcal{L})$ is the linear manifold of $\mathcal{L}$-step real-valued functions on I. We equip the linear space $\mathbb{B}(I)$ of all bounded real-valued functions on I with the traditional sup-norm $\|\cdot\|_I$; see [8, Chapter IV]. Then we denote by $B(I, \mathcal{L})$ the closure of $B_0(I, \mathcal{L})$ in the topology of sup-norm $\|\cdot\|_I$ of the set $\mathbb{B}(I)$. Then $B(I, \mathcal{L})$ is a Banach space similar to the space $B(S, \Sigma)$ of [8, Chapter IV]. The space $B^*(I, \mathcal{L})$ topologically conjugate with respect to $B(I, \mathcal{L})$ is isometric isomorphic to $\mathbb{A}(\mathcal{L})$ with the strong norm defined by the total variation. So, $(B(I, \mathcal{L}), \mathbb{A}(\mathcal{L}))$ is a duality; therefore, we can introduce the standard $*$-weak topology $\tau_*(\mathcal{L})$ of the set $\mathbb{A}(\mathcal{L})$. Recall that

$$(\mathbb{A}(\mathcal{L}), \tau_*(\mathcal{L}))$$

is the locally convex σ- compactum (see [8, Chapter IV]). The corresponding compactness conditions are defined by Alaoglu Theorem.

We denote by $\tau_\mathbb{R}$ (by τ_∂) the usual (discrete) topology of the real line $\mathbb{R}$. Then [1, p. 80], by $\otimes^{\mathcal{L}}(\tau_\mathbb{R})$ (by $\otimes^{\mathcal{L}}(\tau_\partial)$) we denote the natural topology of the Tichonoff power of samples $(\mathbb{R}, \tau_\mathbb{R})$ (of samples $(\mathbb{R}, \tau_\partial)$). Moreover, following to [1, (4.2.8), (4.2.9)] we suppose that

$$\tau_\otimes(\mathcal{L}) \triangleq \otimes^{\mathcal{L}}(\tau_\mathbb{R})|_{\mathbb{A}(\mathcal{L})}, \quad \tau_0(\mathcal{L}) \triangleq \otimes^{\mathcal{L}}(\tau_\partial)|_{\mathbb{A}(\mathcal{L})};$$

we obtain two TS for which (see [2, p. 45]) $\tau_\otimes(\mathcal{L}) \subset \tau_*(\mathcal{L})$ and $\tau_\otimes(\mathcal{L}) \subset \tau_0(\mathcal{L})$. It is know that (see [1, §4.4]) topologies $\tau_*(\mathcal{L})$ and $\tau_0(\mathcal{L})$ are incongruent, but [2, (3.4.19), (3.5.6)]

$$\tau_*(\mathcal{L})|_K = \tau_\otimes(\mathcal{L})|_K \subset \tau_0(\mathcal{L})|_K \ \forall K \in (\tau_*(\mathcal{L}) - \text{comp})[\mathbb{A}(\mathcal{L})]. \tag{3.2}$$

So, for a $*$-weak compact sets the corresponding "narrowing" of $\tau_*(\mathcal{L})$ and $\tau_0(\mathcal{L})$ are comparable. By $B_0^+(I,\mathcal{L})$ and $B^+(I,\mathcal{L})$ we denote the sets of all nonnegative functions of $B_0(I,\mathcal{L})$ and $B(I,\mathcal{L})$ respectively.

If $f : I \to \mathbb{R}$, then $\parallel f \parallel$ is by definition the mapping

$$x \mapsto |f(x)| : \ I \to [0,\infty[;$$

if $f \in B_0(I,\mathcal{L})$ (respectively $f \in B(I,\mathcal{L})$), then $\parallel f \parallel \in B_0^+(I,\mathcal{L}) \ \ (\parallel f \parallel \in B^+(I,\mathcal{L}))$. We use the simplest integration scheme of Section 3.4 of [1]. We note that

$$fg \in B(I,\mathcal{L}) \ \forall f \in B(I,\mathcal{L}) \ \forall g \in B(I,\mathcal{L}).$$

Using this property, we define (see [1, p. 24]) the indefinite η-integral of arbitrary function $f \in B(I,\mathcal{L})$; namely, $f * \eta \in \mathbb{A}_\eta[\mathcal{L}]$ has the following property:

$$\int_I sf\,d\eta = \int_I s\,d(f*\eta) \ \forall s \in B(I,\mathcal{L}). \tag{3.3}$$

If $\mu \in \mathbb{A}(\mathcal{L})$, then $\mathbf{v}_\mu \in (\text{add})_+[\mathcal{L}]$ is the variation of μ as a set function. Moreover, we note that

$$\mu \mapsto \mathbf{v}_\mu(I) : \mathbb{A}(\mathcal{L}) \to [0,\infty[$$

is the strong norm of $\mathbb{A}(\mathcal{L})$. We suppose that

$$\mathbf{B}_*(\mathcal{L},c) \triangleq \{\mu \in \mathbb{A}(\mathcal{L}) \mid \mathbf{v}_\mu(I) \leqslant c\} \ \ \forall c \in [0,\infty[.$$

In the following generalized moment constraints on the choice of elements of $B(I,\mathcal{L})$ are used. If $m \in \mathbb{N}$, $(h_i)_{i\in\overline{1,m}} \in B(I,\mathcal{L})^m$, and $X \in \mathcal{P}'(\mathbb{R}^m)$, then we consider the constraints of the form

$$\left(\int_I h_i f\,d\eta\right)_{i\in\overline{1,m}} \in X. \tag{3.4}$$

Moreover, we consider weakened analogs of (3.4), when the set X is replaced by ε-neighborhoods $O_\varepsilon^{(m)}[X]$, $\varepsilon > 0$. This parameter ε is not fixed; therefore, we obtain a constraint of asymptotic character. In the following, we suppose that in (3.4) X is a closed $\left(\text{in } \left(\mathbb{R}^m, \tau_{\mathbb{R}}^{(m)}\right)\right)$ set.

Now, we consider a model of such that (in the following) will use under description of the behavior of players. Therefore, now we fix a set

$$W \in \mathcal{P}'(B(I,\mathcal{L}))$$

with the following property of the integral boundedness:

$$\exists\, c \in [0, \infty[: \int_I \| w \| \, d\eta \leqslant c \quad \forall w \in W. \tag{3.5}$$

For the definiteness, we fix the number $\mathbf{c} \in [0, \infty[$ for witch

$$\int_I \| w \| \, d\eta \leqslant \mathbf{c} \quad \forall w \in W. \tag{3.6}$$

The elements of W are considered as controls in the problem about the validity of the X-constraint (see (3.4)), where X is a nonempty closed set in $\mathbb{R}^m$, under $m \in \mathbb{N}$; so, $X \subset \mathbb{R}^m$.

Moreover, we introduce the space of generalized elements in the form

$$\widetilde{W} \triangleq \mathrm{cl}(\{f * \eta : f \in W\}, \tau_*(\mathcal{L})). \tag{3.7}$$

By (3.5) and the know Alaoglu Theorem we have the inclusion

$$\widetilde{W} \in (\tau_*(\mathcal{L}) - \mathrm{comp})[\mathbb{A}(\mathcal{L})]. \tag{3.8}$$

We consider the following constraint on the choice of a FAM $\mu \in \widetilde{W}$:

$$\left(\int_I h_i \, d\mu \right)_{i \in \overline{1,m}} \in X \tag{3.9}$$

(recall that $(h_i)_{i \in \overline{1,m}} \in B(I, \mathcal{L})^m$ corresponds to (3.4)). We compare (3.9) and the weakened analogs of (3.4). For this, we introduce the sets

$$W_\partial[\text{æ}] \triangleq \left\{ w \in W \mid \left(\int_I h_i w \, d\eta \right)_{i \in \overline{1,m}} \in O^{(m)}_{\text{æ}}[X] \right\} \in \mathcal{P}(W) \quad \forall\, \text{æ} \in]0, \infty[. \tag{3.10}$$

In connection with (3.9), we introduce the set

$$\widetilde{W}_\partial \triangleq \left\{ \mu \in \widetilde{W} \mid \left(\int_I h_i \, d\mu \right)_{i \in \overline{1,m}} \in X \right\}. \tag{3.11}$$

Proposition 3.1. *The following equality is valid:*

$$\widetilde{W}_\partial = \bigcap_{\text{æ} \in]0, \infty[} \mathrm{cl}(\{f * \eta : f \in W_\partial[\text{æ}]\}, \tau_*(\mathcal{L})). \tag{3.12}$$

The proof is similar to the proof of Theorem 5.3.1 of monograph [1]. But, we consider the scheme of this proof denoting by $\widehat{W}$ the set on the right side of (3.12).

Let $\mu_* \in \widetilde{W}_\partial$. Then, $\mu_* \in \widetilde{W}$ and the inclusion

$$\left(\int_I h_i\, d\mu_*\right)_{i\in\overline{1,m}} \in X \tag{3.13}$$

is valid. With the employment of (3.7), we choose a net $(D, \preceq, g)$ in W for which

$$(D, \preceq, (g(\delta) * \eta)_{\delta\in D}) \overset{\tau_*(\mathcal{L})}{\longrightarrow} \mu_* \tag{3.14}$$

(in addition, we use axiom of choice). From (3.14) and [3, (4.6.16)] we have the convergence properties

$$\left(D, \preceq, \left(\int_I h_i\, d(g(\delta) * \eta)\right)_{\delta\in D}\right) \overset{\tau_{\mathbb{R}}}{\longrightarrow} \int_I h_i\, d\mu_* \quad \forall\, i \in \overline{1,m}. \tag{3.15}$$

From (3.3), (3.15) and the definition of $\tau_{\mathbb{R}}^{(m)}$, the convergence follows:

$$\left(D, \preceq, \left(\left(\int_I h_i g(\delta)\, d\eta\right)_{i\in\overline{1,m}}\right)_{\delta\in D}\right) \overset{\tau_{\mathbb{R}}^{(m)}}{\longrightarrow} \left(\int_I h_i\, d\mu_*\right)_{i\in\overline{1,m}}. \tag{3.16}$$

By (2.2) $O_{æ}^{(m)}[X] \in \mathcal{N}^0_{\tau_{\mathbb{R}}^{(m)}}[X]$ $\forall\, æ \in]0,\infty[$. Using (3.13), we obtain the following useful property. Namely, if $\zeta \in]0,\infty[$, then by (3.16)

$$O_\zeta^{(m)}[X] \in \mathcal{N}^0_{\tau_{\mathbb{R}}^{(m)}}\left(\left(\int_I h_i\, d\mu_*\right)_{i\in\overline{1,m}}\right),$$

and, for some $\delta_\zeta \in D$, the obvious statement is valid: under $\delta \in D$ with the property $\delta_\zeta \preceq \delta$

$$\left(\int_I h_i g(\delta)\, d\eta\right)_{i\in\overline{1,m}} \in O_\zeta^{(m)}[X]$$

and, as a corollary, $g(\delta) \in W_\partial[\zeta]$ and $g(\delta) * \eta \in \mathfrak{W}_\zeta$, where $\mathfrak{W}_\zeta \triangleq \{f * \eta : f \in W_\partial[\zeta]\}$. Therefore, by (3.14) $\mu_* \in \mathrm{cl}(\mathfrak{W}_\zeta, \tau_*(\mathcal{L}))$. Since the choice of ζ was arbitrary, then

$$\mu_* \in \mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})) \quad \forall\, æ \in]0,\infty[.$$

In other words, $\mu_* \in \widehat{W}$. So, the inclusion

$$\widetilde{W}_\partial \subset \widehat{W} \tag{3.17}$$

is established. Let $\mu^* \in \widehat{W}$. Then, $\mu^* \in \mathbb{A}(\mathcal{L})$ and

$$\mu^* \in \mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})) \quad \forall\, æ \in]0,\infty[.$$

Fix $æ^0 \in]0,\infty[$. Then, $æ_0 \triangleq \frac{æ^0}{2} \in]0,\infty[$ and by [3, (4.6.1), (4.6.2), (4.6.5)]

$$\mathfrak{N}_0 \triangleq \left\{ \nu \in \mathbb{A}(\mathcal{L}) \mid \left| \int_I h_i \, d\mu^* - \int_I h_i \, d\nu \right| < æ_0 \ \forall i \in \overline{1,m} \right\} \in \mathcal{N}^0_{\tau_*(\mathcal{L})}(\mu^*). \tag{3.18}$$

Therefore, $\{f * \eta : f \in W_\partial[æ_0]\} \cap \mathfrak{N}_0 \neq \varnothing$. Then, for some function $w \in W_\partial[æ_0]$, the inclusion $w * \eta \in \mathfrak{N}_0$ is valid. Therefore, by (3.3) and (3.18) $\left|\int_I h_i \, d\mu^* - \int_I h_i w \, d\eta\right| < æ_0 \ \ \forall i \in \overline{1,m}$. Using the definition of $\|\cdot\|^{(m)}$, we obtain that

$$\left\| \left(\int_I h_i \, d\mu^* \right)_{i \in \overline{1,m}} - \left(\int_I h_i w \, d\eta \right)_{i \in \overline{1,m}} \right\|^{(m)} < æ_0.$$

By the choice of w we have (see (2.2) and (3.10)) $\left\| \left(\int_I h_i w \, d\eta\right)_{i\in\overline{1,m}} - \mathbf{x} \right\|^{(m)} < æ_0$ for some $\mathbf{x} \in X$. From the two last inequalities, we obtain that

$$\left\| \left(\int_I h_i \, d\mu^* \right)_{i \in \overline{1,m}} - \mathbf{x} \right\|^{(m)} < æ^0.$$

By (2.2) and (3.10) the inclusion

$$\left(\int_I h_i \, d\mu^* \right)_{i \in \overline{1,m}} \in O_{æ^0}[X].$$

Since the choice of $æ^0$ was arbitrary, the following property takes place:

$$\left(\int_I h_i \, d\mu^* \right)_{i \in \overline{1,m}} \in O_{æ}[X] \ \ \forall\, æ \in]0,\infty[.$$

This means that (see (2.2)) $\forall\, æ \in]0,\infty[\ \exists\, \mathbf{x}_æ \in X$:

$$\left\| \left(\int_I h_i \, d\mu^* \right)_{i \in \overline{1,m}} - \mathbf{x}_æ \right\|^{(m)} < æ.$$

We obtain that $\left(\int_I h_i \, d\mu^*\right)_{i\in\overline{1,m}} \in \mathrm{cl}\left(X, \tau_{\mathbb{R}}^{(m)}\right)$. Then,

$$\left(\int_I h_i \, d\mu^* \right)_{i \in \overline{1,m}} \in X, \tag{3.19}$$

since X is a closed set. In addition, $\mu^* \in \mathrm{cl}(\{f * \eta : f \in W_\partial[1]\}, \tau_*(\mathcal{L}))$, where $W_\partial[1] \subset W$ according to (3.10). In addition, by (3.7) $\mathrm{cl}(\{f * \eta : f \in W_\partial[1]\}, \tau_*(\mathcal{L})) \subset \widetilde{W}$. Therefore, $\mu^* \in \widetilde{W}$. Using (3.11) and (3.19), we obtain that $\mu^* \in \widetilde{W}_\partial$. Since the choice of μ^* was arbitrary, the inclusion $\widehat{W} \subset \widetilde{W}_\partial$ is established. With regard of (3.17), the equality $\widetilde{W}_\partial = \widehat{W}$ is established.

Corollary 3.1. *The following two conditions are equivalent:* (1) $\widetilde{W}_\partial \neq \varnothing$; (2) $W_\partial[æ] \neq \varnothing \ \forall\, æ \in]0,\infty[$.

Proof. Let (1) be fulfilled. Then by Proposition 3.1

$$\mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})) \neq \varnothing \ \forall\, æ \in]0,\infty[.$$

Since $\mathrm{cl}(\varnothing, \tau_*(\mathcal{L})) = \varnothing$, the property (2) is valid. So, (1) $\Rightarrow$ (2).

Let (2) be fulfilled. Then, by (3.7) and (3.10), under $æ \in]0,\infty[$, the set $\mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L}))$ is closed (in the sense of $\tau_*(\mathcal{L})$) and not empty. So, the family

$$\mathfrak{F} \triangleq \{\mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})) : æ \in]0,\infty[\} \tag{3.20}$$

is a centered system of closed subsets of $\mathbb{A}(\mathcal{L})$ (we use the obvious property: under $0 < æ_1 \leqslant æ_2$, the inclusion $W_\partial[æ_1] \subset W_\partial[æ_2]$ is valid; see (3.10)). Now, we recall (3.8). For this, we note that under $æ \in]0,\infty[$

$$\begin{aligned}\mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})|_{\widetilde{W}}) &= \mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})) \cap \widetilde{W} \\ &= \mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})),\end{aligned} \tag{3.21}$$

since by (3.10) $W_\partial[æ] \subset W$ and, as a corollary (see (3.7)),

$$\{f * \eta : f \in W_\partial[æ]\} \subset \{f * \eta : f \in W\}$$

and by (3.7) and the monotonicity of the closure operation

$$\mathrm{cl}(\{f * \eta : f \in W_\partial[æ]\}, \tau_*(\mathcal{L})) \subset \widetilde{W}.$$

So, by (3.20) and (3.21) $\mathfrak{F}$ is the centered system of closed sets in the compact TS

$$\left(\widetilde{W}, \tau_*(\mathcal{L})|_{\widetilde{W}}\right).$$

Then, the intersection of all sets of $\mathfrak{F}$ is not empty (see [9, § 3.1])

$$\widetilde{W}_\partial = \bigcap_{F \in \mathfrak{F}} F \neq \varnothing.$$

So, (1) is valid. The implication (2) $\Rightarrow$ (1) is established. □

Now, we recall a simple variant of Proposition 5.2.1 of monograph [2]. For this we fix $s \in \mathbb{N}$ and

$$(\pi_i)_{i \in \overline{1,s}} \in B(I, \mathcal{L})^s.$$

So, $\pi_1 \in B(I, \mathcal{L}), \ldots, \pi_s \in B(I, \mathcal{L})$. Later, we introduce the mapping

$$w \mapsto \left(\int_I \pi_i w \, d\eta\right)_{i \in \overline{1,s}} : W \to \mathbb{R}^s. \tag{3.22}$$

For brevity, we denote this mapping (3.22) by Π; then $\Pi : W \to \mathbb{R}^s$ and

$$\Pi(w) = \left(\int_I \pi_i w\, d\eta\right)_{i\in\overline{1,s}} \quad \forall\, w \in W. \tag{3.23}$$

Moreover, let $\widetilde{\Pi} : \widetilde{W} \to \mathbb{R}^s$ is defined by the rule:

$$\widetilde{\Pi}(\mu) \triangleq \left(\int_I \pi_i\, d\mu\right)_{i\in\overline{1,s}} \quad \forall\, \mu \in \widetilde{W}. \tag{3.24}$$

Finally, let $\mathcal{J} : W \to \widetilde{W}$ be the mapping such that

$$\mathcal{J}(w) \triangleq w * \eta \quad \forall\, w \in W. \tag{3.25}$$

With regard of (3.3), (3.24), and (3.25), we have the equality chain

$$(\widetilde{\Pi}\circ\mathcal{J})(w) = \widetilde{\Pi}(\mathcal{J}(w)) = \left(\int_I \pi_i\, d(w*\eta)\right)_{i\in\overline{1,s}} = \left(\int_I \pi_i w\, d\eta\right)_{i\in\overline{1,s}} = \Pi(w) \quad \forall\, w \in W.$$

So, $\widetilde{\Pi} \circ \mathcal{J} = \Pi$. From (3.16) and the definition of $\tau_*(\mathcal{L})$, we obtain that

$$\widetilde{\Pi} \in C\left(\widetilde{W}, \tau_*(\mathcal{L})|_{\widetilde{W}}, \mathbb{R}^s, \tau_{\mathbb{R}}^{(s)}\right); \tag{3.26}$$

of course, in (3.26), we use the known properties of $\tau_*(\mathcal{L})$ and the contraction operation (see [3, (2.5.30), (4.6.16)]).

