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# LIPSCHITZ STABILITY FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH "SUPREMUM"

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**Abstract:** This paper investigates Lipschitz stability for nonlinear impulsive differential equations with "supremum". Sufficient conditions for Lipschitz stability, uniformly Lipschitz stability, uniformly eventually Lipschitz stability are obtained. Piecewise Lyapunov like functions has been applied. Comparison result, required for the application of Razumikhin method, is proved.

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**Key Words:** Lipschitz stability, Lyapunov function, impulsive differential equations with "supremum"

## 1. Introduction

Differential equations are a basic yet powerful mathematical apparatus for studying real world objects and phenomena. These equations, when used as models and combined with information technology tools, allow us to conduct theoretical investigations and examinations and to predict the behavior of real systems. However, since real life processes are quite intricate, complex mathematical equations are required to study them. One natural setting is the case of evolutionary equations using past history, specially, when the evolutionary equations using the maximum of the studied function on a past time interval. For example, in the theory of automatic control in various technical systems often the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals. This requires the use of differential equations with "maxima" in the optimal control theory (see, for example, V. Plotnikov and O. Kichmarenko [17], [21]). Since the

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maximum function has very specific properties, it makes the equations strongly non-linear and requires independent study of various properties of differential equations with "maxima" (see [1], [2], [4], [6], [9], [16], [22]).

Differential equations with "maxima" have some very specific properties than known in the literature other types of equations. For example, let us consider the following two types of scalar equations:

(A) *Differential equation with delay:*

$$x' = \Big(x(\tau(t))\Big)^2,$$

where the function $\tau \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\tau(t) \leq t$, $\tau(t) \not\equiv t$;

(B) *Differential equation with "maxima":*

$$x' = \Big(\max_{s\in[t-h,t]} x(s)\Big)^2,$$

where $h$ is a positive constant.

Note that equation (A) seems to be very similar to (B), specially in the case of $\tau(t) \equiv t-h$. At the same time if we look more carefully to the considered equations, equation (B) is reduced to an ordinary differential equation and the initial condition is required only at one single point. At the same time no function $\tau(t)$ could reduce the equation (A) to an ordinary differential equation.

The above example demonstrates some differences between the well known in the literature differential equations with delay and the comparatively new differential equations with "maxima" and proves the necessity of independent deeply study of equations, containing maximum of the unknown function.

At the beginning of the 90's, D.D. Bainov and his collaborators S. Hristova and S. Milusheva combined the ideas of impulsive differential equations with differential equations with "maxima" and initiated the investigations of these type of equations (see [3], [13], [14], [20]). These equations are adequate models of real processes whose present states have instantaneous changes at certain moments and they depend significantly on the maximal value of the state on a past time interval.

One of the main problems in the qualitative theory of differential equations is stability of the solutions. Several types of stability have been investigated in the past, applying various methods such as first and second method of Lyapunov. One type of stability, very useful in real world problems, deals with so called Lipschitz stability. This stability for nonlinear ordinary differential equations was introduced and studied by F.M. Dannan, S. Elaydi (see [5]). As it is mentioned in this paper the uniform Lipschitz stability lies somewhere between uniform stability on one side and the notions of asymptotic stability in variation of Brauer and uniform stability in variation of Brauer and Strauss on the other side. Furthermore, uniform Lipschitz stability neither implies asymptotic stability nor it is implied by it. Lipschitz stability is also studied for impulsive differential equations in [18], [19], [23], [24], [25].

In the present paper, we study Lipschitz stability of impulsive differential equations with "supremum". The method of Lyapunov functions combined by Razumikhin metohd is applied. A comparison result is proved and it is used as a base of the main results.

Note that several types of stability properties for impulsive differential equations with "supremum" as well as boundedness are studied recently in [7]-[12].

## 2. Preliminary Notes and Definitions

Let $\{\tau_k\}_1^\infty$ be a sequence of fixed points in $\mathbb{R}$ such that $\tau_{k+1} > \tau_k$ and $\lim_{k\to\infty} \tau_k = \infty$. Let $r > 0$ be a fixed constant.

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space with norm $||x||$, $\Omega$ be a bounded domain in $\mathbb{R}^n$ containing origin and $\mathbb{R}_+ = [0, \infty)$.

Consider the system of nonlinear impulsive differential equations with "supremum"

$$x' = f(t, x(t), \sup_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \geq t_0, \quad t \neq \tau_k \tag{1}$$

$$x(\tau_k + 0) = I_k(x(\tau_k - 0)) \quad \text{for} \quad k = 1, 2, \dots, \tag{2}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, 3, \dots$.

