International Electronic Journal of Pure and Applied Mathematics

Volume 1 No. 4 2010, 359-364

SOME RESULTS ON GRACEFUL GRAPHS

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Abstract: A tree T with n vertices and a perfect matching M is strongly graceful if T admits a graceful labeling f such that $f(u)+f(v) = n-1$ for every edge $uv \in M$. Broersma and Hoede conjectured that every tree containing a perfect matching is strongly graceful in 1999. We prove that (A)-tree with a defect strongly tree results to a strongly graceful tree. Also, one type of olive tree is graceful and K_5 is not graceful by definition of (5;2,3)-free set.

AMS Subject Classification: 26A33
Key Words: strongly graceful, graceful labeling, sum-free set

1. Introduction

All graphs mentioned in this paper are simple, undirected and finite. The undefined terminologies will follow [2]. For the sake of simplicity, the shorthand $[m, n]$ stands for the set $\{m, m+1, ..., n\}$, where m and n are non-negative integers with $m \leq n$.

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in 1960s. Graceful labelings is very important part of labeling problem which has been introduced firstly by Rose.

Conjecture 1. (see [6]) *Each tree is graceful.*

Definition 1. (see [5]) A graceful labeling of a simple graph G with q edges is assignment of distinct labels from $\{1, 2, ...q\}$ to vertices of G, where edges are label from $\{1, 2, ..., q\}$ is used exactly once as an edge label.

Received: May 18, 2010

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Among the trees known to be graceful are: caterpillar (caterpillar is a tree with the property that the removal of endpoints leaves a path); tree with at more 4 end-vertices; tree with at most 27 vertices; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) and so on. Although the tremendous work of many literatures, GTC which has called the effort to prove it a "disease" by Kotzig is still open up now. In this paper we mainly discuss the gracefulness of graphs.

Definition 2. (see [7]) Let T be a tree with n vertices and a perfect matching M. T is strongly graceful if T admits a graceful labeling if such that $f(u)+f(v)=n-1$ for every edge $uv \in M$. We define a bipartite graceful tree T if it admits a graceful labeling f such that $f(u) < f(v)$ for all $u \in V_1$ and $v \in V_2$, where $V(T) = V_1 \cup V_2$, and both V_1, V_2 are independent.

Conjecture 2. (see Broersma and Hoede [4]) *Every tree containing a perfect matching is strongly graceful.*

Definition 3. (see [3]) A bipartite labeling of a tree T on p vertices is a bipartition $f : V \longrightarrow [0, n-1]$ for which there exists a positive number k such that whenever$f(u) \leq k \leq f(v)$, then u and v have different colors.

2. Lemmas and Theorems

Let w be a vertex of a tree T. A defect matching M^* of T is a matching that saturates each vertex of $V(T)\backslash\{w\}$. We say that T is defect strongly graceful if T has a graceful labeling h such that each $h(x)+h(y) = |T|-1$ for each edge xy of the defect matching M^*. Also, h is called its own neighbor set $N(u) = \{v, u_1, u_2, ..., u_{d_T(u)-1}\}$, where every u_i is a leaf of T for $1 \leq i \leq d_T(u)-1$, and the degree $d_T(v) \geq 2$.

Lemma 1. (see [7]) *Let T be a tree with a perfect matching M and n vertices. Suppose that f is a strongly labeling of T. Then*

(i) The maximum degree $\Delta \leq n/2$; each end-node of T has degree 2; and $|S| = |U|$, where S and U are independent sets of T such that $V(T) = S \cup U$.

(ii) For $uv \in M$ there are $f(u) = n-1-k$ and $f(v) = k$ for $1 \leq k \leq n/2$.

(iii) There is a path $P = uvxy$ in T such that $uv, xy \in M$ and $f(u) = 0$, $f(v) = n-1$ and $f(x) = 1$ and $f(y) = n-2$, or $f(u) = n-1$, $f(v) = 0$, $f(x) = n-2$ and $f(y) = 1$.

(iv) There are no two different perfect matching in T.

Lemma 2. (see [7]) *Let T be a tree with a perfect matching M and n vertices. Then T is strongly graceful if it satisfies one of the following conditions:*

(i) The diameter $D(T) \leq 5$, and

(ii) The maximum degree $\Delta \geq n/2$.