Proposition 3.2. *The following chain of equalities are valid:*

$$\bigcap_{æ\in]0,\infty[} \mathrm{cl}\left(\left\{\left(\int_I \pi_i w\, d\eta\right)_{i\in\overline{1,s}} : w \in W_\partial[æ]\right\}, \tau_{\mathbb{R}}^{(s)}\right) = \bigcap_{æ\in]0,\infty[} \mathrm{cl}\left(\Pi^1(W_\partial[æ]), \tau_{\mathbb{R}}^{(s)}\right)$$

$$= \widetilde{\Pi}^1\left(\widetilde{W}_\partial\right) = \left\{\left(\int_I \pi_i\, d\mu\right)_{i\in\overline{1,s}} : \mu \in \widetilde{W}_\partial\right\}. \tag{3.27}$$

The proof is a particular case of the proof of Proposition 5.2.1 in [2]. But, we consider the corresponding scheme supposing that A and B are the first and last sets in (3.27) respectively. Then, by (3.23) we obtain the coincidence of A and the second set in (3.27). Moreover, by (3.24) the third set in (3.27) and B coincide too. So, we should establish the coincidence of the second and third sets in (3.27). For this we use Proposition 5.2.1 in [2]. In addition, we note that by (3.8)

$$\left(\widetilde{W}, \tau_*(\mathcal{L})|_{\widetilde{W}}\right) \tag{3.28}$$

is a nonempty compactum and $\left(\mathbb{R}^s, \tau_{\mathbb{R}}^{(s)}\right)$ is a nonempty Hausdorff TS.

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By (3.26) $\widetilde{\Pi}$ is the continuous mapping from TS (3.28) into $\left(\mathbb{R}^s, \tau_{\mathbb{R}}^{(s)}\right)$. Finally,

$$\Pi = \widetilde{\Pi} \circ \mathcal{J}. \tag{3.29}$$

Now, we use the above-mentioned Proposition 5.2.1 of [2] under the following specific system:

$$X = W,\ Y = \mathbb{R}^s,\ K = \widetilde{W},\ \tau = \tau_{\mathbb{R}}^{(s)},\ \vartheta = \tau_*(\mathcal{L})|_{\widetilde{W}},\ s = \Pi,\ m = \mathcal{J},\ g = \widetilde{\Pi}.$$

By (3.29) we obtain (in particular [2, p.147]) the equality

$$\bigcap_{H \in \mathcal{X}} \mathrm{cl}\left(\Pi^1(H), \tau_{\mathbb{R}}^{(s)}\right) = \widetilde{\Pi}^1\left(\bigcap_{H \in \mathcal{X}} \mathrm{cl}\left(\mathcal{J}^1(H), \tau_*(\mathcal{L})|_{\widetilde{W}}\right)\right) \tag{3.30}$$

under $\mathcal{X} \in \mathcal{B}[W]$, where $\mathcal{B}[W]$ is defined by [2, (3.3.8)] (the set of all directed dually with respect to inclusion nonempty subfamilies of $\mathcal{P}(W)$). Now, we note the following obvious property: by (3.10)

$$\{W_\partial[æ] : æ \in]0, \infty[\,\} \in \mathcal{B}[W]. \tag{3.31}$$

So, we use the family (3.31) instead of $\mathcal{X}$ in (3.30). Then, from (3.25), (3.30) and (3.31)

$$\bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\Pi^1(W_\partial[æ]), \tau_{\mathbb{R}}^{(s)}\right) = \widetilde{\Pi}^1\left(\bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})|_{\widetilde{W}}\right)\right)$$
$$= \widetilde{\Pi}^1\left(\bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\{w * \eta : w \in W_\partial[æ]\}, \tau_*(\mathcal{L})|_{\widetilde{W}}\right)\right). \tag{3.32}$$

In addition, by the definition of $\mathcal{J}$, for any $æ \in]0, \infty[$, $\mathcal{J}^1(W_\partial[æ]) \in \mathcal{P}\left(\widetilde{W}\right)$ and, as a corollary,

$$\mathrm{cl}\left(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})|_{\widetilde{W}}\right) = \mathrm{cl}\left(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})\right) \cap \widetilde{W}, \tag{3.33}$$

where by (3.7) $\widetilde{W}$ is a closed set in $(\mathbb{A}(\mathcal{L}), \tau_*(\mathcal{L}))$ with the property: $\mathcal{J}^1(W_\partial[æ]) \subset \widetilde{W}$. As a corollary, $\mathrm{cl}(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})) \subset \widetilde{W}$ and by (3.33)

$$\mathrm{cl}(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})|_{\widetilde{W}}) = \mathrm{cl}(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})) \tag{3.34}$$

for any $æ \in]0, \infty[$. From (3.25), (3.32) and (3.34), the equality

$$\bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\Pi^1(W_\partial[æ]), \tau_{\mathbb{R}}^{(s)}\right) = \widetilde{\Pi}^1\left(\bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\mathcal{J}^1(W_\partial[æ]), \tau_*(\mathcal{L})\right)\right)$$

$$= \widetilde{\Pi}^1 \left(\bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\{w * \eta : w \in W_\partial[æ]\}, \tau_*(\mathcal{L})\right) \right). \tag{3.35}$$

From (3.35) and Proposition 3.1, we have the chain of equalities:

$$A = \bigcap_{æ \in]0,\infty[} \mathrm{cl}\left(\Pi^1(W_\partial[æ]), \tau_{\mathbb{R}}^{(s)}\right) = \widetilde{\Pi}^1\left(\widetilde{W}_\partial\right) = B. \qquad \square$$

Now, we use Proposition 3.6.1 of monograph [3]. For this, we note that by (2.2)

$$O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right] \in \mathcal{N}^0_{\tau_{\mathbb{R}}^{(s)}}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right] \ \forall \zeta \in]0,\infty[. \tag{3.36}$$

By Proposition 3.2, (3.36), and [3, (3.6.4)] the following statement is valid.

Proposition 3.3. *If $\zeta \in]0,\infty[$, then for some $\xi \in]0,\infty[$,*

$$\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right) \subset \mathrm{cl}\left(\Pi^1(W_\partial[\xi]), \tau_{\mathbb{R}}^{(s)}\right) \subset O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right].$$

Proof. The proof follows from [3, (3.2.8),(3.6.4)] immediately. But, for a completeness, we give the corresponding scheme. Namely, by Proposition 3.2 we obtain that $\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right) = (\tau - \mathrm{LIM})[\mathcal{U} \mid f]$ of [3, (3.2.8)], where $\tau = \tau_{\mathbb{R}}^{(s)}$, $f = \Pi$, and

$$\mathcal{U} = \{W_\partial[æ] : æ \in]0,\infty[\,\}. \tag{3.37}$$

In addition, in (3.37), $\mathcal{U} \in \mathcal{P}'(\mathcal{P}(W))$ has the property:

$$\forall B_1 \in \mathcal{U}\ \forall B_2 \in \mathcal{U}\ \exists B_3 \in \mathcal{U} :\ B_3 \subset B_1 \cap B_2 \tag{3.38}$$

(really, the dependence $æ \mapsto W_\partial[æ] :\ ]0,\infty[\to \mathcal{P}(W)$ is monotone). Then, by [3, (3.2.1)] and (3.38), for $\mathcal{U}$ (3.37), the property $\mathcal{U} \in \mathcal{B}[W]$ takes place. Now, we recall (3.5). Then, for $w \in W$ and $i \in \overline{1,s}$

$$|\,\Pi(w)(i)| = \left|\int_I \pi_i w\, d\eta\right| \leqslant \int_I \|\,\pi_i w\,\|\, d\eta \leqslant \|\pi_i\| \int_I \|\,w\,\|\, d\eta \leqslant \mathbf{c}\|\pi_i\|.$$

Then, by (3.38) we obtain that the number $\mathbf{c}_\pi \triangleq \mathbf{c} \max_{i \in \overline{1,s}} \|\pi_i\| \in [0,\infty[$ has the following property:

$$\|\Pi(w)\|_s \leqslant \mathbf{c}_\pi \quad \forall w \in W.$$

In particular, for $æ \in]0,\infty[$, we obtain the inequalities $\|\Pi(w)\|_s \leqslant \mathbf{c}_\pi \ \ \forall w \in W_\partial[æ]$. As a corollary,

$$\|y\|_s \leqslant \mathbf{c}_\pi\ \forall æ \in]0,\infty[\ \forall y \in \mathrm{cl}\left(\Pi^1(W_\partial[æ]), \tau_{\mathbb{R}}^{(s)}\right).$$

So, under $æ \in]0,\infty[$, the set $\mathrm{cl}\left(\Pi^1(W_\partial[æ]),\tau_{\mathbb{R}}^{(s)}\right)$ is bounded and closed in $\left(\mathbb{R}^s,\tau_{\mathbb{R}}^{(s)}\right)$. So, any such set is compact in the sense of $\tau_{\mathbb{R}}^{(s)}$.

Recall that in our case by (3.36) and [3, (3.2.8)]

$$O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right] = O_\zeta^{(s)}[(\tau - \mathrm{LIM})[\mathcal{U} \mid f]] \in \mathcal{N}^0_{\tau_{\mathbb{R}}^{(s)}}[(\tau - \mathrm{LIM})[\mathcal{U} \mid \Pi]] \tag{3.39}$$

under designations formulated in the proof beginning. Then, by (3.39) and Proposition 3.6.1 of [3] (with the employment of the above-mentioned compactness property) we obtain that, for some $H \in \mathcal{U}$, where $\mathcal{U}$ is defined in (3.37), the inclusion

$$O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right] \in \mathcal{N}^0_{\tau_{\mathbb{R}}^{(s)}}\left[\mathrm{cl}\left(\Pi^1(H),\tau_{\mathbb{R}}^{(s)}\right)\right] \tag{3.40}$$

is valid. From (3.37) and (3.40), we obtain the property

$$O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right] \in \mathcal{N}^0_{\tau_{\mathbb{R}}^{(s)}}\left[\mathrm{cl}\left(\Pi^1(W_\partial[\xi]),\tau_{\mathbb{R}}^{(s)}\right)\right], \tag{3.41}$$

where $\xi \in]0,\infty[$. From definitions of Section 2 and (3.41), for such number ξ,

$$\mathrm{cl}\left(\Pi^1\left(W_\partial[\xi]\right),\tau_{\mathbb{R}}^{(s)}\right) \subset O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right]. \tag{3.42}$$

By Proposition 3.2 and (3.42) we obtain the required chain of inclusions:

$$\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right) \subset \mathrm{cl}\left(\Pi^1\left(W_\partial[\xi]\right),\tau_{\mathbb{R}}^{(s)}\right) \subset O_\zeta^{(s)}\left[\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)\right]. \qquad \square$$

We note that Corollary 3.1, Propositions 3.2, and 3.3 will be used in two concrete variants corresponding to the employment in constructions of the first and second players. General scheme of this employment corresponds to [7].

4. Main Game Problem and its Relaxations

In this section we return to the game problem (1.1). Now, we make the more precise statement with the employment of designations of Section 3. Namely, we fix the measurable spaces $(I_1,\mathcal{L}_1)$, $I_1 \neq \varnothing$, and $(I_2,\mathcal{L}_2)$, $I_2 \neq \varnothing$, with semialgebras of sets ($\mathcal{L}_1$ is semialgebra of subsets of I_1 and $\mathcal{L}_2$ is semialgebra of subsets of I_2). Moreover, let

$$(\eta_1 \in (\mathrm{add})_+[\mathcal{L}_1]) \ \& \ (\eta_2 \in (\mathrm{add})_+[\mathcal{L}_2]). \tag{4.1}$$

So, by (4.1) we obtain two finitely-additive measure spaces:

$$(I_1,\mathcal{L}_1,\eta_1) \ \& \ (I_2,\mathcal{L}_2,\eta_2). \tag{4.2}$$

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As a result, in (4.2) we have two variants of the space $(I, \mathcal{L}, \eta)$ of Section 3. Fix the following two sets:

$$(U \in \mathcal{P}'(B(I_1, \mathcal{L}_1))) \ \& \ (V \in \mathcal{P}'(B(I_2, \mathcal{L}_2))).$$

We suppose that for some $c_U \in [0, \infty[$ and $c_V \in [0, \infty[$

$$\left(\int_{I_1} \| u \| \, d\eta_1 \leqslant c_U \ \forall \, u \in U\right) \ \& \ \left(\int_{I_2} \| v \| \, d\eta_2 \leqslant c_V \ \forall \, v \in V\right). \tag{4.3}$$

Now, we fix $k \in \mathbb{N}$, $l \in \mathbb{N}$, $(\alpha_i)_{i \in \overline{1,k}} \in B(I_1, \mathcal{L}_1)^k$, and $(\beta_j)_{j \in \overline{1,l}} \in B(I_2, \mathcal{L}_2)^l$. Then, for some number $m_0 \in [0, \infty[$ we obtain that

$$\left(\|\alpha_i\|_{I_1} \leqslant m_0 \ \forall \, i \in \overline{1,k}\,\right) \ \& \ \left(\|\beta_j\|_{I_2} \leqslant m_0 \ \forall \, j \in \overline{1,l}\,\right).$$

Then, we have the following two systems of inequalities:

$$\left|\int_{I_1} \alpha_i u \, d\eta_1\right| \leqslant \int_{I_1} \| \alpha_i u \| \, d\eta_1 \leqslant \|\alpha_i\|_{I_1} \int_{I_1} \| u \| \, d\eta_1 \leqslant m_0 c_U, \quad \forall \, u \in U, \ \forall \, i \in \overline{1,k}; \tag{4.4}$$

$$\left|\int_{I_2} \beta_j v \, d\eta_2\right| \leqslant \int_{I_2} \| \beta_j v \| \, d\eta_2 \leqslant \| \ \beta_j\|_{I_2} \int_{I_2} \| v \| \, d\eta_2 \leqslant m_0 c_V, \quad \forall \, v \in V, \ \forall \, j \in \overline{1,l}. \tag{4.5}$$

From (4.4) and (4.5), we obtain (by definitions of $\| \cdot \|^{(k)}$ and $\| \cdot \|^{(l)}$) under

$$\left(\mathbf{K} \triangleq \left\{y \in \mathbb{R}^k \mid \|y\|^{(k)} \leqslant m_0 c_U\right\}\right) \ \& \ \left(\mathbf{L} \triangleq \left\{z \in \mathbb{R}^l \mid \|z\|^{(l)} \leqslant m_0 c_V\right\}\right),$$

the following two systems of inclusions

$$\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}} \in \mathbf{K} \ \forall \, u \in U\right) \ \& \ \left(\left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j \in \overline{1,l}} \in \mathbf{L} \ \forall \, v \in V\right). \tag{4.6}$$

Of course, $\mathbf{K} \in \left(\tau_{\mathbb{R}}^{(k)} - \text{comp}\right)\left[\mathbb{R}^k\right]$ and $\mathbf{L} \in \left(\tau_{\mathbb{R}}^{(l)} - \text{comp}\right)\left[\mathbb{R}^l\right]$. With the employment of (4.6), we have the properties:

$$\left(\mathfrak{U} \triangleq \left\{\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}} : u \in U\right\} \in \mathcal{P}'(\mathbf{K})\right) \ \& \\ \& \ \left(\mathfrak{V} \triangleq \left\{\left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j \in \overline{1,l}} : v \in V\right\} \in \mathcal{P}'(\mathbf{L})\right). \tag{4.7}$$

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For brevity, in the following, we suppose that

$$\left(\overline{\mathfrak{U}} \triangleq \mathrm{cl}\left(\mathfrak{U}, \tau_{\mathbb{R}}^{(k)}\right)\right) \ \& \ \left(\overline{\mathfrak{V}} \triangleq \mathrm{cl}\left(\mathfrak{V}, \tau_{\mathbb{R}}^{(l)}\right)\right). \tag{4.8}$$

By (4.7) and (4.8) we obtain that $\left(\overline{\mathfrak{U}} \in \mathcal{P}'(\mathbf{K})\right) \ \& \ \left(\overline{\mathfrak{V}} \in \mathcal{P}'(\mathbf{L})\right)$; since $\overline{\mathfrak{U}}$ and $\overline{\mathfrak{V}}$ are closed sets (see (4.8)), then

$$\left(\overline{\mathfrak{U}} \in \left(\tau_{\mathbb{R}}^{(k)} - \mathrm{comp}\right)\left[\mathbb{R}^k\right]\right) \ \& \ \left(\overline{\mathfrak{V}} \in \left(\tau_{\mathbb{R}}^{(l)} - \mathrm{comp}\right)\left[\mathbb{R}^l\right]\right). \tag{4.9}$$

Of course, $\overline{\mathfrak{U}} \times \overline{\mathfrak{V}}$ is a nonempty metrizable compactum. Namely, we introduce the metric
$\rho : \left(\overline{\mathfrak{U}} \times \overline{\mathfrak{V}}\right) \times \left(\overline{\mathfrak{U}} \times \overline{\mathfrak{V}}\right) \to [0, \infty[$ by the obvious rule: for $x' \in \overline{\mathfrak{U}},\ y' \in \overline{\mathfrak{V}},\ x'' \in \overline{\mathfrak{U}}$, and $y'' \in \overline{\mathfrak{V}}$

$$\rho\left(\left(x', y'\right), \left(x'', y''\right)\right) \triangleq \sup\left(\left\{\left\|x' - x''\right\|^{(k)}; \left\|y' - y''\right\|^{(l)}\right\}\right);$$

then by (4.9) the topology τ_ρ^0 of $\overline{\mathfrak{U}} \times \overline{\mathfrak{V}}$ generated by ρ is compact. As a result,

$$\left(\overline{\mathfrak{U}} \times \overline{\mathfrak{V}}, \tau_\rho^0\right) \tag{4.10}$$

is a nonempty compactum. In the following, we fix a function

$$\mathbf{f}_0 : \overline{\mathfrak{U}} \times \overline{\mathfrak{V}} \to \mathbb{R} \tag{4.11}$$

continuous in the sense of (4.10) (so, $\mathbf{f}_0$ is a function on $\overline{\mathfrak{U}} \times \overline{\mathfrak{V}}$ continuous in the totality of variables). From (4.11) we have (in particular) that, for any $y \in \mathfrak{U}$ and $z \in \mathfrak{V}$, the value $\mathbf{f}_0(y, z) \in \mathbb{R}$ is defined. From (4.7) we obtain that

$$\mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, \left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j \in \overline{1,l}}\right) \in \mathbb{R} \ \forall\, u \in U \ \forall\, v \in V.$$

Using this obvious property, we suppose that

$$\Phi : U \times V \to \mathbb{R} \tag{4.12}$$

is defined by the following rule: if $u \in U$ and $v \in V$, then

$$\Phi(u, v) \triangleq \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, \left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j \in \overline{1,l}}\right). \tag{4.13}$$

We use Φ (4.12), (4.13) as the cost function in our game problem.