Note that for $x : \ [t-r, t] \to \mathbb{R}^n, \ x = (x_1, x_2, \dots, x_n)$ we denote

$$\sup_{s\in[t-r,t]} x(s) = \Big( \sup_{s\in[t-r,t]} x_1(s), \sup_{s\in[t-r,t]} x_2(s), \dots, \sup_{s\in[t-r,t]} x_n(s) \Big).$$

Denote by $PC(X, Y)$ $(X \subset \mathbb{R}, Y \subset \mathbb{R}^n)$ the set of all functions $u : X \to Y$ which are piecewise continuous in $X$ with points of discontinuity of the first kind at the points $\tau_k \in X$ and which are continuous from the left at the points $\tau_k \in X$, $u(\tau_k) = u(\tau_k - 0)$.

We denote by $PC^1(X, Y)$ the set of all functions $u \in PC(X, Y)$ which are continuously differentiable for $t \in X, \quad t \neq \tau_k$ .

Let $X \subset \mathbb{R}$. Denote by $Z(X)$ the set of all integers $k$ such that $\tau_k \in X$.

Let $t_0 \in \mathbb{R}_+$ be a fixed point and $\phi \in PC([t_0-r, t_0], \mathbb{R}^n)$. We denote by $x(t; t_0, \phi)$ the solution of system (1), (2) with initial conditions

$$x(t; t_0, \phi) = \phi(t), \quad t \in [t_0 - r, t_0], \qquad x(t_0 + 0; t_0, \phi) = \phi(t_0). \tag{3}$$

Introduce the following notations

$$G_k = \{(t, x) \in [-r, \infty) \times \mathbb{R}^n : \ t \in (\tau_{k-1}, \tau_k)\}, \quad k = 1, 2, \dots,$$

$$\mathcal{G} = \bigcup_{k=1}^{\infty} G_k.$$

We will introduce the class $\Lambda$ of Lyapunov functions.

**Definition 1.** We will say that the function $V(t,x) : [-r,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to class $\Lambda$ if:

1. $V(t,x)$ is a continuous function in $\mathcal{G}$ and $V(t,0) \equiv 0$ for $t \in [-r,\infty)$;

2. For each $k = 1,2,\dots$ and $x \in \mathbb{R}^n$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{t\uparrow\tau_k} V(t,x), \qquad V(\tau_k + 0, x) = \lim_{t\downarrow\tau_k} V(t,x)$$

and

$$V(\tau_k, x) = V(\tau_k - 0, x) = \lim_{t\uparrow\tau_k} V(t,x);$$

3. Function $V(t,x)$ is Lipschitz with respect to its second argument in each of the sets $G_k$, $k = 1,2,\dots$.

Let $V \in \Lambda$. For any $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k = 1,2,\dots$ and any function $\psi \in PC([t-r,t],\mathbb{R}^n)$ we will define a derivative of the function $V$ along a trajectory of the solution of (1), (2) as follows:

$$D_{(1),(2)}V(t,\psi) = \lim_{\epsilon\to 0}\sup \frac{1}{\epsilon}\Big[V\Big(t, \psi(t)+\epsilon f(t,\psi(t), \max_{s\in[-r,0]} \psi(t+s))\Big) - V(t,\psi(t))\Big]. \quad (4)$$

**Remark 1.** Note that the derivative defined by (4) is a functional.

Let $PC([t_0-r,t_0],\mathbb{R}^n)$. We will use the following notation:

$$||\varphi(t_0)||_r \;=\; \sup_{t\in[-r,0]} ||\varphi(t_0+t)||.$$

Let $\rho > 0$ be a given number. Consider the following sets:

$K = \{a \in C(\mathbb{R}_+,\mathbb{R}_+) :\ a(r) \text{ is strictly increasing and } a(0) = 0\}$;
$\mathcal{K} = \{a \in C(\mathbb{R}_+,\mathbb{R}_+) :\ a(r) \text{ is strictly increasing and } a(s) \geq s,\ a(0) = 0\}$;
$S_\rho = \{x \in \mathbb{R}^n : ||x|| < \rho\}$.