Lemma 3. (see [7], String Linking Lemma) *Let T be a bipartite graceful tree and let H be a defect strongly graceful tree with a defect matching M(H). There exists vertex $u \in T$ such that identifying u with $v \in V(H)$ results a strongly graceful tree.*

Theorem 4. *Let T be a bipartite tree and let H be a defect strongly graceful tree with a defect matching $M(H)$. There exist a vertex $u \in T$ such that identified u and $v \in V(H)$ results a strongly graceful tree.*

Proof. For the sake of simplicity, we pick up a (A)-tree from [7] in the following:

A (A)-tree is a bipartite graceful tree T with n vertices and $V(T) = U \cup V$, where $U \cap V = \emptyset$ and any edge xy holds $x \in U$ and $y \in V$. Let $U = u_1, u_2, ..., u_s$ and $V = v_1, v_2, ..., v_t$. Then T admits a graceful labeling f such that $f(u_i) = i - 1$ for $u_i \in U$ and $1 \leq i \leq s$, and $f(v_j) = s + j - 1$ for $v_j \in V$ and $1 \leq j \leq t$. Here we are ready to prove.

Let H be a tree with m vertices that admits a defect strongly graceful labeling with a defect matching $M^*(H)$. Assume that $h(w) = |H| - 1$. We construct G^* by identifying v_1 and w into one which replaced by w^*, and make a labeling g as $g(u_i) = f(u_i)$ for $1 \leq i \leq s$ and $g(v_j) = f(v_j)+|H|-1$ for $2 \leq j \leq t$, $g(w^*) = f(v_1)+|H|-1$; as a result, $\{|g(x)-g(y)| : xy \in V(T) \subset E(G^*)\} = [|H|+|T|-1]$ and $g(x) = h(x)+s$, it contributes $\{|g(x) - g(y)| : xy \in E(H) \subset E(G^*)\} = [1, |H| - 1]$. Obviously, there has a perfect matching $M(G^*) = M(T) \cup M(T)$, $f(u) + f(v) = |G^*| - 1$ for each $uv \in G^*$. Therefore, g is exactly a strongly graceful labeling of G^*. □

Theorem 5. *A tree T with n vertices and a perfect matching M admits a strongly graceful labeling f. For each $uv \in E(T)\backslash M$, both integers $f(x) + f(y)$ and n have same obevity.*

Proof. There is perfect matching M in tree T with n vertices, which means $|T| = n$ is even. Furthermore, $f(x) + f(y) = n - 1$ for each $xy \in M$, $|f(x) - f(y)|$ is even too. Every element of the set $\{|f(x) + f(y)| : xy \in M\}$ is odds. Obviously, by observing the definition of strongly gracefulness, every element of $\{f(x) - f(y) : xy \in E(T)\backslash M\}$ is even. □

3. Contributing One Type of Olive Tree

Let T be a tree with n vertices and perfect matching M. Since $\Delta \leq n/2$ and each end-node of T has degree 2, we can make path seed [7] in the following:

Path seeds contain two classes of path:

(1) P_{2m} is a path with $2m$ vertices depicted by $P_{2m} = u_1u_2...u_m$. Thus, $V(p_{2m}) = S \cup U$, where $S = \{u_{2i-1} : 1 \leq i \leq \ m\}$ and $S = \{u_(2i) : 1 \leq i \leq m\}$. The path P_{2m} contains a perfect matching $M = \{u_{2i-1}u_{2i} : 1 \leq i \leq m\}$, and has a strongly graceful labeling defined in this way: $f(u_{2i-1}) = i - 1$ and $f(u_{2i}) = 2m - i$ for $1 \leq i \leq m$.

(2) P_{2m-1} stands for a path $P_{2m+1} = u_1u_2...u_{2m+1}$ on $2m+1$ vertices. It contains a defect matching $M = \{u_{2i-1}u_{2i} : 1 \le i \le m\}$, and admits a defect graceful labeling h defined by setting

$$h(u_{2i-1}) = i - 1 \text{ for } 1 \le i \le m \;\text{ and }\; f(u_{2j}) = 2m + 1 - j \text{ for } 1 \le i \le m.$$

We just defined a type of olive tree is made of path seeds P_{2^n-1}. We choose one vertex which has degree 1 from every distinct path and then identified these vertices into one vertex w. Then we obtain a type of olive tree, denoted by $P^*_{2^n}$.

Theorem 6. *$P^*_{2^n}$ is strongly graceful.*

Proof. Firstly, we use notation $h(P_{2^i}) = \{x|x \in \mathbf{N}\}$ to denote the labelings of every path. The order of labels to vertices admits the distance to w to be changed from large to small. Then define the labeling h for $P^*_{2^n}$ in the following:

$$h(w) = 2^{n-1}, \qquad h(P_{2^0}) = \{2^{n+1} - 1 - 2^{n-1}\},$$

$$h(P_{2^1}) = \{2^{n-1} + 2^0, 2^{n+1} - 2^{n-1} - 2\},$$

$$h(P_{2^2}) = \{2^{n-1} + 2, 2^{n+1} - 2^{n-1} - 2 - 1, 2^{n-1} + 2^2 - 1, 2^{n+1} - 2^{n-1} - 2^2\},$$