In the following we fix $p \in \mathbb{N},\ q \in \mathbb{N},\ (\gamma_i)_{i \in \overline{1,p}} \in B(I_1, \mathcal{L}_1)^p$, and $(\omega_j)_{j \in \overline{1,q}} \in B(I_2, \mathcal{L}_2)^q$. Moreover (see Section 1), we fix closed sets

$$\left(Y \in \mathcal{P}'\left(\mathbb{R}^p\right)\right) \ \& \ \left(Z \in \mathcal{P}'\left(\mathbb{R}^q\right)\right).$$

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We consider the constraints (1.2) on the choice of $u \in U$ and $v \in V$. In connection with the possible instability of arising game problem (see example of Section 1), we consider under $\varepsilon > 0$ and $\delta > 0$ the relaxations

$$\left(\left(\int_{I_1} \gamma_i u \, d\eta_1\right)_{i \in \overline{1,p}} \in O_\varepsilon^{(p)}[Y]\right) \ \& \ \left(\left(\int_{I_2} \omega_j v \, d\eta_2\right)_{j \in \overline{1,q}} \in O_\delta^{(q)}[Z]\right), \tag{4.14}$$

where $u \in U$ and $v \in V$. For the constraints (4.14), we introduce the corresponding attainable sets. For this, we suppose that under $\zeta \in]0, \infty[$

$$U_\partial[\zeta] \triangleq \left\{u \in U \mid \left(\int_{I_1} \gamma_i u \, d\eta_1\right)_{i \in \overline{1,p}} \in O_\zeta^{(p)}[Y]\right\}, \tag{4.15}$$

$$V_\partial[\zeta] \triangleq \left\{v \in V \mid \left(\int_{I_2} \omega_j v \, d\eta_2\right)_{j \in \overline{1,q}} \in O_\zeta^{(q)}[Z]\right\}. \tag{4.16}$$

We use (4.15) and (4.16) under consideration of the maximin Φ (4.12) under constraints (4.14), where $\varepsilon > 0, \ \delta > 0, \ \varepsilon \approx 0, \ \delta \approx 0$.

Moreover, a generalized problem is required; in this problem, the precise constraints generated by Y and Z are used. Introduce the corresponding spaces of generalized elements supposing

$$\left(\widetilde{U} \triangleq \mathrm{cl}(\{u * \eta_1 : u \in U\}, \tau_*(\mathcal{L}_1))\right) \ \& \ \left(\widetilde{V} \triangleq \mathrm{cl}(\{v * \eta_2 : v \in V\}, \tau_*(\mathcal{L}_2))\right). \tag{4.17}$$

Elements of $\widetilde{U}$ and $\widetilde{V}$ we will consider as generalized controls of the first and second players respectively.

Some Concrete Cases. In general case, the determination of $\widetilde{U}$ and $\widetilde{V}$ by (4.17) is very difficult problem. But, for many useful variants of U and V, the concrete representations of $\widetilde{U}$ and $\widetilde{V}$ are known (see [1, 2, 3, 4]). Now we recall only several natural examples (moreover, see [7]). Namely, the pair $(U, \widetilde{U})$ assumes following realizations:

1′) $U = \left\{u \in B_0(I_1, \mathcal{L}_1) \mid \int_{I_1} \| u \| \, d\eta_1 \leqslant b_1\right\}$ and
$\widetilde{U} = \{\mu \in \mathbb{A}_{\eta_1}[\mathcal{L}_1] \mid \mathbf{v}_\mu(I_1) \leqslant b_1\}, \quad b_1 \in [0, \infty[$;

2′) $U = \left\{u \in B_0^+(I_1, \mathcal{L}_1) \mid \int_{I_1} u \, d\eta_1 \leqslant b_1\right\}$ and
$\widetilde{U} = \{\mu \in (\mathrm{add})^+[\mathcal{L}_1; \eta_1] \mid \mu(I_1) \leqslant b_1\}, \quad b_1 \in [0, \infty[$;

3′) $U = \left\{u \in B_0^+(I_1, \mathcal{L}_1) \mid \int_{I_1} u \, d\eta_1 = b_1\right\}$ and
$\widetilde{U} = \{\mu \in (\mathrm{add})^+[\mathcal{L}_1; \eta_1] \mid \mu(I_1) = b_1\}, \quad b_1 \in [0, \infty[$;

4′) if $r \in \mathbb{N}, \quad (L_i)_{i \in \overline{1,r}} : \overline{1,r} \to \mathcal{L}, \quad (c_i)_{i \in \overline{1,r}} : \overline{1,r} \to [0, \infty[$ and $I_1 = \bigcup_{i=1}^{r} L_i$, then

$U = \left\{ u \in B_0(I_1, \mathcal{L}_1) \mid \int_{L_k} \| u \| \, d\eta_1 \leqslant c_k \ \forall k \in \overline{1,r} \right\}$, $\widetilde{U} = \{\mu \in \mathbb{A}_{\eta_1}[\mathcal{L}_1] \mid \mathbf{v}_\mu(L_k) \leqslant c_k \ \forall k \in \overline{1,r}\}$.

Analogously, the pair $(V, \widetilde{V})$ can be realized in the following forms:

1″) $V = \left\{ v \in B_0(I_2, \mathcal{L}_2) \mid \int_{I_2} \| v \| \, d\eta_2 \leqslant b_2 \right\}$ and

$\widetilde{V} = \{\nu \in \mathbb{A}_{\eta_2}[\mathcal{L}_2] \mid \mathbf{v}_\nu(I_2) \leqslant b_2\}, \ b_2 \in [0, \infty[$;

2″) $V = \left\{ v \in B_0^+(I_2, \mathcal{L}_2) \mid \int_{I_2} v \, d\eta_2 \leqslant b_2 \right\}$ and $\widetilde{V} = \{\nu \in (\mathrm{add})^+[\mathcal{L}_2; \eta_2] \mid \nu(I_2) \leqslant b_2\}, \ b_2 \in [0, \infty[$;

3″) $V = \left\{ v \in B_0^+(I_2, \mathcal{L}_2) \mid \int_{I_2} v \, d\eta_2 = b_2 \right\}$ and $\widetilde{V} = \{\nu \in (\mathrm{add})^+[\mathcal{L}_2; \eta_2] \mid \nu(I_2) = b_2\}, \ b_2 \in [0, \infty[$;

4″) if $s \in \mathbb{N}$, $(\mathbb{L}_i)_{i \in \overline{1,s}} : \overline{1,s} \to \mathcal{L}$, $(\widetilde{c}_i)_{i \in \overline{1,s}} : \overline{1,s} \to [0, \infty[$ and $I_2 = \bigcup_{i=1}^{s} \mathbb{L}_i$, then $V = \left\{ v \in B_0(I_2, \mathcal{L}_2) \mid \int_{\mathbb{L}_k} \| v \| \, d\eta_2 \leqslant \widetilde{c}_k \ \forall k \in \overline{1,s} \right\}$, $\widetilde{V} = \{\nu \in \mathbb{A}_{\eta_2}[\mathcal{L}_2] \mid \mathbf{v}_\nu(\mathbb{L}_k) \leqslant \widetilde{c}_k \ \forall k \in \overline{1,s}\}$.

In this connection, see [5, §15]. In $1') - 4')$ and $1'') - 4'')$, we consider usual controls (elements of U and V) as step-functions. This is natural with point of view of applications. But, elements of $B_0(I_1, \mathcal{L}_1)$ and $B_0^+(I_1, \mathcal{L}_1)$ can be replaced by elements of $B(I_1, \mathcal{L}_1)$ and $B^+(I_1, \mathcal{L}_1)$ respectively. Analogous remark takes place in connection with the possible replacement of elements of $B_0(I_2, \mathcal{L}_2)$ and $B_0^+(I_2, \mathcal{L}_2)$ by elements of $B(I_2, \mathcal{L}_2)$ and $B^+(I_2, \mathcal{L}_2)$ respectively.

We note that by (4.3) and [3, (3.7.7)]

$$(u * \eta_1 \in \mathbf{B}_*(\mathcal{L}_1, c_U) \ \forall u \in U) \ \& \ (v * \eta_2 \in \mathbf{B}_*(\mathcal{L}_2, c_V) \ \forall v \in V).$$

Therefore, from (4.17) the following inclusions are valid:

$$(\{u * \eta_1 : u \in U\} \subset \mathbf{B}_*(\mathcal{L}_1, c_U) \ \& \ (\{v * \eta_2 : v \in V\} \subset \mathbf{B}_*(\mathcal{L}_2, c_V)).$$

From [3, (3.4.19), (3.7.6)] and (4.17), we obtain the obvious inclusions:

$$\left(\widetilde{U} \subset \mathbf{B}_*(\mathcal{L}_1, c_U)\right) \ \& \ \left(\widetilde{V} \subset \mathbf{B}_*(\mathcal{L}_2, c_V)\right); \tag{4.18}$$

moreover, $\widetilde{U} \neq \varnothing$ and $\widetilde{V} \neq \varnothing$. Then, by (4.17), (4.18), and [3, (3.4.19)]

$$\left(\widetilde{U} \in (\tau_*(\mathcal{L}_1) - \mathrm{comp})[\mathbb{A}(\mathcal{L}_1)]\right) \ \& \ \left(\widetilde{V} \in (\tau_*(\mathcal{L}_2) - \mathrm{comp})[\mathbb{A}(\mathcal{L}_2)]\right). \tag{4.19}$$

Using (4.19), we introduce the corresponding induced compact topologies

$$\left(\widetilde{\tau}_U^*(\mathcal{L}_1) \triangleq \tau_*(\mathcal{L}_1)|_{\widetilde{U}}\right) \ \& \ \left(\widetilde{\tau}_V^*(\mathcal{L}_2) \triangleq \tau_*(\mathcal{L}_2)|_{\widetilde{V}}\right)$$

of the sets $\widetilde{U}$ and $\widetilde{V}$ respectively. As a corollary, we obtain the following nonempty compacta:

$$\left(\widetilde{U}, \widetilde{\tau}^*_U(\mathcal{L}_1)\right) \ \& \ \left(\widetilde{V}, \widetilde{\tau}^*_V(\mathcal{L}_2)\right). \tag{4.20}$$

With compacta (4.20), we connect the problem about the precise validiting of moment constrains

$$\left(\int_{I_1} \gamma_i \, d\mu\right)_{i\in\overline{1,p}} \in Y, \ \left(\int_{I_2} \omega_j \, d\nu\right)_{j\in\overline{1,q}} \in Z$$

on the choice of $\mu \in \widetilde{U}$ and $\nu \in \widetilde{V}$ respectively. Namely, we suppose that

$$\widetilde{U}_\partial \triangleq \left\{\mu \in \widetilde{U} \mid \left(\int_{I_1} \gamma_i \, d\mu\right)_{i\in\overline{1,p}} \in Y\right\}, \tag{4.21}$$

$$\widetilde{V}_\partial \triangleq \left\{\nu \in \widetilde{V} \mid \left(\int_{I_2} \omega_j \, d\nu\right)_{j\in\overline{1,q}} \in Z\right\}. \tag{4.22}$$

Proposition 4.1. *The following equality is valid*

$$\widetilde{U}_\partial = \bigcap_{\varepsilon\in]0,\infty[} \mathrm{cl}(\{f * \eta_1 : f \in U_\partial[\varepsilon]\}, \tau_*(\mathcal{L}_1)). \tag{4.23}$$

Proof. The proof follows from Proposition 3.1. We use $(I_1, \mathcal{L}_1, \eta_1)$ instead of $(I, \mathcal{L}, \eta)$, U instead of W, p instead of m, Y instead of X, $(\gamma_i)_{i\in\overline{1,p}}$ instead of $(h_i)_{i\in\overline{1,m}}$. Under these replacements, we use $U_\partial[\xi]$ instead of $W_\partial[\xi]$, where $\xi \in]0,\infty[$. Moreover, by (3.11) and (4.21) we can be use $\widetilde{U}_\partial$ instead of $\widetilde{W}_\partial$. Then, by Proposition 3.1 (see (3.12)) we obtain the required equality (4.23). □

From Corollary 3.1, we obtain that

$$\left(\widetilde{U}_\partial \neq \varnothing\right) \Leftrightarrow (U_\partial[\varepsilon] \neq \varnothing \ \forall \varepsilon \in]0,\infty[\,). \tag{4.24}$$

In connection with (4.24), we should be used the replacements mentioned after Proposition 4.1.

Introduce in consideration the mapping

$$u \mapsto \left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i\in\overline{1,k}} : U \to \mathbb{R}^k$$

denoted by Π_1. So, $\Pi_1 : U \to \mathbb{R}^k$ is mapping such that

$$\Pi_1(u) = \left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i\in\overline{1,k}} \ \forall u \in U. \tag{4.25}$$

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By analogy with Section 3 we introduce the generalized variant of Π_1; namely, let $\widetilde{\Pi}_1 : \widetilde{U} \to \mathbb{R}^k$ is defined by the following rule:

$$\widetilde{\Pi}_1(\mu) \triangleq \left(\int_{I_1} \alpha_i \, d\mu\right)_{i\in\overline{1,k}} \quad \forall \mu \in \widetilde{U}. \tag{4.26}$$

Using (4.26), we introduce the mapping $\mathcal{J}_1 : U \to \widetilde{U}$ by the rule:

$$\mathcal{J}_1(u) \triangleq u * \eta_1 \; \forall u \in U.$$

Of course, by (3.3), (4.25), and (4.26) we obtain the following equality:

$$\Pi_1 = \widetilde{\Pi}_1 \circ \mathcal{J}_1.$$

Finally, we note that $\widetilde{\Pi}_1$ is a continuous mapping: $\widetilde{\Pi}_1 \in C\left(\widetilde{U}, \tau_U^*(\mathcal{L}_1), \mathbb{R}^k, \tau_{\mathbb{R}}^{(k)}\right)$ (of course, we use [3, (2.5.30)]). Now, we obtain the following

Proposition 4.2. *The following equality is valid:*

$$\bigcap_{\varepsilon\in]0,\infty[} \mathrm{cl}\left(\{\Pi_1(u) : u \in U_\partial[\varepsilon]\}, \tau_{\mathbb{R}}^{(k)}\right) = \left\{\widetilde{\Pi}_1(\mu) : \mu \in \widetilde{U}_\partial\right\}. \tag{4.27}$$

This proposition is a variant of Proposition 3.2: we use the replacements mentioned after Proposition 4.1. Moreover, we use $\mathcal{J}_1$ instead of $\mathcal{J}$, and $\widetilde{\Pi}_1$ instead of $\widetilde{\Pi}$. Under these concrete definitions, (4.27) is extracted from (3.27).

Proposition 4.3. *If* $\zeta \in]0,\infty[$, *then for some* $\xi \in]0,\infty[$

$$\left\{\widetilde{\Pi}_1(\mu) : \mu \in \widetilde{U}_\partial\right\} \subset \mathrm{cl}\left(\{\Pi_1(u) : u \in U_\partial[\xi]\}, \tau_{\mathbb{R}}^{(k)}\right) \subset O_\zeta^{(k)}\left[\left\{\widetilde{\Pi}_1(\mu) : \mu \in \widetilde{U}_\partial\right\}\right].$$

The corresponding proof follows from Proposition 3.3. Indeed, $\widetilde{\Pi}^1\left(\widetilde{W}_\partial\right)$ of Proposition 3.3 is replaced by $\left\{\widetilde{\Pi}_1(\mu) : \mu \in \widetilde{U}_\partial\right\}$ (recall that $\widetilde{W}_\partial$ is replaced by $\widetilde{U}_\partial$ too; see (3.11) and (4.21)) and $\Pi_1(W_\partial[\xi])$ is replaced by $\{\Pi_1(u) : u \in U_\partial[\xi]\}$ (see (3.10) and (4.15)).

Later, we use the following now designations. Namely,

$$G'_\varepsilon \triangleq \{\Pi_1(u) : u \in U_\partial[\varepsilon]\} = \left\{\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i\in\overline{1,k}} : u \in U_\partial[\varepsilon]\right\} \in \mathcal{P}(\mathbb{R}^k) \; \forall \varepsilon \in]0,\infty[. \tag{4.28}$$

Moreover, in the following

$$\mathbb{G}_1 \triangleq \left\{\widetilde{\Pi}_1(\mu) : \mu \in \widetilde{U}_\partial\right\} = \left\{\left(\int_{I_1} \alpha_i \, d\mu\right)_{i\in\overline{1,k}} : \mu \in \widetilde{U}_\partial\right\} \in \mathcal{P}(\mathbb{R}^k). \tag{4.29}$$

Then, by (4.28), (4.29), and Proposition 4.2 we obtain that

$$\mathbb{G}_1 = \bigcap_{\varepsilon \in]0,\infty[} \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right). \tag{4.30}$$

Finally, from (4.28), (4.29), and Proposition 4.3 the following property is extracted; namely, $\forall \zeta \in]0,\infty[\ \exists \xi \in]0,\infty[$:

$$\mathbb{G}_1 \subset \mathrm{cl}\left(G'_\xi, \tau_{\mathbb{R}}^{(k)}\right) \subset O_\zeta^{(k)}[\mathbb{G}_1]. \tag{4.31}$$

Proposition 4.4. *The following equality is valid:*

$$\widetilde{V}_\partial = \bigcap_{\delta \in]0,\infty[} \mathrm{cl}\left(\{f * \eta_2 : f \in V_\partial[\varepsilon]\}, \tau_*(\mathcal{L}_2)\right). \tag{4.32}$$

Proof. The proof follows from Proposition 3.1. In addition, we use the space $(I_2, \mathcal{L}_2, \eta_2)$ instead of $(I, \mathcal{L}, \eta)$ of Section 3, V instead of W, q instead of m, Z instead of X, $(\omega_j)_{j\in\overline{1,q}}$ instead of $(h_i)_{i\in\overline{1,m}}$. Under these replacements, we use $V_\partial[\xi]$ instead of $W_\partial[\xi]$, where $\xi \in]0,\infty[$. Moreover, by (3.11) and (4.22) we can be used $\widetilde{V}_\partial$ instead of $\widetilde{W}_\partial$. Now, from Proposition 3.1 (see (3.12)) we obtain the required equality (4.32). □

Using the obvious replacements mentioned after Proposition 4.4, we obtain that

$$\left(\widetilde{V}_\partial \neq \varnothing\right) \Leftrightarrow \left(V_\partial[\delta] \neq \varnothing\ \forall \delta \in]0,\infty[\right). \tag{4.33}$$

For the concrete variant of Proposition 3.2, we introduce the mapping

$$v \mapsto \left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j\in\overline{1,l}} : V \to \mathbb{R}^l$$

denoted by Π_2. So, $\Pi_2 : V \to \mathbb{R}^l$ is the mapping such that

$$\Pi_2(v) = \left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j\in\overline{1,l}} \quad \forall v \in V. \tag{4.34}$$

Now, we introduce the mapping $\widetilde{\Pi}_2 : \widetilde{V} \to \mathbb{R}^l$ by the following rule

$$\widetilde{\Pi}_2(\nu) \triangleq \left(\int_{I_2} \beta_j \, d\nu\right)_{j\in\overline{1,l}} \quad \forall \nu \in \widetilde{V}. \tag{4.35}$$

Moreover, using (4.17), we introduce the mapping $\mathcal{J}_2 : V \to \widetilde{V}$ by the rule

$$\mathcal{J}_2(v) \triangleq v * \eta_2 \ \forall v \in V.$$

Then, by (3.3), (4.34), and (4.35) we have the equality

$$\Pi_2 = \widetilde{\Pi}_2 \circ \mathcal{J}_2. \tag{4.36}$$

In addition, the mapping $\widetilde{\Pi}_2$ is continuous: $\widetilde{\Pi}_2 \in C\left(\widetilde{V}, \tau_V^*(\mathcal{L}_2), \mathbb{R}^l, \tau_{\mathbb{R}}^{(l)}\right)$ (see [3, (2.5.30)]). From Proposition 3.2, the following statement follows.