**Definition 2.** The zero solution of the system of impulsive differential equations with "supremum" (1), (2) is said to be:

1) *uniformly eventually stable*, if for $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ and $\tau(\epsilon) > 0$ such that for any initial point $t_0 \geq \tau(\epsilon)$ and any initial function $\varphi \in PC([t_0 - r, t_0],\mathbb{R}^n)$ such that $||\varphi(t_0)||_r \leq \delta$ the inequality

$$||x(t;t_0,\varphi)|| \leq \epsilon \qquad t \geq t_0$$

holds.

2) *Lipschitz stable*, if for any initial point $t_0 \in R_+$ there exist $M > 0$ and $\delta = \delta(t_0) > 0$ such that for any initial function $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that $||\varphi(t_0)||_r \leq \delta$ the inequality

$$||x(t; t_0, \varphi)|| \leq M||\varphi(t_0)||_r, \qquad t \geq t_0$$

holds.

3) *uniformly Lipschitz stable*, if there exist constants $M > 0$ and $\delta > 0$ such that for any initial point $t_0 \in R_+$ and for any initial function $\varphi \in PC([t_0-r, t_0], \mathbb{R}^n)$ such that $||\varphi(t_0)||_r \leq \delta$ the inequality

$$||x(t; t_0, \varphi)|| \leq M||\varphi(t_0)||_r, \qquad t \geq t_0$$

holds.

4) *uniformly eventually Lipshitz stable*, if for $\epsilon > 0$ there exist $M > 0$, $\delta(\epsilon) > 0$ and $\tau(\epsilon) > 0$ such that for any initial point $t_0 \geq \tau(\epsilon)$ and any initial function $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ such that $||\varphi(t_0)||_r \leq \delta$ the inequality

$$||x(t; t_0, \varphi)|| \leq M||\varphi(t_0)||_r, \qquad t \geq t_0$$

holds.

**Lemma 1.** *If the zero solution of impulsive differential equations with "supremum" (1), (2) is uniformly eventually Lipshitz stable, then the zero soltion of (1), (2) is uniformly eventually stable.*

*Proof.* Let $\epsilon > 0$ there exist $M > 0$, $\delta_1 = \delta_1(\epsilon) > 0$ and $\tau(\epsilon) > 0$. Choose $\delta = \min\left(\delta_1, \frac{\epsilon}{M}\right)$. Then from the uniformly eventually Lipshitz stability we get for $||\varphi(t_0)||_r \leq \delta \leq \delta_1$ the validity of $||x(t; t_0, \varphi)|| \leq M||\varphi(t_0)||_r \leq \epsilon$. ☐

In our further investigations we will use the initial value problem for the comparison scalar ordinary differential equation

$$\begin{aligned} &u' = g(t, u), \quad t \neq \tau_k, \\ &u(\tau_k + 0) = \xi_k(u(\tau_k)), \quad k = 1, 2, \dots, \\ &u(t_0) = u_0, \end{aligned} \tag{5}$$

where $u, u_0 \in \mathbb{R}$, $g \in C[\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}]$, $t_0 \in \mathbb{R}_+$.

In the further investigations we will use the following comparison result:

**Lemma 2.** *Let the following conditions be fulfilled:*

1. *The functions $f \in PC([t_0, T] \times \Omega \times \Omega, \mathbb{R}^n)$ and $I_k \in C(\Omega, \Omega)$ for $k \in Z([t_0, T))$, where $\Omega \subset \mathbb{R}^n$, and $t_0, T: \ 0 \leq t_0 < T < \infty$ are constants.*

2. *The function* $\varphi \in PC([t_0 - r, t_0], \Omega)$.

3. *The initial value problem (1), (2), (3) has a solution* $x(t) = x(t; t_0, \varphi)$, *such that* $x(t) \in \Omega$ *on* $[t_0 - r, T]$.

4. *The functions* $g \in PC([t_0, T] \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t, 0) \equiv 0$ *for* $t \in [t_0, T]$ *and* $\xi_k \in \mathcal{K}$, $k \in Z((t_0, T))$.

5. *For any initial point* $u_0 \in \mathbb{R}_+$ *the initial value problem for the scalar impulsive differential equation (5) has a maximal solution* $u^*(t) = u^*(t; t_0, u_0)$, *which is defined for* $t \in [t_0, T]$.