$$\begin{aligned} h(P_{2^3}) = \{&2^{n-1} + 2^2, 2^{n+1} - 2^{n-1} - 2^2 - 1, 2^{n-1} + 2^2 + 1, 2^{n+1} - 2^{n-1} - 2^2 - 2, \\ &2^{n-1} + 2^2 + 2, 2^{n+1} - 2^{n-1} - 2^2 - 3, 2^{n-1} + 2^2 + 3, 2^{n+1} - 2^{n-1} - 2^3\}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} h(P_{2^j}) = \{&2^{n-1}+2^{j-1}, 2^{n+1}-2^{n-1}-2^{j-1}-1, 2^{n-1}+2^{j-1}+1, 2^{n+1}-2^{n-1}-2^{j-1}-2, \\ &..., 2^{n-1} + 2^j - 1, 2^{n+1} - 2^{n-1} - 2^j\}, \end{aligned}$$

$$h(P_{2^n}) = \{0, 2^{n+1} - 1, 1, 2^{n+1} - 2, 2, 2^{n+1} - 3, ..., 2^{n-1} - 1, 2^{n+1} - 2^{n-1}\}.$$

Obviously there is a perfect matching in $P^*_{2^n}$, and $f(u) + f(v) = 2^n - 1$ for every $uv \in M$, $|f(u) - f(v)| \in \{2k - 1 : 1 \le k \le 2^{n-1}, k \in \mathbf{N}\}$, for each $uv \in E(P^*_{2^n})$, $|f(u) - f(v)| \in \{2k - 2 : 1 \le k \le 2^{n-1}\}$. Therefore f is a strongly graceful labeling of $P^*_{2^n}$. □

4. Another Proof of Complete Graph K_5 is not Graceful

In 1965 [1], Erdös has investigated the sum-free problem. A subset S of an Abelian group G is *sum-free* if $(S+S)\cap S=\emptyset$, i.e. if there are no $a,b,c\in S$ such that $a+b=c$. Bing Yao [8] generalized the sum-free problem. A set S is called a k-set if it contains k elements, also k is cardinality $|S|$. The largest integer and the smallest integer in S are denoted by $\max(S)$ and $\min(S)$.

Definition 4. (see [8]) For integer $k,\theta\geq 3$ and $\beta\geq 1$, a k-set S is $(k;\beta,\theta)$-free if it does not contain distinct elements a_{i_j} $(1\leq j\leq\theta)$ such that $\sum_{j=i_1}^{i_\theta-1}a_j=\beta a_{i_\theta}$.

By observing this and definition of gracefulness, we found there is a relationship between them when $\beta=2$, $\theta=3$. So it is also useful to proof whether complete graph K_n is graceful.

Theorem 7. *Complete graph K_5 is not graceful.*

Proof. According to the definition of K_5, we obtain $|K_5|=10$. So, if f is graceful labeling of K_5, each vertex of K_5 is assigned a number $f(u)\in[0,10]$ such that $f(u)\neq f(v)$ for $u\neq v$ in T, and the set of all edge labels is equal to $[1,10]$. Therefore, we must choose 0 and 10 as elements of vertex labels set, otherwise it contradicts with the definition of gracefulness. Furthermore, if the set we choose is not (5;2,3)-free, it contradicts with definition of gracefulness too. Thus, based on above two keys, we can only have 5-set in the following: $\{0,1,3,7,10\}$, $\{0,1,3,8,10\}$, $\{0,1,3,9,10\}$, $\{0,1,4,9,10\}$, $\{0,1,6,7,10\}$, $\{0,1,6,8,10\}$, $\{0,1,6,9,10\}$, $\{0,1,7,8,10\}$, $\{0,1,7,9,10\}$, $\{0,2,3,7,10\}$, $\{0,2,3,8,10\}$, $\{0,2,3,9,10\}$, $\{0,3,4,7,10\}$, $\{0,3,4,9,10\}$, $\{0,4,6,7,10\}$, $\{0,4,7,9,10\}$, $\{0,6,7,8,10\}$, $\{0,6,7,9,10\}$.

There is no one of these 16 (5;2,30)-free sets by checking out every one of those sets satisfied this conditions: labeling all elements of edge set is equal to [1,10]. Thus K_5 is not graceful. □

Question. What is the minimal number of vertices and edges of $K_n (n\geq 5)$ such that the resulting graph will be connected and graceful?

Acknowledgments

This research is supported by the project from Inner Mongolian University for Nationalities (MDX2009044). We wish to thank sincerely the referees for their valuable suggestions and improvements to the paper.

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