Proposition 4.5. *The following equality is valid:*

$$\bigcap_{\delta \in]0,\infty[} \mathrm{cl}\left(\{\Pi_2(v) : v \in V_\partial[\delta]\}, \tau_{\mathbb{R}}^{(l)}\right) = \left\{\widetilde{\Pi}_2(\nu) : \nu \in \widetilde{V}_\partial\right\}. \tag{4.37}$$

In connection with the proof, we use Π_2 instead of Π, $\mathcal{J}_2$ instead of $\mathcal{J}$, $\widetilde{V}_\partial$ instead of $\widetilde{W}_\partial$, and $\widetilde{\Pi}_2$ instead of $\widetilde{\Pi}$. Moreover, we keep in mind the above-mentioned concrete variants of $(I, \mathcal{L}, \eta), W, m, X$ and $(h_i)_{i \in \overline{1,m}}$. Then, (4.37) is extracted from (3.27).

Proposition 4.6. *If $\zeta \in]0, \infty[$, then for some $\xi \in]0, \infty[$*

$$\left\{\widetilde{\Pi}_2(\nu) : \nu \in \widetilde{V}_\partial\right\} \subset \mathrm{cl}\left(\{\Pi_2(v) : v \in V_\partial[\xi]\}, \tau_{\mathbb{R}}^{(l)}\right) \subset O_\zeta^{(l)}\left[\left\{\widetilde{\Pi}_2(\nu) : \nu \in \widetilde{V}_\partial\right\}\right].$$

In connection with the proof, we note the following concrete variant of Proposition 3.3. Namely, $\Pi^1\left(\widetilde{W}_\partial\right)$ of Proposition 3.3 is replaced by $\left\{\widetilde{\Pi}_2(\nu) : \nu \in \widetilde{V}_\partial\right\}$ and $W_\partial[\xi]$ is replaced by $V_\partial[\xi]$. Other clarifications are analogous to remarks after Proposition 4.3.

Now, we introduce some new designations. Namely, let

$$G''_\delta \triangleq \{\Pi_2(v) : v \in V_\partial[\delta]\} = \left\{\left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j \in \overline{1,l}} : v \in V_\partial[\delta]\right\} \in \mathcal{P}(\mathbb{R}^l) \ \forall \delta \in]0, \infty[. \tag{4.38}$$

Moreover, we suppose that

$$\mathbb{G}_2 \triangleq \left\{\widetilde{\Pi}_2(\nu) : \nu \in \widetilde{V}_\partial\right\} = \left\{\left(\int_{I_2} \beta_j \, d\nu\right)_{j \in \overline{1,l}} : \nu \in \widetilde{V}_\partial\right\} \in \mathcal{P}(\mathbb{R}^l). \tag{4.39}$$

Then, from (4.38), (4.39), and Proposition 4.5 the following equality is valid

$$\mathbb{G}_2 = \bigcap_{\delta \in]0,\infty[} \mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right). \tag{4.40}$$

From Proposition 4.6, (4.38), and (4.39) the important property follows; namely, $\forall \zeta \in]0, \infty[\ \exists \xi \in]0, \infty[$:

$$\mathbb{G}_2 \subset \mathrm{cl}\left(G''_\xi, \tau_{\mathbb{R}}^{(l)}\right) \subset O_\zeta^{(l)}[\mathbb{G}_2]. \tag{4.41}$$

In the following, we use (4.31) for obtaining of representation connecting generalized game problem and game problems with weakened constraints. Now, we note that the cost function

$$\widetilde{\Phi} : \widetilde{U} \times \widetilde{V} \to \mathbb{R}$$

of generalized game problem is defined by the rule

$$(\mu, \nu) \mapsto \mathbf{f}_0 \left(\left(\int_{I_1} \alpha_i \, d\mu \right)_{i \in \overline{1,k}}, \left(\int_{I_2} \beta_j \, d\nu \right)_{j \in \overline{1,l}} \right) : \widetilde{U} \times \widetilde{V} \to \mathbb{R}. \quad (4.42)$$

Using (4.26), (4.35), and (4.42) we obtain that

$$\widetilde{\Phi}(\mu, \nu) = \mathbf{f}_0 \left(\widetilde{\Pi}_1(\mu), \widetilde{\Pi}_2(\nu) \right) \ \forall \mu \in \widetilde{U} \ \forall \nu \in \widetilde{V}. \quad (4.43)$$

The generalized problem is defined as the maximin problem with the cost function $\widetilde{\Phi}$ under the constraints

$$\left(\int_{I_1} \gamma_i \, d\mu \right)_{i \in \overline{1,p}} \in Y, \ \left(\int_{I_2} \omega_j \, d\nu \right)_{j \in \overline{1,q}} \in Z$$

on the choice of $\mu \in \widetilde{U}$ and $\nu \in \widetilde{V}$ respectively.

5. Asymptotic of Maximin under Weakened Constraints

In the following, taking into account (4.24) and (4.33), we suppose that

$$\left(\widetilde{U}_\partial \neq \varnothing \right) \ \& \ \left(\widetilde{V}_\partial \neq \varnothing \right). \quad (5.1)$$

So, in the following, we consider the case of the compatible generalized problem. Then, by (4.24) and (4.33)

$$\left(U_\partial[\varepsilon] \neq \varnothing \ \forall \varepsilon \in]0, \infty[\, \right) \ \& \ \left(V_\partial[\delta] \neq \varnothing \ \forall \delta \in]0, \infty[\, \right). \quad (5.2)$$

Recall that by (4.7), (4.15), and (4.28) $G'_\varepsilon \subset \mathfrak{U} \subset \overline{\mathfrak{U}} \ \forall \varepsilon \in]0, \infty[$. Since $\overline{\mathfrak{U}}$ is a closed set, we obtain that

$$\mathrm{cl} \left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)} \right) \subset \overline{\mathfrak{U}} \ \forall \varepsilon \in]0, \infty[. \quad (5.3)$$

On the other hand, by (4.28) and (5.2) $G'_\varepsilon \neq \varnothing \ \forall \varepsilon \in]0, \infty[$. As a corollary, we obtain that, in our case, the closure of any set (4.28) is a nonempty closed subset of $\overline{\mathfrak{U}}$. The above-mentioned closure is a nonempty bounded closed subset of $\mathbb{R}^k$. Then,

$$\mathrm{cl} \left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)} \right) \in \left(\tau_{\mathbb{R}}^{(k)} - \mathrm{comp} \right) \left[\mathbb{R}^k \right] \ \forall \varepsilon \in]0, \infty[.$$

It is convenient to consider the sets $\mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)$, $\varepsilon \in]0,\infty[$, as (closed) subsets of $\overline{\mathfrak{U}}$ with the relative topology. Namely, we introduce the topology

$$\mathbf{t}_1 \triangleq \tau_{\mathbb{R}}^{(k)}|_{\overline{\mathfrak{U}}} \tag{5.4}$$

of the set $\overline{\mathfrak{U}}$. From (5.4), the following simple property is valid: the topology $\mathbf{t}_1$ is generated by the metric

$$(x,y) \mapsto \|x-y\|^{(k)} : \overline{\mathfrak{U}} \times \overline{\mathfrak{U}} \to [0,\infty[. \tag{5.5}$$

Of course, $\left(\overline{\mathfrak{U}}, \mathbf{t}_1\right)$ is a nonempty metrizable compactum. In addition, for any $\varepsilon \in]0,\infty[$, the set $\mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)$ is closed in $\left(\overline{\mathfrak{U}}, \mathbf{t}_1\right)$; indeed, by (5.4) $\mathrm{cl}\left(G'_\varepsilon, \mathbf{t}_1\right) = \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right) \cap \overline{\mathfrak{U}} = \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)$ $\Big($so, $\mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)$ is a nonempty closed set in the space $\left(\overline{\mathfrak{U}}, \mathbf{t}_1\right)\Big)$.

By (4.7), (4.16), and (4.35) $G''_\delta \subset \mathfrak{V} \subset \overline{\mathfrak{V}}\ \forall \delta \in]0,\infty[$. Since $\overline{\mathfrak{V}}$ is a closed set, we have the obvious corollary:

$$\mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right) \subset \overline{\mathfrak{V}}\ \forall \delta \in]0,\infty[. \tag{5.6}$$

By (4.7) and (5.2) we obtain that (in our case) $G''_\delta \neq \varnothing\ \forall \delta \in]0,\infty[$. So, under $\delta \in]0,\infty[$ $\mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right)$ is a nonempty bounded closed set of $\mathbb{R}^{(l)}$. Therefore,

$$\mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right) \in \left(\tau_{\mathbb{R}}^{(l)} - \mathrm{comp}\right)\left[\mathbb{R}^{(l)}\right]. \tag{5.7}$$

We introduce $\mathbf{t}_2 \triangleq \tau_{\mathbb{R}}^{(l)}|_{\overline{\mathfrak{V}}}$. Then we obtain the nonempty compactum $(\overline{\mathfrak{V}}, \mathbf{t}_2)$. In addition, the topology $\mathbf{t}_2$ is generated by the metric

$$(x,y) \mapsto \|x-y\|^{(l)} : \overline{\mathfrak{V}} \times \overline{\mathfrak{V}} \to [0,\infty[.$$

So, $(\overline{\mathfrak{V}}, \mathbf{t}_2)$ is a nonempty metrizable compactum. In addition, for any $\delta \in]0,\infty[$, the set $\mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right)$ is closed in this compactum $(\overline{\mathfrak{V}}, \mathbf{t}_2)$. Indeed,

$$\mathrm{cl}\left(G''_\delta, \mathbf{t}_2\right) = \mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right) \cap \overline{\mathfrak{V}} = \mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right).$$

Now, we return to (4.7). Then, by definition of Π_1 and Π_2

$$(\mathfrak{U} = \{\Pi_1(u) : u \in U\})\ \&\ (\mathfrak{V} = \{\Pi_2(v) : v \in V\}). \tag{5.8}$$

We recall (4.9) and (4.11). Then, by definition of ρ we obtain that $\mathbf{f}_0$ is uniformly continuous real-valued function. Namely $\forall \varepsilon \in]0,\infty[\ \exists \delta \in]0,\infty[\ \forall x' \in \overline{\mathfrak{U}}\ \forall y' \in \overline{\mathfrak{V}}\ \forall x'' \in \overline{\mathfrak{U}}\ \forall y'' \in \overline{\mathfrak{V}}$

$$\left(\rho\left(\left(x', y'\right), \left(x'', y''\right)\right) < \delta\right) \Rightarrow \left(\left|\mathbf{f}_0\left(x', y'\right) - \mathbf{f}_0\left(x'', y''\right)\right| < \varepsilon\right).$$

Moreover, the function $\mathbf{f}_0$ is bounded. We note some obvious corollaries. If $K_* \in \mathcal{P}'\left(\overline{\mathfrak{U}}\right)$ and $z \in \overline{\mathfrak{V}}$, then the set $\{\mathbf{f}_0(y,z) : y \in K_*\}$ is not empty and bounded; therefore, the number

$$\inf_{y\in K_*} \mathbf{f}_0(y,z) = \inf(\{\mathbf{f}_0(y,z) : y \in K_*\}) \in \mathbb{R}$$

is defined. Moreover, by the boundedness of $\mathbf{f}_0$ we have the boundedness of the function

$$z \mapsto \inf_{y\in K_*} \mathbf{f}_0(y,z) : \overline{\mathfrak{V}} \to \mathbb{R}.$$

Therefore, we have the value

$$\sup_{z\in L_*} \inf_{y\in K_*} \mathbf{f}_0(y,z) \in \mathbb{R}\ \forall K_* \in \mathcal{P}'\left(\overline{\mathfrak{U}}\right)\ \forall L_* \in \mathcal{P}'\left(\overline{\mathfrak{V}}\right). \tag{5.9}$$

The values (5.9) are connected with the game problem with the cost function (4.12), (4.13). For this, we recall that $\mathfrak{U} \subset \overline{\mathfrak{U}}$ and $\mathfrak{V} \subset \overline{\mathfrak{V}}$. In addition, by (4.13), (4.25), and (4.34)

$$\Phi(u,v) = \mathbf{f}_0(\Pi_1(u), \Pi_2(v))\ \forall u \in U\ \forall v \in V. \tag{5.10}$$

From (5.8) and (5.10), we obtain that Φ is a bounded functional on $U \times V$. By (4.15) and (5.2) we have the property

$$U_\partial[\varepsilon] \in \mathcal{P}'(U)\ \forall \varepsilon \in]0,\infty[. \tag{5.11}$$

Analogously, by (4.16) and (5.2) the property

$$V_\partial[\delta] \in \mathcal{P}'(V)\ \forall \delta \in]0,\infty[\tag{5.12}$$

is valid. Then, by the boundedness of Φ, under $\varepsilon \in]0,\infty[$ and $v \in V$, the set $\{\Phi(u,v) : u \in U_\partial[\varepsilon]\}$ is bounded. Therefore, we have the following values

$$\inf_{u\in U_\partial[\varepsilon]} \Phi(u,v) \in \mathbb{R}\ \forall \varepsilon \in]0,\infty[\ \forall v \in V. \tag{5.13}$$

Moreover, by the boundedness of Φ we obtain that, under $\varepsilon \in]0,\infty[$, the (real-valued) function $v \mapsto \inf_{u\in U_\partial[\varepsilon]} \Phi(u,v) : V \to \mathbb{R}$ is bounded too. Therefore (see (5.12)), the values

$$\mathbf{v}(\varepsilon,\delta) \triangleq \sup_{v\in V_\partial[\delta]} \inf_{u\in U_\partial[\varepsilon]} \Phi(u,v) \in \mathbb{R}\ \forall \varepsilon \in]0,\infty[\ \forall \delta \in]0,\infty[. \tag{5.14}$$

In the following, we consider asymptotics of values (5.14) under $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. But, first, we consider a representation of the values (5.14) in terms of the sets G'_ε, $\varepsilon > 0$, and G''_δ, $\delta > 0$. For this, we note that by (4.28) and (5.10), under $\varepsilon \in]0,\infty[$ and $v \in V$

$$\{\Phi(u,v) : u \in U_\partial[\varepsilon]\} = \{\mathbf{f}_0(\Pi_1(u), \Pi_2(v)) : u \in U_\partial[\varepsilon]\} = \{\mathbf{f}_0(y, \Pi_2(v)) : y \in G'_\varepsilon\}$$

is a nonempty bounded set; therefore,

$$\inf_{u\in U_\partial[\varepsilon]} \Phi(u,v) = \inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,\Pi_2(v)). \tag{5.15}$$

In connection with (5.15), we recall (5.13). Using (5.15) under $\varepsilon \in]0,\infty[$, we note that $\left(\inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,\Pi_2(v))\right)_{v\in V} = \left(\inf_{u\in U_\partial[\varepsilon]} \Phi(u,v)\right)_{v\in V}$ is a bounded function. As a corollary, under $\varepsilon \in]0,\infty[$ and $\delta \in]0,\infty[$, the function

$$\left(\inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,\Pi_2(v))\right)_{v\in V_\partial[\delta]} = \left(\inf_{u\in U_\partial[\varepsilon]} \Phi(u,v)\right)_{v\in V_\partial[\delta]} \tag{5.16}$$

is bounded. Then, in the particular, under $\varepsilon \in]0,\infty[$ and $\delta \in]0,\infty[$, by (4.35) we have the boundedness property of the function

$$z \mapsto \inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,z) : G''_\delta \to \mathbb{R}.$$

And what is more, the following equality

$$\left\{\inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,z) : z \in G''_\delta\right\} = \left\{\inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,\Pi_2(v)) : v \in V_\partial[\delta]\right\},$$

where $\varepsilon \in]0,\infty[$ and $\delta \in]0,\infty[$. As a corollary, by (5.16) we obtain that

$$\left\{\inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,z) : z \in G''_\delta\right\} = \left\{\inf_{u\in U_\partial[\varepsilon]} \Phi(u,v) : v \in V_\partial[\delta]\right\}$$

is a nonempty bounded set for any $\varepsilon \in]0,\infty[$ and $\delta \in]0,\infty[$. Therefore, by (5.14)

$$\mathbf{v}(\varepsilon,\delta) = \sup_{z\in G''_\delta} \inf_{y\in G'_\varepsilon} \mathbf{f}_0(y,z) \ \forall \varepsilon \in]0,\infty[\ \forall \delta \in]0,\infty[. \tag{5.17}$$

So, in (5.16) we obtain the finite-dimensional representation of the realizable maximins (5.14), in this connection, see (5.9).