6. *The function* $V : [t_0 - r, T] \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$ *is such that:*

   *(i) for any number* $t \in [t_0, T]: \ t \neq \tau_k$ *and any function* $\psi \in PC([t - r, t], \Omega)$ *such that* $V(t, \psi(t)) \geq V(t + s, \psi(t + s))$ *for* $s \in [-r, 0)$ *the inequality*

$$D_{(1),(2)} V(t, \psi(t)) \leq g(t, V(t, \psi(t)))$$

   *holds.*

   *(ii)* $V(\tau_k + 0, I_k(x)) \leq \xi_k(V(\tau_k, x)), \quad k \in Z((t_0, T)), x \in \Omega.$

*Then the inequality* $\sup_{s \in [-r,0]} V(t_0 + s, \varphi(t_0 + s)) \leq u_0$ *implies the inequality* $V(t, x(t)) \leq u^*(t)$ *for* $t \in [t_0, T]$.

*Proof.* Let the inequality $\sup_{s \in [-r,0]} V(t_0 + s, \varphi(t_0 + s)) \leq u_0$ holds.

For any natural number $n$ consider the initial value problem for the scalar impulsive differential equation

$$\begin{aligned} & u' = g(t, u) + \frac{1}{n}, \quad t \in [t_0, T], \ t \neq \tau_k, \\ & u(\tau_k + 0) = \xi_k(u(\tau_k)) + \frac{1}{n}, \quad k \in Z((t_0, T)), \\ & u(t_0) = u_0 + \frac{1}{n}. \end{aligned} \tag{6}$$

Denote by $v_n(t)$ the maximal solution of the initial value problem (6), which is defined for $t \in [t_0, T]$.

From $g(t, u) + \frac{1}{n} > 0$ on $[t_0, T] \times \mathbb{R}_+$ and $\xi_k(u) + \frac{1}{n} > u$ for $k \in Z((t_0, T))$ it follows function $v_n(t)$ is increasing in $[t_0, T]$.

Define a function $m(t) \in PC([t_0 - r, T], \mathbb{R}_+) : \quad m(t) = V(t, x(t))$.

Because of the fact that $u^*(t; t_0, u_0) = \lim_{n \to \infty} v_n(t)$ it is enough to prove that for any natural number $n$ the inequality

$$m(t) \leq v_n(t) \quad \text{for} \ \ t \in [t_0, T] \tag{7}$$

holds.

Note that for any natural number $n$ inequality $m(t_0) \leq \sup_{s\in[-r,0]} V(t_0+s, \varphi(t_0+s)) \leq u_0 < v_n(t_0)$ holds.

Assume inequality (7) is not true. There exists a natural number $n$ and a point $t_1 \in (t_0, T)$: $m(t_1) > v_n(t_1)$. For that natural number $n$ consider the point

$$t^* = \sup\{t \in [t_0, T] : m(s) < v_n(s) \text{ for } s \in [t_0, t)\}.$$

Note that $t^* > t_0$ and $t^* < T$.

Consider following two cases:

*Case 1.* Let $t^* \neq \tau_k,\ k \in Z([t_0, T))$. Therefore

$$m(t^*) = v_n(t^*), \quad m(t) < v_n(t) \text{ for } t \in [t_0, t^*). \tag{8}$$

From inequality (8) it follows that

$$D_- m(t^*) \geq v_n'(t^*) = g(t^*, v_n(t^*)) + \frac{1}{n} = g(t^*, m(t^*)) + \frac{1}{n} > g(t^*, m(t^*)). \tag{9}$$

*Case 1.1.* Let $t^* - r \geq t_0$. From monotonicity of function $v_n(t)$ on $[t_0, T]$ it follows that $m(t^*) = v_n(t^*) > v_n(s) > m(s)$ for $s \in [t^* - r, t^*)$.

*Case 1.2.* Let $t^* - r < t_0$. As in Case 1.1 we obtain $m(t^*) \geq m(s)$ for $s \in [t_0, t^*)$.

Let $s \in [t^* - r, t_0)$. Then from the monotonicity of the function $v_n(t)$ we get $m(t^*) = v_n(t^*) \geq v_n(t_0 + 0) = u_0 + \frac{1}{n} > u_0 \geq \sup_{s\in[-r,0]} V(t_0 + s, \phi(t_0 + s)) \geq m(s)$.

Therefore, $m(t^*) \geq m(s)$ for $s \in [t^* - r, t^*)$.

From condition 6($i$) it follows $D_- m(t^*) \leq g(t^*, m(t^*))$ which contradicts (9).

Therefore the inequality (7) is true.

*Case 2.* Let there exists a natural number $k \in Z((t_0, T))$ such that $t^* = \tau_k$.