We recall the following above-mentioned properties:

$$\begin{aligned} &\left(\mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right) \in (\mathbf{t}_1 - \mathrm{comp})\left[\overline{\mathfrak{U}}\right] \ \forall \varepsilon \in]0,\infty[\right) \\ \& &\left(\mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right) \in (\mathbf{t}_2 - \mathrm{comp})\left[\overline{\mathfrak{V}}\right] \ \forall \delta \in]0,\infty[\right). \end{aligned} \tag{5.18}$$

In connection with (5.18), we note that by the continuity property of $\mathbf{f}$, for any $z \in \overline{\mathfrak{V}}$, the function

$$y \mapsto \mathbf{f}_0(y,z) : \overline{\mathfrak{U}} \to \mathbb{R} \tag{5.19}$$

is continuous in the sense of $\mathbf{t}_1$ generated by the metric (5.5). As a corollary by (5.18) the function (5.19) attains the minimum on the nonempty compact set $\mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)$, where $\varepsilon \in]0,\infty[$. Namely,

$$\min_{y \in \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)} \mathbf{f}_0(y,z) \in \mathbb{R} \ \forall \varepsilon \in]0,\infty[\ \forall z \in \overline{\mathfrak{V}}.$$

Therefore, under $\varepsilon \in]0,\infty[$, the function

$$\Psi_\varepsilon \triangleq \left(\min_{y \in \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)} \mathbf{f}_0(y,z)\right)_{z \in \overline{\mathfrak{V}}},$$

$\Psi_\varepsilon : \overline{\mathfrak{V}} \to \mathbb{R}$, is defined correctly. In addition,

$$\Psi_\varepsilon(z) = \min_{y \in \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)} \mathbf{f}_0(y,z) \ \forall z \in \overline{\mathfrak{V}}. \tag{5.20}$$

Since under $z \in \overline{\mathfrak{V}}$ the function $\mathbf{f}_0(\cdot, z)$ of the type

$$y \mapsto \mathbf{f}_0(y,z) : \overline{\mathfrak{U}} \to \mathbb{R} \tag{5.21}$$

is $\mathbf{t}_1$-continuous, then by (5.3) and (5.20), for $\varepsilon \in]0,\infty[$

$$\Psi_\varepsilon(z) = \inf_{y \in G'_\varepsilon} \mathbf{f}_0(y,z).$$

Using the boundedness of $\mathbf{f}_0$, under $\varepsilon \in]0,\infty[$, we have the boundness property of the mapping Ψ_ε; therefore, the values

$$\sup_{z \in \mathbb{G}''_\delta} \Psi_\varepsilon(z) = \sup\left(\left\{\Psi_\varepsilon(z) : z \in \mathbb{G}''_\delta\right\}\right) = \sup_{z \in G''_\delta} \inf_{y \in G'_\varepsilon} \mathbf{f}_0(y,z) \in \mathbb{R} \ \forall \delta \in]0,\infty[$$

are defined correctly. And what is more, by (5.17)

$$\mathbf{v}(\varepsilon,\delta) = \sup_{z \in G''_\delta} \Psi_\varepsilon(z) \ \forall \varepsilon \in]0,\infty[\ \forall \delta \in]0,\infty[. \tag{5.22}$$

Since $\mathbf{f}_0$ is the uniformly continuous function, the function Ψ_ε is a $\mathbf{t}_2$-continuous. Then, by (5.22)

$$\max_{z \in \mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right)} \Psi_\varepsilon(z) = \sup_{z \in G''_\delta} \Psi_\varepsilon(z) = \mathbf{v}(\varepsilon,\delta) \ \forall \varepsilon \in]0,\infty[\ \forall \delta \in]0,\infty[. \tag{5.23}$$

From (5.20) and (5.23) the equality follows:

$$\mathbf{v}(\varepsilon,\delta) = \max_{z \in \mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right)} \min_{y \in \mathrm{cl}\left(G'_\varepsilon, \tau_{\mathbb{R}}^{(k)}\right)} \mathbf{f}_0(y,z) \ \forall \varepsilon \in]0,\infty[\ \forall \delta \in]0,\infty[.$$

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We recall (4.31) and (4.41). Moreover, from (4.30) and (5.3), we obtain that $\mathbb{G}_1 \subset \overline{\mathfrak{U}}$. Finally, by (4.29) and (5.1) $\mathbb{G}_1 \neq \varnothing$. Then $\mathbb{G}_1 \in \mathcal{P}'\left(\overline{\mathfrak{U}}\right)$. Now, we note that by (4.30) and axioms of the family of closed sets, $\mathbb{G}_1$ is closed in $\left(\overline{\mathfrak{U}}, \mathbf{t}_1\right)$. Therefore,

$$\mathbb{G}_1 \in (\mathbf{t}_1 - \text{comp})\left[\overline{\mathfrak{U}}\right]. \tag{5.24}$$

By (5.24) and the continuity property of the function (5.21) the values

$$\min_{y \in \mathbb{G}_1} \mathbf{f}_0(y, z) \in \mathbb{R} \;\forall\, z \in \overline{\mathfrak{V}} \tag{5.25}$$

is defined correctly. Using (5.25), we introduce the mapping

$$\Psi : \overline{\mathfrak{V}} \to \mathbb{R} \tag{5.26}$$

by the following rule: if $z \in \overline{\mathfrak{V}}$, then

$$\Psi(z) \triangleq \min_{y \in \mathbb{G}_1} \mathbf{f}_0(y, z). \tag{5.27}$$

With the employment of the uniform continuity of $\mathbf{f}_0$, from (5.26) and (5.27), we obtain that the mapping Ψ is $\mathbf{t}_2$-continuous:

$$\Psi \in C\left(\overline{\mathfrak{V}}, \mathbf{t}_2, \mathbb{R}, \tau_{\mathbb{R}}\right). \tag{5.28}$$

By (4.40) and (5.6) we have the inclusion $\mathbb{G}_2 \subset \overline{\mathfrak{V}}$. Moreover, by (4.39) and (5.1) $\mathbb{G}_2 \neq \varnothing$. So, $\mathbb{G}_2 \in \mathcal{P}'\left(\overline{\mathfrak{V}}\right)$. From (4.40), we obtain that $\mathbb{G}_2$ is closed in $\left(\overline{\mathfrak{V}}, \mathbf{t}_2\right)$ and, as a corollary,

$$\mathbb{G}_2 \in (\mathbf{t}_2 - \text{comp})\left[\overline{\mathfrak{V}}\right]. \tag{5.29}$$

Using (5.28) and (5.29), we obtain that the value

$$\mathbf{V} \triangleq \max_{z \in \mathbb{G}_2} \Psi(z) \in \mathbb{R} \tag{5.30}$$

is defined correctly. From (5.27) and (5.30) the obvious equality

$$\mathbf{V} = \max_{z \in \mathbb{G}_2} \min_{y \in \mathbb{G}_1} \mathbf{f}_0(y, z)$$

follows. Using (4.29) and (4.39), we obtain that

$$\mathbf{V} = \max_{\nu \in \widetilde{V}_\partial} \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0(\widetilde{\Pi}_1(\mu), \widetilde{\Pi}_2(\nu)) \in \mathbb{R}.$$

By (4.21), (4.22), and (4.43) the equality

$$\mathbf{V} = \max_{\nu \in \widetilde{V}_\partial} \min_{\mu \in \widetilde{U}_\partial} \widetilde{\Phi}(\mu, \nu) \tag{5.31}$$

is realized. Now, we compare the functions Ψ and $\Psi_\varepsilon,\ \varepsilon > 0$.

Proposition 5.1. *The function* Ψ *assumes the following approximate representation:*
$\forall\,\zeta\in]0,\infty[\ \exists\,\theta\in]0,\infty[$:

$$\Psi(z)\in[\Psi_\varepsilon(z),\Psi_\varepsilon(z)+\zeta[\ \forall\,\varepsilon\in]0,\theta[\ \forall\,z\in\overline{\mathfrak{V}}.$$

Proof. We fix $\zeta\in]0,\infty[$. Since (4.10) is the nonempty compactum metrizable by ρ and $\left((\overline{\mathfrak{U}}\times\overline{\mathfrak{V}},\rho\right)$ is a compact metric space), then $\mathbf{f}_0$ (4.11) is an uniformly continuous function. Therefore, for some $\overline{\delta}\in]0,\infty[$, the following property takes place: $\forall\,y'\in\overline{\mathfrak{U}}\ \forall\,z'\in\overline{\mathfrak{V}}\ \forall\,y''\in\overline{\mathfrak{U}}\ \forall\,z''\in\overline{\mathfrak{V}}$

$$\left(\rho\left(\left(y',z'\right),\left(y'',z''\right)\right)<\overline{\delta}\right)\Rightarrow\left(\left|\mathbf{f}_0\left(y',z'\right)-\mathbf{f}_0\left(y'',z''\right)\right|<\zeta\right).\tag{5.32}$$

By (4.31) for some $\overline{\varepsilon}\in]0,\infty[$, the inclusion chain

$$\mathbb{G}_1\subset\mathrm{cl}\left(G'_{\overline{\varepsilon}},\tau_{\mathbb{R}}^{(k)}\right)\subset O_{\overline{\delta}}^{(k)}[\mathbb{G}_1].\tag{5.33}$$

is realized. By definition of the metric ρ we obtain that $\forall\,y'\in\overline{\mathfrak{U}}\ \forall\,y''\in\overline{\mathfrak{U}}$

$$\left(\left\|y'-y''\right\|^{(k)}<\overline{\delta}\right)\Rightarrow\left(\rho\left(\left(y',z\right),\left(y'',z\right)\right)<\overline{\delta}\ \forall\,z\in\overline{\mathfrak{V}}\right).$$

Therefore, from (5.32) we have the property: $\forall\,y'\in\overline{\mathfrak{U}}\ \forall\,y''\in\overline{\mathfrak{U}}$

$$\left(\left\|y'-y''\right\|^{(k)}<\overline{\delta}\right)\Rightarrow\left(\left|\mathbf{f}_0\left(y',z\right)-\mathbf{f}_0\left(y'',z\right)\right|<\zeta\ \forall\,z\in\overline{\mathfrak{V}}\right).\tag{5.34}$$

Fix æ $\in]0,\overline{\varepsilon}[$ and $z_*\in\overline{\mathfrak{V}}$. Then by (5.20) we obtain that

$$\Psi_{æ}(z_*)=\min_{y\in\mathrm{cl}\left(G'_{æ},\tau_{\mathbb{R}}^{(k)}\right)}\mathbf{f}_0(y,z_*).\tag{5.35}$$

Moreover, from (5.27), the equality

$$\Psi(z_*)=\min_{y\in\mathbb{G}_1}\mathbf{f}_0(y,z_*).\tag{5.36}$$

Since æ $<\overline{\varepsilon}$, from (4.15) and (4.16), the obvious inclusion

$$U_\partial[æ]\subset U_\partial\left[\overline{\varepsilon}\right]$$

follows and, as a corollary, by (4.28) $G'_{æ}\subset G'_{\overline{\varepsilon}}$; then

$$\mathrm{cl}\left(G'_{æ},\tau_{\mathbb{R}}^{(k)}\right)\subset\mathrm{cl}\left(G'_{\overline{\varepsilon}},\tau_{\mathbb{R}}^{(k)}\right).$$

With the employment of (4.30) and (5.33), we obtain that

$$\mathbb{G}_1\subset\mathrm{cl}\left(G'_{æ},\tau_{\mathbb{R}}^{(k)}\right)\subset O_{\overline{\delta}}^{(k)}[\mathbb{G}_1].\tag{5.37}$$

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Now, from (5.35)-(5.37), the inequality

$$\Psi_{æ}(z_*) \leqslant \Psi(z_*) \tag{5.38}$$

follows. Using (5.35), we choose $y_* \in \mathrm{cl}\left(G'_{æ}, \tau_{\mathbb{R}}^{(k)}\right)$ such that

$$\mathbf{f}_0(y_*, z_*) = \Psi_{æ}(z_*). \tag{5.39}$$

By (2.2) and (5.37), for some $y^* \in \mathbb{G}_1$, the inequality

$$\|y_* - y^*\|^{(k)} < \overline{\delta}$$

is realized; in addition, by (5.36) the estimate

$$\Psi(z_*) \leqslant \mathbf{f}_0\left(y^*, z_*\right) \tag{5.40}$$

takes place. Using (5.34), we obtain that

$$|\mathbf{f}_0(y_*, z_*) - \mathbf{f}_0\left(y^*, z_*\right)| < \zeta.$$

From (5.40) and the last inequality, the inequality $\Psi(z_*) < \mathbf{f}_0(y_*, z_*) + \zeta$ is valid. As a corollary, by (5.39) we have the estimate

$$\Psi(z_*) < \Psi_{æ}(z_*) + \zeta.$$

Using (5.38), we obtain the following chain of inequalities:

$$\Psi_{æ}(z_*) \leqslant \Psi(z_*) < \Psi_{æ}(z_*) + \zeta.$$

Since the choice of æ and z_* was arbitrary, we obtain that

$$\Psi(z) \in [\Psi_\varepsilon(z), \Psi_\varepsilon(z) + \zeta[\ \forall \varepsilon \in]0, \overline{\varepsilon}[\ \forall z \in \overline{\mathfrak{Y}} \tag{5.41}$$

(we recall that $\overline{\varepsilon} \in]0, \infty[$). But, the choice of ζ was arbitrary too. Therefore, from (5.41), the required statement follows. □

Returning to (5.30), we note that by (5.27)

$$\mathbf{V} = \max_{z \in \mathbb{G}_2} \min_{y \in \mathbb{G}_1} \mathbf{f}_0(y, z). \tag{5.42}$$

With the employment of (4.29) and (4.39), we obtain that (see (5.1), (5.42))

$$\mathbf{V} = \max_{\nu \in \widetilde{V}_\partial} \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0\left(\widetilde{\Pi}_1(\mu), \widetilde{\Pi}_2(\nu)\right) = \max_{z \in \mathbb{G}_2} \min_{y \in \mathbb{G}_1} \mathbf{f}_0(y, z). \tag{5.43}$$

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In addition, the mapping $\widetilde{\Phi}$ is defined (see (4.43)) by the rule

$$(\mu,\nu) \mapsto \mathbf{f}_0\left(\widetilde{\Pi}_1(\mu),\widetilde{\Pi}_2(\nu)\right) :\ \widetilde{U}\times\widetilde{V}\to\mathbb{R},$$

$\widetilde{\Phi}$ is the cost function of the generalized game problem; this function is continuous:

$$\widetilde{\Phi}\in C\left(\widetilde{U}\times\widetilde{V}, \widetilde{\tau}^*_U(\mathcal{L}_1)\otimes\widetilde{\tau}^*_V(\mathcal{L}_2),\mathbb{R},\tau_{\mathbb{R}}\right),$$

where $\widetilde{\tau}^*_U(\mathcal{L}_1)\otimes\widetilde{\tau}^*_V(\mathcal{L}_2)$ is the natural topology of $\widetilde{U}\times\widetilde{V}$ corresponding to the product of the TS (4.20).

Theorem 5.1. *The generalized maximin* $\mathbf{V}$ *(see (5.42), (5.43)) defines the asymptotics of values of the realizable maximins; namely,* $\forall\zeta\in]0,\infty[\ \exists\,\theta_\zeta\in]0,\infty[$:

$$|\mathbf{v}(\varepsilon,\delta)-\mathbf{V}|<\zeta\ \forall\varepsilon\in]0,\theta_\zeta[\ \forall\,\delta\in]0,\theta_\zeta[. \tag{5.44}$$

Proof. We will use (5.22), (5.30), and Proposition 5.1. Using (5.30), we choose $z_0\in\mathbb{G}_2$ such that

$$\mathbf{V}=\Psi(z_0). \tag{5.45}$$

Fix $æ\in]0,\infty[$. By Proposition 5.1, for some $\delta^*_1\in]0,\infty[$, we have the property:

$$\Psi(z)\in[\Psi_\varepsilon(z),\Psi_\varepsilon(z)+æ[\ \ \forall\,\varepsilon\in]0,\delta^*_1[\ \ \forall\,z\in\overline{\mathfrak{V}}. \tag{5.46}$$

Since $z_0\in\overline{\mathfrak{V}}$ (see (5.37)), from (5.46), the inclusion system follows:

$$\Psi(z_0)\in[\Psi_\varepsilon(z_0),\Psi_\varepsilon(z_0)+æ[\ \ \forall\,\varepsilon\in]0,\delta^*_1[. \tag{5.47}$$

By the choice of z_0 and (4.40) we obtain that $z_0\in\mathrm{cl}\left(G''_\delta,\tau^{(l)}_{\mathbb{R}}\right)\ \forall\,\delta\in]0,\infty[$. In addition, by (5.47) under $\varepsilon\in]0,\delta^*_1[$ and $\delta\in]0,\infty[$,

$$\Psi(z_0)<\max_{z\in\mathrm{cl}\left(G''_\delta,\tau^{(l)}_{\mathbb{R}}\right)}\Psi_\varepsilon(z)+æ. \tag{5.48}$$

Using (5.23), we have the following inequality

$$\Psi(z_0)<\mathbf{v}(\varepsilon,\delta)+æ\ \forall\varepsilon\in]0,\delta^*_1[\ \forall\delta\in]0,\infty[.$$

So, we have the inequalities:

$$\mathbf{V}-\mathbf{v}(\varepsilon,\delta)<æ\ \forall\varepsilon\in]0,\delta^*_1[\ \forall\delta\in]0,\infty[. \tag{5.49}$$

Using the uniform continuity of $\mathbf{f}_0$, we choose $\delta^*_2\in]0,\infty[$ such that $\forall\,x'\in\overline{\mathfrak{U}}\ \forall\,y'\in\overline{\mathfrak{V}}\ \forall\,x''\in\overline{\mathfrak{U}}\ \forall\,y''\in\overline{\mathfrak{V}}$

$$\left(\rho\left((x',y'),(x'',y'')\right)<\delta^*_2\right)\Rightarrow\left(\left|\mathbf{f}_0\left(x',y'\right)-\mathbf{f}_0\left(x'',y''\right)\right|<æ\right). \tag{5.50}$$

Using (4.41), we choose $\delta_3^* \in]0, \infty[$ for which

$$\mathrm{cl}\left(G''_{\delta_3^*}, \tau_{\mathbb{R}}^{(l)}\right) \subset O_{\delta_2^*}^{(l)}[\mathbb{G}_2]. \tag{5.51}$$

We note that by (4.16), under $\delta \in]0, \delta_3^*[$, the inclusion $V_\partial[\delta] \subset V_\delta[\delta_3^*]$ and, as a corollary, $G''_\delta \subset G''_{\delta_3^*}$; therefore,

$$\mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right) \subset \mathrm{cl}\left(G_{\delta_3^*}, \tau_{\mathbb{R}}^{(l)}\right) \subset O_{\delta_2^*}^{(l)}[\mathbb{G}_2]; \tag{5.52}$$

see (4.38) and (5.51). Let $\delta^0 \triangleq \inf\left(\left\{\delta_i^* : i \in \overline{1,3}\right\}\right)$; of course, $\delta^0 \in]0, \infty[$. Choose arbitrary members

$$\left(\varepsilon_0 \in \left]0, \delta^0\right[\right) \ \& \ \left(\delta_0 \in \left]0, \delta^0\right[\right). \tag{5.53}$$

Since $\varepsilon_0 < \delta_1^*$, by (5.53) and (5.49) the obvious inequality

$$\mathbf{V} - \mathbf{v}(\varepsilon_0, \delta_0) < \text{æ} \tag{5.54}$$

is realized. From (5.48) the following estimate is realized:

$$\Psi(z_0) < \max_{z \in \mathrm{cl}\left(G''_\delta, \tau_{\mathbb{R}}^{(l)}\right)} \Psi_{\varepsilon_0}(z) + \text{æ}$$

(we use the above-mentioned inequality $\varepsilon_0 < \delta_1^*$).

In addition,

$$\mathbf{v}(\varepsilon_0, \delta_0) = \max_{z \in \mathrm{cl}\left(G''_{\delta_0}, \tau_{\mathbb{R}}^{(l)}\right)} \Psi_{\varepsilon_0}(z).$$

Recall that by (5.45) and (5.54), $\Psi(z_0) < \mathbf{v}(\varepsilon_0, \delta_0) + \text{æ}$. Using (5.7) and continuity property of Ψ_{ε_0}, we choose

$$\widehat{z} \in \mathrm{cl}\left(G''_{\delta_0}, \tau_{\mathbb{R}}^{(l)}\right)$$

such that $\Psi_{\varepsilon_0}(\widehat{z}) = \mathbf{v}(\varepsilon_0, \delta_0)$. Recall that $\delta_0 \leqslant \delta_3^*$. Then, by (5.52)

$$\mathrm{cl}\left(G''_{\delta_0}, \tau_{\mathbb{R}}^{(l)}\right) \subset O_{\delta_2^*}^{(l)}[\mathbb{G}_2].$$

Using this inclusion, we choose $z^0 \in \mathbb{G}_2$ for which

$$\left\|\widehat{z} - z^0\right\| < \delta_2^*.$$

Using the last inequality, we obtain that by (5.50)

$$\left|\mathbf{f}_0\left(x, \widehat{z}\right) - \mathbf{f}\left(x, z^0\right)\right| < \text{æ} \ \forall\, x \in \mathfrak{X}. \tag{5.55}$$

Recall that by the choice of $\widehat{z}$ and (5.20)

$$\mathbf{v}(\varepsilon_0, \delta_0) = \Psi_{\varepsilon_0}\left(\widehat{z}\right) = \min_{y \in \mathrm{cl}\left(G'_{\varepsilon_0}, \tau_{\mathbb{R}}^{(k)}\right)} \mathbf{f}_0\left(y, \widehat{z}\right). \tag{5.56}$$

By (4.30) we obtain the inclusion $\mathbb{G}_1 \subset \mathrm{cl}\left(G'_{\varepsilon_0}, \tau_{\mathbb{R}}^{(k)}\right)$. Therefore, by (5.56)

$$\mathbf{v}(\varepsilon_0, \delta_0) \leqslant \min_{y \in \mathbb{G}_1} \mathbf{f}_0\left(y, \widehat{z}\right). \tag{5.57}$$

From (5.27) and (5.57), the inequality $\mathbf{v}(\varepsilon_0, \delta_0) \leqslant \Psi\left(\widehat{z}\right)$ is realized. Moreover, by (5.27)

$$\Psi\left(z^0\right) = \min_{y \in \mathbb{G}_1} \mathbf{f}_0\left(y, z^0\right).$$

Choose $y^0 \in \mathbb{G}_1$ such that $\Psi\left(z^0\right) = \mathbf{f}_0\left(y^0, z^0\right)$. Of course, from (5.57), the inequality $\mathbf{v}(\varepsilon_0, \delta_0) \leqslant \mathbf{f}_0\left(y^0, \widehat{z}\right)$ follows, where by (5.55)

$$\left|\mathbf{f}_0\left(y^0, \widehat{z}\right) - \mathbf{f}_0\left(y^0, z^0\right)\right| < æ$$

(we use the obvious inclusion $y^0 \in \overline{\mathfrak{U}}$; see (5.24)). As a corollary,

$$\mathbf{v}(\varepsilon_0, \delta_0) < \mathbf{f}_0\left(y^0, z^0\right) + æ = \Psi\left(z^0\right) + æ.$$

From (5.30) the inequality $\Psi\left(z^0\right) \leqslant \mathbf{V}$ follows. Therefore, $\mathbf{v}(\varepsilon_0, \delta_0) < \mathbf{V} + æ$ and, as a corollary,

$$\mathbf{v}(\varepsilon_0, \delta_0) - \mathbf{V} < æ.$$

With the employment of (5.54), we obtain that

$$|\mathbf{v}(\varepsilon_0, \delta_0) - \mathbf{V}| < æ.$$

Since the choice of ε_0 and δ_0 was arbitrary, from (5.53), we obtain that

$$\delta^0 \in]0, \infty[: \ |\mathbf{v}(\varepsilon, \delta) - \mathbf{V}| < æ \ \forall \varepsilon \in]0, \delta^0[\ \forall \delta \in]0, \delta^0[. \tag{5.58}$$

But, the choice of æ was arbitrary too. Therefore, by (5.58) we have the required property (5.44). □

By Theorem 5.1 we obtain the possibility of the exhausting representation of asymptotics of the realized maximins in terms of the generalized game problem. Namely, the maximin (5.43) of our generalized problem defines the above-mentioned asymptotics.

6. Some Conditions Sufficient for a Result Stability

In this section we consider the question about the coincidence of usual and generalized maximins. In connection with Theorem 5.1, this question is concerning the stability property.

Now, we refuse from the supposition (5.1) and consider the general case of our basic problem. For this general case, we suppose that

$$U^{(\partial)} \triangleq \left\{ u \in U \mid \left(\int_{I_1} \gamma_i u \, d\eta_1 \right)_{i \in \overline{1,p}} \in Y \right\}, \tag{6.1}$$

$$V^{(\partial)} \triangleq \left\{ v \in V \mid \left(\int_{I_2} \omega_j v \, d\eta_2 \right)_{j \in \overline{1,q}} \in Z \right\}. \tag{6.2}$$

In (6.1) and (6.2), we introduce the sets of admissible usual controls. Of course,

$$(U^{(\partial)} \subset U_\partial[\varepsilon] \ \forall \varepsilon \in]0, \infty[) \ \& \ (V^{(\partial)} \subset V_\partial[\delta] \ \forall \delta \in]0, \infty[).$$

Remark 6.1. Consider arbitrary measurable space $(I, \mathcal{L})$ with a semialgebra of sets, $I \neq \varnothing$. We use general constructions of Section 3 (see (3.1)-(3.42)). Fix $\eta \in (\text{add})_+[\mathcal{L}]$.

As in Section 3, in this remark, we fix a set $W \in \mathcal{P}'(B(I, \mathcal{L}))$ such that the condition (3.5) is valid. For the definiteness we fix $\mathbf{c} \in [0, \infty[$ such that (3.6) takes place. We preserve (3.7) and obtain (3.8). Moreover, we fix a (nonempty) closed set $X \in \mathcal{P}'(\mathbb{R}^m)$, where $m \in \mathbb{N}$. But, with respect to $(h_i)_{i \in \overline{1,m}}$ of Section 3, we suppose that

$$h_j \in B_0(I, \mathcal{L}) \ \forall j \in \overline{1,m}. \tag{6.3}$$

So, (6.3) is our new condition. We note the natural particular case of (6.3). Namely, it is possible to consider the variant

$$(h_i)_{i \in \overline{1,n}} = (\chi_{L_i})_{i \in \overline{1,n}}, \tag{6.4}$$

where $(L_i)_{i \in \overline{1,n}} \in \mathcal{L}^n$ and, for $\Lambda \in \mathcal{L}$,

$$\chi_\Lambda : I \to \mathbb{R}$$

is the indicator [15, §II.2] of Λ: $\chi_\Lambda(x) \triangleq 1$ under $x \in \Lambda$ and $\chi_\Lambda(y) \triangleq 0$ under $y \in I \backslash \Lambda$. The case (6.4) corresponds to the following variant of constraints

$$\left(\int_{L_i} f \, d\eta \right)_{i \in \overline{1,m}} \in X \tag{6.5}$$

(we keep in mind the concrete version of (3.4)). We note that the constraint (6.5) is used in control problems. Now, we recall examples of [1, §1.3] and [2, §2.4].

Of course, in this remark we fix $\eta \in (\text{add})_+[\mathcal{L}]$ and postulate that

$$\widetilde{W} = \text{cl}(\{f * \eta : f \in W\}, \tau_0(\mathcal{L})). \tag{6.6}$$

So, we consider the case, when

$$\widetilde{W} = \mathrm{cl}(\{f * \eta : f \in W\}, \tau_*(\mathcal{L})) = \mathrm{cl}(\{f * \eta : f \in W\}, \tau_0(\mathcal{L})). \quad (6.7)$$

In connection with (6.7), we note that this condition includes, in particular, the following concrete cases similar to $1') - 4')$ and $1'') - 4'')$:

$1^*)$ $W = \left\{ f \in B_0(I, \mathcal{L}) \mid \int_I \| f \| \, d\eta \leqslant b \right\}$ and $\widetilde{W} = \{\mu \in \mathbb{A}_\eta[\mathcal{L}] \mid \mathbf{v}_\mu(I) \leqslant b\}$, $b \in [0, \infty[$;

$2^*)$ $W = \left\{ f \in B_0^+(I, \mathcal{L}) \mid \int_I f \, d\eta \leqslant b \right\}$ and $\widetilde{W} = \{\mu \in (\mathrm{add})^+[\mathcal{L}; \eta] \mid \mu(I) \leqslant b\}$, $b \in [0, \infty[$;

$3^*)$ $W = \left\{ f \in B_0^+(I, \mathcal{L}) \mid \int_I f \, d\eta = b \right\}$ and $\widetilde{W} = \{\mu \in (\mathrm{add})^+[\mathcal{L}; \eta] \mid \mu(I) = b\}$, $b \in [0, \infty[$;

$4^*)$ if $r \in \mathbb{N}$, $(L_i)_{i \in \overline{1,r}} : \overline{1,r} \to \mathcal{L}$, $(c_i)_{i \in \overline{1,r}} : \overline{1,r} \to [0, \infty[$ and $I = \bigcup_{i=1}^{r} L_i$, then $W = \left\{ f \in B_0(I, \mathcal{L}) \mid \int_{L_k} \| f \| \, d\eta \leqslant c_k \ \forall k \in \overline{1,r} \right\}$ and $\widetilde{W} = \{\mu \in \mathbb{A}_\eta[\mathcal{L}] \mid \mathbf{v}_\mu(L_k) \leqslant c_k \ \forall k \in \overline{1,r}\}$.

We return to the general case, using $\widetilde{W}_\partial$ (3.11), also

$$W^{(\partial)} \triangleq \left\{ f \in W \mid \left(\int_I h_i f \, d\eta \right)_{i \in \overline{1,m}} \in X \right\}. \quad (6.8)$$

Elements of $W^{(\partial)}$ (of $\widetilde{W}_\partial$) are usual (generalized) precise solutions.

The Basic Property. The following equality takes place:

$$\widetilde{W}_\partial = \mathrm{cl}\left(\left\{ f * \eta : f \in W^{(\partial)} \right\}, \tau_*(\mathcal{L}) \right). \quad (6.9)$$

It is easy to see that ((6.8), [1, (3.4.11)])

$$\widetilde{W^{(\partial)}} \triangleq \left\{ f * \eta : f \in W^{(\partial)} \right\} \in \mathcal{P}(\widetilde{W}_\partial). \quad (6.10)$$

By closedness of Y we have from (6.10) by definition of $*$-weak topology that the set $\widetilde{W}_\partial$ is closed in topology $\tau_*(\mathcal{L})$. Therefore $\mathrm{cl}\left(\widetilde{W^{(\partial)}}, \tau_*(\mathcal{L}) \right) \subset \widetilde{W}_\partial$. We will establish opposite inclosure. Let $\mu_0 \in \widetilde{W}_\partial$. We choice (see (6.6),[9, p. 89]) a net $(D, \preceq, h)$ in W with the property

$$(D, \preceq, (h(\delta) * \eta)_{\delta \in D}) \xrightarrow{\tau_0(\mathcal{L})} \mu_0. \quad (6.11)$$

From (6.11) and definition of elementary integral [1, p. 81] it follows that $\exists \delta_1 \in D$ $\forall \delta_2 \in D$

$$(\delta_1 \preceq \delta_2) \Rightarrow \left(\left(\int_I h_i h(\delta_2) \, d\eta \right)_{i \in \overline{1,m}} = \left(\int_I h_i \, d\mu_0 \right)_{i \in \overline{1,m}} \right);$$

we use also the axioms of directed set. By the choice of μ_0 we have the property:

$$\left(\int_I h_i h(\delta)\, d\eta\right)_{i\in\overline{1,m}} \in X \tag{6.12}$$

at some moment. This means, that the net $(D, \preceq, h)$ is contained in $W^{(\partial)}$ at some moment.

Note, that the net in left side of (6.11) takes values in $\mathbf{B}_*(\mathcal{L}, \mathbf{c})$ (see Section 3), and, besides, $\mu_0 \in \mathbf{B}_*(\mathcal{L}, \mathbf{c})$.

The topologies

$$\tau_W^* \triangleq \tau_*(\mathcal{L}) \mid_{\mathbf{B}_*(\mathcal{L},\mathbf{c})}, \quad \tau_W^0 \triangleq \tau_0(\mathcal{L}) \mid_{\mathbf{B}_*(\mathcal{L},\mathbf{c})}$$

generate comparable ([2, (3.5.6)]) TS

$$(\mathbf{B}_*(\mathcal{L}, \mathbf{c}), \tau_W^*), \quad (\mathbf{B}_*(\mathcal{L}, \mathbf{c}), \tau_W^0); \quad \tau_W^* \subset \tau_W^0. \tag{6.13}$$

From (6.11) and the choice of $(D, \preceq, h)$, the convergence

$$(D, \preceq, (h(\delta) * \eta)_{\delta\in D}) \xrightarrow{\tau_W^0} \mu_0$$

is realized. We use the comparability of TS in (6.13); this means that

$$(D, \preceq, (h(\delta) * \eta)_{\delta\in D}) \xrightarrow{\tau_W^*} \mu_0.$$

Using [3, (2.3.9)], we obtain the property

$$(D, \preceq, (h(\delta) * \eta)_{\delta\in D}) \xrightarrow{\tau_*(\mathcal{L})} \mu_0. \tag{6.14}$$

In the other hand, the net in the left side of (6.14) has values in $\widetilde{W}^{(\partial)}$ at some moment (see (6.12)). Then from (6.14) and [9, p. 89] it follows that $\mu_0 \in \mathrm{cl}(\widetilde{W}^{(\partial)}, \tau_*(\mathcal{L}))$. We complete the proof of the inclusion $\widetilde{W}_\partial \subset \mathrm{cl}(\widetilde{W}^{(\partial)}, \tau_*(\mathcal{L}))$.

By the above-mentioned basic property the following equivalence is valid:

$$\left(W^{(\partial)} \neq \varnothing\right) \Leftrightarrow \left(\widetilde{W}_\partial \neq \varnothing\right). \tag{6.15}$$

We use Remark 6.1 in our game problem. Namely, we use two natural variants of constructions of the above-mentioned remark. The first variant corresponds to the stipulations:

$$I = I_1,\ \mathcal{L} = \mathcal{L}_1,\ \eta = \eta_1,\ W = U,\ m = p.$$

Moreover, in this case, using the equality $m = p$, we suppose that

$$(h_i)_{i\in\overline{1,m}} = (\gamma_i)_{i\in\overline{1,p}}.$$

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Then, the following concrete variants are realized: $\widetilde{W} = \widetilde{U},\ \widetilde{W}_\partial = \widetilde{U}_\partial,\ W^{(\partial)} = U^{(\partial)}$. Moreover, in correspondence with (6.3), in the following, we suppose that

$$\gamma_i \in B_0(I_1, \mathcal{L}_1)\ \forall\, i \in \overline{1, p}. \tag{6.16}$$

Finally, in correspondence with (6.6), in the following, we suppose that

$$\widetilde{U} = \mathrm{cl}(\{f * \eta_1 : f \in U\}, \tau_0(\mathcal{L}_1)). \tag{6.17}$$

Then, in this case, we obtain the following concrete variant of (6.9):

$$\widetilde{U}_\partial = \mathrm{cl}(\{u * \eta_1 : u \in U^{(\partial)}\}, \tau_*(\mathcal{L}_1)). \tag{6.18}$$

The second variant of constructions of Remark 6.1 corresponds to the stipulations:

$$I = I_2,\ \mathcal{L} = \mathcal{L}_2,\ \eta = \eta_2,\ W = V,\ m = q.$$

Moreover, using the equality $m = q$, we suppose that

$$(h_i)_{i \in \overline{1,m}} = (\omega_j)_{j \in \overline{1,q}}.$$

Then, the following concrete variants are realized:

$$\widetilde{W} = \widetilde{V},\ \widetilde{W}_\partial = \widetilde{V}_\partial,\ W^{(\partial)} = V^{(\partial)}.$$

Moreover, in correspondence with (6.3), in the following, we suppose that

$$\omega_j \in B_0(I_2, \mathcal{L}_2)\ \ \forall\, j \in \overline{1, q}. \tag{6.19}$$

Finally, in correspondence with (6.6), in the following, we suppose that

$$\widetilde{V} = \mathrm{cl}(\{f * \eta_2 : f \in V\}, \tau_0(\mathcal{L})). \tag{6.20}$$

Then, in the considered second case, we obtain the following concrete variant of (6.9):

$$\widetilde{V}_\partial = \mathrm{cl}\left(\{v * \eta_2 : v \in V^{(\partial)}\}, \tau_*(\mathcal{L}_2)\right). \tag{6.21}$$

So, in the following, we suppose that (6.16), (6.17), (6.19), and (6.20) are valid. Therefore, the equalities (6.18) and (6.21) are valid (namely, (6.18) and (6.21) are two concrete variants of (6.9)).

From (6.18), the following equivalence takes place:

$$\left(U^{(\partial)} \neq \varnothing\right) \Leftrightarrow \left(\widetilde{U}_\partial \neq \varnothing\right). \tag{6.22}$$

Analogously, by (6.21) we obtain that

$$\left(V^{(\partial)} \neq \varnothing\right) \Leftrightarrow \left(\widetilde{V}_\partial \neq \varnothing\right). \tag{6.23}$$

In (6.22) and (6.23), we have the "exhausting" conditions of the compatibility.

In the following, we suppose that

$$\left(U^{(\partial)} \neq \varnothing\right) \ \& \ \left(V^{(\partial)} \neq \varnothing\right). \tag{6.24}$$

So, later we have restricted to the "compatible" case. Of course, from (6.22)-(6.24), the relations

$$\left(\widetilde{U}_\partial \neq \varnothing\right) \ \& \ \left(\widetilde{V}_\partial \neq \varnothing\right) \tag{6.25}$$

follow (under the conditions (6.24)). We recall that $u*\eta_1 \in \widetilde{U}_\partial \ \forall\, u \in U^{(\partial)}$. Therefore, under $u \in U^{(\partial)}$

$$\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}} = \left(\int_{I_1} \alpha_i \, d(u * \eta_1)\right)_{i \in \overline{1,k}} \in \mathbb{G}_1$$

and, as a corollary, the inequalities

$$\Psi(z) \leqslant \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, z\right) \ \forall\, z \in \overline{\mathfrak{V}}. \tag{6.26}$$

From (6.26), we obtain that, for any $z \in \overline{\mathfrak{V}}$, the value

$$\inf_{u \in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, z\right) \in [\Psi(z), \infty[\tag{6.27}$$

is defined correctly. And what is more, the following statement takes place.

Proposition 6.1. *If* $z \in \overline{\mathfrak{V}}$, *then*

$$\Psi(z) = \inf_{u \in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, z\right). \tag{6.28}$$

Proof. Here we suppose that $\widetilde{U}^{(\partial)} \triangleq \left\{u * \eta_1 : u \in U^{(\partial)}\right\}$; then by (6.18) we obtain that

$$\widetilde{U}_\partial = \mathrm{cl}\left(\widetilde{U}^{(\partial)}, \tau_*(\mathcal{L}_1)\right). \tag{6.29}$$

Moreover, from (6.27) we have the inequality

$$\Psi(z) \leqslant \inf_{u \in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, z\right).$$

Using (5.27), we choose $y_0 \in \mathbb{G}_1$ such that

$$\Psi(z) = \mathbf{f}_0(y_0, z).$$

By (4.29) we obtain that $y_0 = \widetilde{\Pi}_1(\mu_0)$ for some $\mu_0 \in \widetilde{U}_\partial$; so, by (4.26)

$$y_0 = \left(\int_{I_1} \alpha_i \, d\mu_0\right)_{i \in \overline{1,k}}. \tag{6.30}$$

Using (6.30), we obtain the obvious equality

$$\Psi(z) = \mathbf{f}_0\left(\left(\left(\int_{I_1} \alpha_i \, d\mu_0\right)_{i \in \overline{1,k}}, z\right)\right). \tag{6.31}$$

With the employment of (6.29) we choose a net $(D, \preceq, \widetilde{h})$ in $\widetilde{U}^{(\partial)}$ such that [3, (2.3.11)]

$$(D, \preceq, \widetilde{h}) \overset{\tau_*(\mathcal{L}_1)}{\longrightarrow} \mu_0. \tag{6.32}$$

Then $\widetilde{h}(d) \in \widetilde{U}^{(\partial)}$ under $d \in D$; by definition of $\widetilde{U}^{(\partial)}$

$$U^{(\partial)}[d] \triangleq \left\{u \in U^{(\partial)} \mid u * \eta_1 = \widetilde{h}(d)\right\} \in \mathcal{P}'\left(U^{(\partial)}\right).$$

Therefore, by axiom of choice we have the property

$$\prod_{d \in D} U^{(\partial)}[d] \neq \varnothing. \tag{6.33}$$

Using (6.33), we choose a selector

$$h \in \prod_{d \in D} U^{(\partial)}[d] \tag{6.34}$$

of the multifunction $d \mapsto U^{(\partial)}[d] : D \to \mathcal{P}'\left(U^{(\partial)}\right)$. From (6.34), we obtain that

$$h : D \to U^{(\partial)}.$$

Moreover, $h(d) \in U^{(\partial)}[d]$ under $d \in D$; therefore, $h(d) * \eta_1 = \widetilde{h}(d)$. So,

$$h(\mathbf{d}) * \eta_1 = \widetilde{h}(\mathbf{d}) \ \forall \, \mathbf{d} \in D. \tag{6.35}$$

From (6.32) and [3, (4.6.16)], we obtain the following properties of convergence:

$$\left(D, \preceq, \left(\int_{I_1} \alpha_i \, d\widetilde{h}(\delta)\right)_{\delta \in D}\right) \to \int_{I_1} \alpha_i \, d\mu_0 \ \forall \, i \in \overline{1,k}. \tag{6.36}$$

By definition of the norm $\|\cdot\|^{(k)}$ and (6.36) we have the convergence

$$\left(D, \preceq, \left(\left(\int_{I_1} \alpha_i \, d\widetilde{h}(\delta)\right)_{i \in \overline{1,k}}\right)_{\delta \in D}\right) \overset{\tau_{\mathbb{R}}^{(k)}}{\longrightarrow} \left(\int_{I_1} \alpha_i \, d\mu_0\right)_{i \in \overline{1,k}}.$$

With the employment of continuity of $\mathbf{f}_0$ and (6.31), the convergence

$$\left(D, \preceq, \left(\mathbf{f}_0\left(\left(\int_{I_1} \alpha_i \, d\widetilde{h}(\delta)\right)_{i\in\overline{1,k}}, z\right)\right)_{\delta\in D}\right) \xrightarrow{\tau_{\mathbb{R}}} \Psi(z). \tag{6.37}$$

On the other hand, by (3.3) and (6.35) the equality system

$$\left(\int_{I_1} \alpha_i h(\delta)\, d\eta_1\right)_{i\in\overline{1,k}} = \left(\int_{I_1} \alpha_i \, d\widetilde{h}(\delta)\right)_{i\in\overline{1,k}} \quad \forall\, \delta \in D.$$

Therefore, by (6.37) we obtain the following convergence

$$\left(D, \preceq, \left(\mathbf{f}_0\left(\left(\int_{I_1} \alpha_i h(\delta)\, d\eta_1\right)_{i\in\overline{1,k}}, z\right)\right)_{\delta\in D}\right) \xrightarrow{\tau_{\mathbb{R}}} \Psi(z).$$

So, under æ $\in]0, \infty[$, for some $\delta_{æ} \in D$, the implication system is realized: $\forall\, \delta \in D$

$$(\delta_{æ} \preceq \delta) \Rightarrow \left(\left|\mathbf{f}_0\left(\left(\int_{I_1} \alpha_i h(\delta)\, d\eta_1\right)_{i\in\overline{1,k}}, z\right) - \Psi(z)\right| < æ\right).$$

In particular, we have the inequality $\left|\mathbf{f}_0\left(\left(\int_{I_1} \alpha_i h(\delta_{æ})\, d\eta_1\right)_{i\in\overline{1,k}}, z\right) - \Psi(z)\right| < æ$, where $h(\delta_{æ}) \in U^{(\partial)}$. As a corollary, we obtain that

$$\inf_{u\in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u\, d\eta_1\right)_{i\in\overline{1,k}}, z\right) \leqslant \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i h(\delta_{æ})\, d\eta_1\right)_{i\in\overline{1,k}}, z\right) < \Psi(z) + æ.$$

Since the choice of æ was arbitrary, the inequality $\inf\limits_{u\in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u\, d\eta_1\right)_{i\in\overline{1,k}}, z\right) \leqslant \Psi(z)$ is valid. Using (6.27) and the last inequality, we have the required equality (6.28). Since the choice of z was arbitrary, our statement is established. □

From (6.21), the obvious inclusions

$$v * \eta_2 \in \widetilde{V}_{\partial} \quad \forall\, v \in V^{(\partial)}. \tag{6.38}$$

Therefore, by (4.36), (4.39) and (6.38) we obtain that

$$\Pi_2(v) = \widetilde{\Pi}_2(v * \eta_2) \in \mathbb{G}_2 \;\forall\, v \in V^{(\partial)}. \tag{6.39}$$

With the employment of (6.39) and Proposition 6.1, the following equality system is realized $\forall\, v \in V$:

$$\Psi\left(\widetilde{\Pi}_2(v * \eta_2)\right) = \inf_{u\in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u\, d\eta_1\right)_{i\in\overline{1,k}}, \Pi_2(v)\right) = \inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u), \Pi_2(v)).$$

Then, from (5.30) and (6.39), we obtain that

$$\inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) \leqslant \mathbf{V} \;\forall\, v \in V^{(\partial)}. \tag{6.40}$$

Using (6.25) and (6.40), we have the following property:

$$\left\{\inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) : v \in V^{(\partial)}\right\} \in \mathcal{P}'(\,]-\infty,\mathbf{V}]);$$

as a corollary, the corresponding supremum is defined correctly:

$$\sup_{v\in V^{(\partial)}}\inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) = \sup\left(\left\{\inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) : v \in V^{(\partial)}\right\}\right) \in]-\infty,\mathbf{V}].$$

In particular, we obtain the inequality

$$\sup_{v\in V^{(\partial)}}\inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) \leqslant \mathbf{V}. \tag{6.41}$$

Now, from (5.10), (6.1), and (6.2) we have the following obvious representation

$$\Phi(u,v) = \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) \;\forall\, u \in U^{(\partial)} \;\forall\, v \in V^{(\partial)}.$$

In addition, we obtain that by (6.41)

$$\sup_{v\in V^{(\partial)}}\inf_{u\in U^{(\partial)}} \Phi(u,v) = \sup_{v\in V^{(\partial)}}\inf_{u\in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u),\Pi_2(v)) \leqslant \mathbf{V}.$$

Theorem 6.1. *The usual and generalized maximins coincidence:*

$$\mathbf{V} = \sup_{v\in V^{(\partial)}}\inf_{u\in U^{(\partial)}} \Phi(u,v). \tag{6.42}$$

Proof. With the employment of (4.43) and (5.30), we choose $\nu_0 \in \widetilde{V}_\partial$ such that

$$\min_{\mu\in\widetilde{U}_\partial} \widetilde{\Phi}(\mu,\nu_0) = \mathbf{V}. \tag{6.43}$$

Then $z_0 \triangleq \widetilde{\Pi}_2(\nu_0) \in \mathbb{G}_2$. In particular, by (5.29) $z_0 \in \overline{\mathfrak{V}}$ and by (5.27)

$$\Psi(z_0) = \min_{y\in\mathbb{G}_1} \mathbf{f}_0(y,z_0) = \min_{\mu\in\widetilde{U}_\partial} \mathbf{f}_0\left(\widetilde{\Pi}_1(\mu),z_0\right); \tag{6.44}$$

here, we use (4.29). By (4.35) we obtain that

$$z_0 = \widetilde{\Pi}_2(\nu_0) = \left(\int_{I_2} \beta_j \, d\nu_0\right)_{j\in\overline{1,l}}. \tag{6.45}$$

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Now, we use (6.21). For this, we suppose that (see (6.24)) $\widetilde{V}^{(\partial)} \triangleq \{v * \eta_2 : v \in V^{(\partial)}\}$; $\widetilde{V}^{(\partial)} \in \mathcal{P}'\left(\widetilde{V}\right)$. Using (6.21), we obtain the equality $\widetilde{V}_\partial = \mathrm{cl}\left(\widetilde{V}^{(\partial)}, \tau_*(\mathcal{L}_2)\right)$ and $\nu_0 \in \widetilde{V}_\partial$. Then, by [3, (2.3.11)], for some net $(D, \preceq, \widetilde{h})$ in $\widetilde{V}^{(\partial)}$, the following convergence takes place: $(D, \preceq, \widetilde{h}) \xrightarrow{\tau_*(\mathcal{L}_2)} \nu_0$. Therefore, by [3, (4.6.16)] we obtain that

$$\left(D, \preceq, \left(\int_E \beta_j \, d\widetilde{h}(\delta)\right)_{\delta \in D}\right) \xrightarrow{\tau_{\mathbb{R}}} \int_E \beta_j \, d\nu_0 \; \forall\, j \in \overline{1,l}.$$

As a corollary, the obvious convergence

$$\left(D, \preceq, \left(\left(\int_E \beta_j \, d\widetilde{h}(\delta)\right)_{j \in \overline{1,l}}\right)_{\delta \in D}\right) \xrightarrow{\tau_{\mathbb{R}}^{(l)}} \left(\int_E \beta_j \, d\nu_0\right)_{j \in \overline{1,l}}.$$

With the employment of (6.45), we have the convergence

$$\left(D, \preceq, \left(\left(\int_E \beta_j \, d\widetilde{h}(\delta)\right)_{j \in \overline{1,l}}\right)_{\delta \in D}\right) \xrightarrow{\tau_{\mathbb{R}}^{(l)}} z_0. \tag{6.46}$$

By the definition of $\widetilde{V}^{(\partial)}$ the following property is realized:

$$\widetilde{V}^{(\partial)}[d] \triangleq \left\{v \in V^{(\partial)} \mid \widetilde{h}(d) = v * \eta_2\right\} \in \mathcal{P}'\left(V^{(\partial)}\right) \; \forall\, d \in D. \tag{6.47}$$

Using axiom of choice, we obtain that $\prod_{d \in D} \widetilde{V}^{(\partial)}[d] \neq \varnothing$. With the employment of this property, we choose arbitrary $h \in \prod_{d \in D} \widetilde{V}^{(\partial)}[d]$. So, $h : D \to V^{(\partial)}$ and (moreover) the inclusion system $h(d) \in \widetilde{V}^{(\partial)}[d] \; \forall\, d \in D$ takes place. This system means (by (6.47)) that $\widetilde{h}(d) = h(d) * \eta_2 \; \forall\, d \in D$. Using (3.3) and (6.46), we obtain that

$$\left(D, \preceq, \left(\left(\int_E \beta_j h(\delta) \, d\eta_2\right)_{j \in \overline{1,l}}\right)_{\delta \in D}\right) \xrightarrow{\tau_{\mathbb{R}}^{(l)}} z_0. \tag{6.48}$$

In addition, by (4.34) we have the following equality system:

$$(\Pi_2 \circ h)(\delta) = \Pi_2(h(\delta)) = \left(\int_E \beta_j h(\delta) \, d\eta_2\right)_{j \in \overline{1,l}} \; \forall\, \delta \in D.$$

Then, from (6.48) we obtain that

$$(D, \preceq, \Pi_2 \circ h) \xrightarrow{\tau_{\mathbb{R}}^{(l)}} z_0. \tag{6.49}$$

We recall that $h : D \to V$. As a corollary, $\Pi_2 \circ h : D \to \mathfrak{V}$, where $\mathfrak{V} \subset \overline{\mathfrak{V}}$ (see (4.8)). By (6.49) and [3, (2.3.9)] the obvious convergence $(D, \preceq, \Pi_2 \circ h) \xrightarrow{\mathbf{t}_2} z_0$. Then, by (5.28) and [3, (2.5.4)] we obtain that

$$(D, \preceq, \Psi \circ \Pi_2 \circ h) \xrightarrow{\tau_{\mathbb{R}}} \Psi(z_0). \tag{6.50}$$

We recall that by (4.43), (6.43) and (6.44)

$$\mathbf{V} = \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0 \left(\widetilde{\Pi}_1(\mu), \widetilde{\Pi}_2(\nu_0) \right) = \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0 \left(\widetilde{\Pi}_1(\mu), z_0 \right) = \Psi(z_0).$$

So, by (6.50) we have the following convergence

$$(D, \preceq, \Psi \circ \Pi_2 \circ h) \xrightarrow{\tau_{\mathbb{R}}} \mathbf{V}. \tag{6.51}$$

In addition, for $\delta \in D$, we have the relation

$$(\Psi \circ \Pi_2 \circ h)(\delta) = \Psi(\Pi_2(h(\delta))) = \inf_{u \in U^{(\partial)}} \mathbf{f}_0 \left(\left(\int_{I_1} \alpha_i u \, d\eta_1 \right)_{i \in \overline{1,k}}, \Pi_2(h(\delta)) \right), \tag{6.52}$$

where $\Pi_2(h(\delta)) \in \mathfrak{V}$ by (4.8) and (5.8); in the other hand, from (4.25), (5.12), and (6.52) it follows that

$$(\Psi \circ \Pi_2 \circ h)(\delta) = \inf_{u \in U^{(\partial)}} \mathbf{f}_0 \left(\Pi_1(u), \Pi_2(h(\delta)) \right) = \inf_{u \in U^{(\partial)}} \Phi(u, h(\delta)),$$

where $h(\delta) \in V^{(\partial)}$. As a corollary,

$$(\Psi \circ \Pi_2 \circ h)(\delta) \leqslant \sup_{v \in V^{(\partial)}} \inf_{u \in U^{(\partial)}} \Phi(u, v) \ \forall \, \delta \in D. \tag{6.53}$$

If $æ \in]0, \infty[$, then by (6.51) we obtain that, for some $\delta_æ \in D$, $\forall \, \delta \in D$ $(\delta_æ \preceq \delta) \Rightarrow (\mathbf{V} - æ < (\Psi \circ \Pi_2 \circ h)(\delta))$; in particular, we have the property

$$\mathbf{V} - æ < (\Psi \circ \Pi_2 \circ h)(\delta_æ) \leqslant \sup_{v \in V^{(\partial)}} \inf_{u \in U^{(\partial)}} \Phi(u, v) \tag{6.54}$$

(see (6.53)). Using (6.41) and (6.54), we obtain the required equality (6.42). □

Corollary 6.1. *The following property of the maximin stability takes place; namely,* $\forall \, \zeta \in]0, \infty[\ \exists \, \theta_\zeta \in]0, \infty[$:

$$\left| \mathbf{v}(\varepsilon, \delta) - \sup_{v \in V^{(\partial)}} \inf_{u \in U^{(\partial)}} \Phi(u, v) \right| < \zeta \ \forall \, \varepsilon \in]0, \theta_\zeta[\ \forall \, \delta \in]0, \theta_\zeta[.$$

The corresponding proof is realized by the immediate combination of Theorems 5.1 and 6.1 (in this connection, see (4.15), (4.16), (5.14), (6.1) and (6.2)).

Remark 6.1. We recall that Theorem 6.1 and Corollary 6.1 were established under conditions (6.16), (6.17), (6.19), and (6.20). We note that many practically interesting variants of U, V satisfy to (6.17) and (6.20); in this connection, see Remark 6.1 and in particular the cases $1^*) - 4^*$). In connection with (6.16) and (6.19), we note the consideration of (6.5) (see Remark 6.1). Of course, we can transform (6.5) in the two analogous conditions on the choice of $u \in U$ and $v \in V$ respectively. Namely, we keep in mind the following variants of (1.2): under $\left(\Lambda_i^{(1)}\right)_{i\in\overline{1,p}} : \overline{1,p} \to \mathcal{L}_1$ and $\left(\Lambda_j^{(2)}\right)_{j\in\overline{1,q}} : \overline{1,q} \to \mathcal{L}_2$, we consider the constraints

$$\left(\left(\int_{\Lambda_i^{(1)}} u\, d\eta_1\right)_{i\in\overline{1,p}} \in Y\right) \ \& \ \left(\left(\int_{\Lambda_j^{(2)}} v\, d\eta_2\right)_{j\in\overline{1,q}} \in Z\right) \tag{6.55}$$

(see examples [1, §1.3] and [2, §2.4]). Of course, for (6.55), the following concrete variants of (6.16) and (6.19) are required: under $i \in \overline{1,p}$ the function γ_i is the indicator [15, §II.2] of the set $\Lambda_i^{(1)} \in \mathcal{L}_1$; under $j \in \overline{1,q}$ the function ω_j is the indicator of the set $\Lambda_j^{(2)} \in \mathcal{L}_2$. In this connection, see definition before (6.5) in Remark 6.1.

7. Some Additions

In this section we consider the two "intermediate" variants. Namely, in contrast to Section 6, we suppose that the conditions of type (6.3) and (6.6) satisfy only for one from players. Therefore, this section consist from the two following parts: 1^0) the part for which (6.16) and (6.17) are fulfilled; 2^0) the part for which (6.19) and (6.20) are fulfilled.

1^0) Unless otherwise stated, we suppose that (6.16) and (6.17) are fulfilled (the validity of conditions (6.19) and (6.20) are supposed not). In this case we use Remark 6.1 under the stipulation (6.15). Therefore, we obtain the following concrete variant of (6.9): the relation (6.18) is valid. So, in our case, we have the validity of (6.18). Then (6.22) is valid (we obtain (6.22) from (6.18) by simplest properties of operations of a closure and an image). Suppose that

$$U^{(\partial)} \neq \varnothing$$

in our item 1^0). We have the compatibility property of the constraint system of the player forming $u \in U$.

In connection with constraints of the player forming $v \in V$, we suppose only asymptotic compatibility. Namely, using (4.33), we suppose in this part that

$$\widetilde{V}_\partial \neq \varnothing. \tag{7.1}$$

So, for the end of item 1^0), we consider the case

$$\left(U^{(\partial)} \neq \varnothing\right) \ \& \left(\widetilde{V}_\partial \neq \varnothing\right). \tag{7.2}$$

Using (6.22) and (7.2), we obtain the obvious corollary: in our case the relation (5.1) is fulfilled. Therefore, the basic statements of Section 5 are valid. In particular, for the considered case (7.2), the function Ψ is defined correctly. But, in this case, Ψ admits the natural realization in the class of usual controls. Recall the relation (6.18). In particular,

$$u * \eta_1 \in \widetilde{U}_\partial \ \forall u \in U^{(\partial)}.$$

Therefore, by (4.29) we obtain that

$$\widetilde{\Pi}(\mathcal{J}_1(u)) = \widetilde{\Pi}_1(u * \eta_1) = \left(\int_{I_1} \alpha_i \, d(u * \eta_1)\right)_{i \in \overline{1,k}} \in \mathbb{G}_1 \ \forall u \in U^{(\partial)}.$$

By definition of Π_1 we have the inclusion system

$$\Pi_1(u) \in \mathbb{G}_1 \ \forall u \in U^{(\partial)}. \tag{7.3}$$

By (5.27) and (7.3), for $z \in \overline{\mathfrak{V}}$, we have the inclusion

$$\left\{\mathbf{f}_0(\Pi_1(u), z) : \ u \in U^{(\partial)}\right\} \subset [\Psi(z), \infty[, \tag{7.4}$$

where the set on the left hand side of (7.4) is not empty (see (7.2)). Then, the infimum of this set is defined as an element of $\mathbb{R}$ and

$$\Psi(z) \leqslant \inf_{u \in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u), z) \ \forall z \in \overline{\mathfrak{V}}. \tag{7.5}$$

Proposition 7.1. *The function Ψ satisfies to the relations*

$$\Psi(z) = \inf_{u \in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u), z) \ \forall z \in \overline{\mathfrak{V}}. \tag{7.6}$$

Proof. Fix $z \in \overline{\mathfrak{V}}$. Using (5.27), we choose $y_0 \in \mathbb{G}_1$ such that

$$\Psi(z) = \mathbf{f}_0(y_0, z). \tag{7.7}$$

Then, by (4.29) we obtain that $y_0 = \widetilde{\Pi}_1(\mu_0)$ for some $\mu_0 \in \widetilde{U}_\partial$. Now, we recall (6.18); then

$$\mu_0 \in \mathrm{cl}(\{u * \eta_1 : u \in U^{(\partial)}\}, \tau_*(\mathcal{L}_1)).$$

Using reasoning similar to the proof of Proposition 6.1, we choose a net $(D, \preceq, h)$ in $U^{(\partial)}$ such that

$$(D, \preceq, (h(d) * \eta_1)_{d \in D}) \stackrel{\tau_*(\mathcal{L}_1)}{\longrightarrow} \mu_0.$$

Since $\mu_0 \in \widetilde{U}$ and $(D, \preceq, \mathcal{J}_1 \circ h)$ is a net in $\widetilde{U}$ by definition of $\mathcal{J}_1$, from [3, (2.3.9)], the convergence

$$(D, \preceq, \mathcal{J}_1 \circ h) \overset{\widetilde{\tau}_U^*(\mathcal{L}_1)}{\longrightarrow} \mu_0 \tag{7.8}$$

follows. Using the continuity of $\widetilde{\Pi}_1$ (see Section 4), by (7.8) we obtain that

$$(D, \preceq, \widetilde{\Pi}_1 \circ \mathcal{J}_1 \circ h) \overset{\tau_{\mathbb{R}}^{(k)}}{\longrightarrow} \widetilde{\Pi}_1(\mu_0).$$

Recall that $\Pi_1 = \widetilde{\Pi}_1 \circ \mathcal{J}_1$. Therefore,

$$(D, \preceq, \Pi_1 \circ h) \overset{\tau_{\mathbb{R}}^{(k)}}{\longrightarrow} y_0. \tag{7.9}$$

Since $(D, \preceq, h)$ is a net in U, then (see (5.8)) $(D, \preceq, \Pi_1 \circ h)$ is a net in $\mathfrak{U}$. In particular, $(D, \preceq, \Pi_1 \circ h)$ is a net in $\overline{\mathfrak{U}}$ (see (4.8)). Moreover, by (5.24), $y_0 \in \overline{\mathfrak{U}}$. By (5.4), (7.9), and [3, (2.3.9)]

$$(D, \preceq, \Pi_1 \circ h) \overset{\mathbf{t}_1}{\longrightarrow} y_0. \tag{7.10}$$

Using the continuity of $\mathbf{f}_0$, from (7.10), we obtain that

$$(D, \preceq, (\mathbf{f}_0((\Pi_1 \circ h)(d), z))_{d \in D}) \overset{\tau_{\mathbb{R}}}{\longrightarrow} \mathbf{f}_0(y_0, z) \tag{7.11}$$

(we recall that $\mathbf{t}_1$ is generated by the metric (5.8) and $\mathbf{f}_0$ is continuous in the sense of τ_ρ^0). Using (7.7) and (7.11), we have the convergence

$$(D, \preceq, (\mathbf{f}_0(\Pi_1(h(d)), z))_{d \in D}) \overset{\tau_{\mathbb{R}}}{\longrightarrow} \Psi(z). \tag{7.12}$$

Let $æ \in]0, \infty[$. Using (7.12), we choose $d_æ \in D$ such that $\forall\, d \in D$

$$(d_æ \preceq d) \Rightarrow (\,|\mathbf{f}_0(\Pi_1(h(d)), z) - \Psi(z)| < æ).$$

In particular, for $d_æ \in D$, we have the inclusion

$$\mathbf{f}_0(\Pi_1(h(d_æ)), z) < \Psi(z) + æ. \tag{7.13}$$

But, $h(d_æ) \in U^{(\partial)}$. Therefore, by (7.13)

$$\inf_{u \in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u), z) < \Psi(z) + æ.$$

Since the choice of æ was arbitrary, we obtain that

$$\inf_{u \in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u), z) \leqslant \Psi(z). \tag{7.14}$$

From (7.5), the inequality inverse to (7.14) follows. As a corollary

$$\Psi(z) = \inf_{u \in U^{(\partial)}} \mathbf{f}_0(\Pi_1(u), z).$$

Since the choice of z was arbitrary, the relation (7.6) is established. □

Using (4.39), (5.30), and Proposition 7.1, we obtain that

$$\mathbf{V} = \max_{\nu \in \widetilde{V}_\partial} \inf_{u \in U^{(\partial)}} \mathbf{f}_0\left(\Pi_1(u), \widetilde{\Pi}_2(\nu)\right)$$

$$= \max_{\nu \in \widetilde{V}_\partial} \inf_{u \in U^{(\partial)}} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i u \, d\eta_1\right)_{i \in \overline{1,k}}, \left(\int_{I_2} \beta_j \, d\nu\right)_{j \in \overline{1,l}}\right). \quad (7.15)$$

Recall that (7.15) was established under the conditions (6.16), (6.17), and (7.2).

2^0) In the following, the validity of (6.16) and (6.17) is not supposed. But, we suppose that the conditions (6.19) and (6.20) are fulfilled. Then, (6.21) is valid (in this connection see Remark 6.1 too). Recall that from (6.21), the relation (6.23) is valid (see (6.15) too).

In the following (in this section) we suppose that

$$V^{(\partial)} \neq \varnothing. \quad (7.16)$$

As a corollary, from (6.23) and (7.16), we obtain (7.1). Moreover, we suppose that

$$\widetilde{U}_\partial \neq \varnothing.$$

So, for the end of item 2^0), we consider the case

$$\left(\widetilde{U}_\partial \neq \varnothing\right) \ \& \ \left(V^{(\partial)} \neq \varnothing\right). \quad (7.17)$$

Of course, the relation (5.1) is valid too. Therefore, we have the validity of basic statements of Section 5. In particular, in our case (7.17), the function Ψ is defined correctly. In addition, under $v \in V^{(\partial)}$

$$\Psi(\Pi_2(v)) = \min_{y \in \mathbb{G}_1} \mathbf{f}_0(y, \Pi_2(v)); \quad (7.18)$$

see (5.8). Moreover, by (6.21) $\overline{v} * \eta_2 \in \widetilde{V}_\partial$ under $\overline{v} \in V^{(\partial)}$. Therefore, by (4.39) $\widetilde{\Pi}_2(\overline{v} * \eta_2) \in \mathbb{G}_2$. In other words

$$\Pi_2(\overline{v}) = \left(\widetilde{\Pi}_2 \circ \mathcal{J}_2\right)(\overline{v}) = \widetilde{\Pi}_2(\mathcal{J}_2(\overline{v})) = \widetilde{\Pi}_2(\overline{v} * \eta) \in \mathbb{G}_2.$$

Since the choice of $\overline{v}$ was arbitrary, we obtain that

$$\Pi_2(v) \in \mathbb{G}_2 \ \forall v \in V^{(\partial)}. \quad (7.19)$$

Therefore, by (5.30) and (7.16) we have the following property:

$$\left\{\Psi(\Pi_2(v)) : v \in V^{(\partial)}\right\} \in \mathcal{P}'(]-\infty, \mathbf{V}])$$

and, as a corollary, the corresponding supremum is defined correctly:

$$\sup_{v \in V^{(\partial)}} \Psi(\Pi_2(v)) \in]-\infty, \mathbf{V}]. \quad (7.20)$$

Proposition 7.2. *The following equality is valid:*

$$\mathbf{V} = \sup_{v \in V^{(\partial)}} \Psi(\Pi_2(v)). \tag{7.21}$$

Proof. By (7.20) we have the following inequality

$$\sup_{v \in V^{(\partial)}} \Psi(\Pi_2(v)) \leqslant \mathbf{V}. \tag{7.22}$$

Choose $z_0 \in \mathbb{G}_2$ such that $\Psi(z_0) = \mathbf{V}$. Using (4.39), we choose $\nu_0 \in \widetilde{V}_\partial$ such that $z_0 = \widetilde{\Pi}_2(\nu_0)$. As a result,

$$\mathbf{V} = \Psi\left(\widetilde{\Pi}_2(\nu_0)\right). \tag{7.23}$$

By (6.9) $\nu_0 \in \mathrm{cl}\left(\left\{v * \eta_2 : v \in V^{(\partial)}\right\}, \tau_*(\mathcal{L}_2)\right)$. Therefore, we use the reasoning similar to the corresponding reasoning in the proof of Theorem 6.1. Namely, with the employment of [3, (2.3.11)], we choose a net $(D, \sqsubseteq, \varphi)$ in $V^{(\partial)}$ for which

$$(D, \sqsubseteq, \mathcal{J}_2 \circ \varphi) \stackrel{\tau_*(\mathcal{L}_2)}{\longrightarrow} \nu_0 \tag{7.24}$$

(of course, here we use axiom of choice). We recall that $\nu_0 \in \widetilde{V}$ and $(D, \sqsubseteq, \mathcal{J}_2 \circ \varphi)$ is a net in $\widetilde{V}$. Therefore, by [3, (2.3.9)] and (7.24)

$$(D, \sqsubseteq, \mathcal{J}_2 \circ \varphi) \stackrel{\widetilde{\tau}^*_V(\mathcal{L}_2)}{\longrightarrow} \nu_0. \tag{7.25}$$

Then, by (7.27) and continuity of $\widetilde{\Pi}_2$ we obtain that [3, (2.5.4)]

$$\left(D, \sqsubseteq, \widetilde{\Pi}_2 \circ \mathcal{J}_2 \circ \varphi\right) \stackrel{\tau^{(l)}_{\mathbb{R}}}{\longrightarrow} \widetilde{\Pi}(\nu_0).$$

As a corollary, $\left(D, \sqsubseteq, \widetilde{\Pi}_2 \circ \mathcal{J}_2 \circ \varphi\right) \stackrel{\tau^{(l)}_{\mathbb{R}}}{\longrightarrow} z_0$. In addition, by (5.29) $z_0 \in \overline{\mathfrak{V}}$. Moreover, by (4.36) $\widetilde{\Pi}_2 \circ\ \mathcal{J}_2 \circ\ \varphi = \Pi_2 \circ \varphi$ and, as a corollary,

$$(D, \sqsubseteq, \Pi_2 \circ \varphi) \stackrel{\tau^{(l)}_{\mathbb{R}}}{\longrightarrow} z_0. \tag{7.26}$$

By (5.29) and (7.19) $(D, \sqsubseteq, \Pi_2 \circ \varphi)$ is a net in $\overline{\mathfrak{V}}$. Then, by definition of $\mathbf{t}_2$ we obtain from (7.26) the following convergence

$$(D, \sqsubseteq, \Pi_2 \circ \varphi) \stackrel{\mathbf{t}_2}{\longrightarrow} z_0. \tag{7.27}$$

Using (5.28) and [3, (2.5.4)], we obtain that

$$(D, \sqsubseteq, \Psi \circ \Pi_2 \circ \varphi) \stackrel{\tau_{\mathbb{R}}}{\longrightarrow} \Psi(z_0). \tag{7.28}$$

From (7.23) and (7.28), the following convergence is realized:

$$(D, \sqsubseteq, \Psi \circ \Pi_2 \circ \varphi) \xrightarrow{\tau_{\mathbb{R}}} \mathbf{V}. \tag{7.29}$$

Let $\zeta \in]0, \infty[$. Then by (7.29), for some $\mathbf{d} \in D$, we obtain that $\forall\, d \in D$

$$(\mathbf{d} \sqsubseteq d) \Rightarrow (\,|(\Psi \circ \Pi_2 \circ \varphi)(d) - \mathbf{V}| < \zeta).$$

In particular, we have the inequality

$$\mathbf{V} - \zeta < (\Psi \circ \Pi_2 \circ \varphi)(\mathbf{d}) = \Psi(\Pi_2(\varphi(\mathbf{d}))),$$

where $\varphi(\mathbf{d}) \in V^{(\partial)}$. Therefore, the inequality chain

$$\mathbf{V} - \zeta < \Psi(\Pi_2(\varphi(\mathbf{d}))) \leqslant \sup_{v \in V^{(\partial)}} \Psi(\Pi_2(v)) \tag{7.30}$$

is valid. Since the choice of ζ was arbitrary, from (7.30) we obtain that

$$\mathbf{V} \leqslant \sup_{v \in V^{(\partial)}} \Psi(\Pi_2(v)). \tag{7.31}$$

From (7.22) and (7.31), the equality (7.21) follows. □

We recall that by (4.29) and (7.18)

$$\Psi(\Pi_2(v)) = \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0\left(\widetilde{\Pi}_1(\mu), \Pi_2(v)\right) \;\; \forall\, v \in V^{(\partial)}. \tag{7.32}$$

From Proposition 7.2 and (7.32) we have in the considered case the equality

$$\begin{aligned} \mathbf{V} &= \sup_{v \in V^{(\partial)}} \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0\left(\widetilde{\Pi}_1(\mu), \Pi_2(v)\right) \\ &= \sup_{v \in V^{(\partial)}} \min_{\mu \in \widetilde{U}_\partial} \mathbf{f}_0\left(\left(\int_{I_1} \alpha_i \, d\mu\right)_{i \in \overline{1,k}}, \left(\int_{I_2} \beta_j v \, d\eta_2\right)_{j \in \overline{1,l}}\right). \end{aligned} \tag{7.33}$$

Recall that (7.33) was established under the following suppositions: (6.19), (6.20), and (7.17).

References

[1] A.G. Chentsov, *Finitely Additive Measures and Relaxations of Extremal Problems*, Plenum Publishing Corporation, New York-London-Moscow (1996).

[2] A.G. Chentsov, *Asymptotic Attainability*, Kluwer Academic Publishers, Dordrecht-Boston-London (1997).

[3] A.G. Chentsov, S.I. Morina, *Extensions and Relaxation*, Kluwer Academic Publishers, Dordrecht-Boston-London (2002).

[4] A.G. Chentsov, To the question about construction of the correct extensions in the class of finitely additive measures, *Izv. Vuzov. Mathematics*, No. 2 (2002), 58-80.

[5] A.G. Chentsov, Finitely additive measures and extensions of abstract control problems, *Journal of Mathematical Sciences*, **133**, No. 2 (Febrary 2006), 1045-1206; *Contemporary Mathematics and its Applications*, **17**, Optimal Controls (2004).

[6] A.G. Chentsov, On the question of correct extension of a problem on the choice of the probability under restrictions on a system of mathematical expectation, *Usp. Mat. Nauk*, **50**, No. 5, 232-242.

[7] A.G. Chentsov, Ju.V. Shapar, About one the game problem with the approximate realization to constraints, *Dokladi Akademii Nauk*, **427**, No. 2 (2009), 170-175.

[8] N. Dunford, J.M. Schwartz, *Linear Operators*, Volume 1, Interscience, New York (1958).

[9] R. Engelking, *General Topology*, PWN, Warszawa (1977).

[10] R.V. Gamkrelidze, *Foundations of Optimal Control Theory*, Izdat. Tbil. Univ., Tbilissi (1977).

[11] J.L. Kelley, *General Topology*, Van Nostrand, Prinston, NJ (1957).

[12] N.N. Krasovskii, *The Theory of the Control of Motion*, Nauka, Moscow (1968).

[13] N.N. Krasovskii, *Dynamic System Control. Problem of the Mimimum of Guaranteed Result*, Nauka, Moscow (1985).

[14] N.N. Krasovskii, A.I. Subbotin, *Game-Theoretical Control Problems*, Springer-Verlag, Berlin (1988).

[15] J. Neveu, *Bases Mathematiques du Calcul des Probabilities*, Masson, Paris (1964).

[16] A.I. Subbotin, A.G. Chentsov, *Otimization of Guarantees in Control Problems*, Nauka, Moscow (1981).

[17] J. Warga, *Optimal Control of Differential and Functional Equations*, Acad. Press, New York (1972).