*Case 2.1.* Let $m(t^*) \leq v_n(t^*)$ and $m(t^* + 0) > v_n(t^* + 0)$.

From the jump conditions for the solution $v_n(t)$ and condition 6(ii) we obtain

$$\xi_k(v_n(\tau_k)) \geq \xi_k(m(\tau_k)) = m(\tau_k + 0) > v_n(\tau_k + 0) = \xi_k(v_n(\tau_k)) + \frac{1}{n} > \xi_k(v_n(\tau_k)).$$

The above contradiction proves the validity of (7).

*Case 2.2.* Let $m(t^*) < v_n(t^*)$, $m(t^* + 0) = v_n(t^* + 0)$ and $m(t) > v_n(t)$ for $t \in (\tau_k, \tau_k + \delta]$, where $\delta$ is sufficiently small number.

Then

$$\xi_k(v_n(\tau_k)) > \xi_k(m(\tau_k)) = m(\tau_k + 0) = v_n(\tau_k + 0) = \xi_k(v_n(\tau_k)) + \frac{1}{n} > \xi_k(v_n(\tau_k)).$$

The above contradiction proves the validity of (7).

From inequality (7) follows that $m(t) \leq u(t)$ for $t \in [t_0, T]$. □

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## 3. Main Results

We will study the defined above types of Lipschitz stability for nonlinear impulsive differential equations with "supremum". Initially we will use Razumikhin method and piecewise continuous Lyapunov functions.

**Theorem 1.** *Let the following conditions be fulfilled:*

1. *Function* $f \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0,0) \equiv 0$.
2. *The functions* $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $I_k(0) = 0$, *and for* $k = 1, 2, \dots$ *there exist increasing functions* $\xi_k \in C([0,\rho_1),[0,\rho_2))$: $\xi(s) \geq s$ *for* $s \in [0, \rho_1)$, $\xi_k(0) = 0$ *and* $||I_k(x)|| \leq \xi_k(||x||)$ *for* $x \in S_{\rho_1}$, *where* $\rho_1, \rho_2 > 0$ *are constants.*
3. *There exists a number* $\rho_3 > 0$ *and a function* $V : S_{\rho_3} \to \mathbb{R}_+$, $V \in \Lambda$ *such that:*

   *(i) for any number* $t \in \mathbb{R}_+ : \ t \neq \tau_k, \ k = 1, 2, \dots$ *and any function* $\psi \in PC([t-r,t], S_{\rho_3})$ *such that* $V(\psi(t)) \geq V(\psi(t+s))$ *for* $s \in [-r, 0)$ *the inequality*

   $$D_{(1),(2)}V(\psi(t)) \leq g(t, V(\psi(t)))$$

   *holds;*

   *(ii)* $V(I_k(x)) \leq \xi_k(V(x))$, *for* $x \in S_{\rho_3}$, $k = 1, 2 \dots$;

   *(iii) there exist constants* $C_1, C_2 > 0$ *such that* $C_1||x|| \leq V(x) \leq C_2||x||$ *for* $x \in S_{\rho_3}$.
4. *For any* $t_0 \in \mathbb{R}_+$ *and* $\varphi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ *the initial value problem (1), (2), (3) has a solution* $x(t) = x(t; t_0, \varphi)$ *defined for* $t \geq t_0$.
5. *For any initial point* $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ *the initial value problem for the scalar impulsive differential equation (5) has a solution, defined for* $t \geq t_0$.

*Then:*

(A) *If the zero solution of (5) is uniformly Lipschitz stable, then the zero solution of (1), (2) is uniformly Lipschitz stable;*

(B) *If the zero solution of (5) is uniformly eventually stable, then the zero solution of (1), (2) is uniformly eventually stable;*

(C) *If the zero solution of (5) is uniformly eventually Lipshitz stable, then the zero solution of (1), (2) is uniformly eventually Lipshitz stable.*

*Proof.* The proof of all three claims (A), (B), (C) are similar and we will give only the proof of (A).

Let the zero solution of (5) is uniformly Lipschitz stable. Then there exist $M > 0$ and $\delta_1 > 0$ such that for any $u_0 \in \mathbb{R}$: $|u_0| < \delta_1$ the following inequality

$$|u(t; t_0, u_0)| \leq M|u_0|, \quad t \geq t_0, \tag{10}$$

holds where $u(t; t_0, u_0)$ is a solution of the initial value problem (5).

Since $V(0) = 0$ there exists a number $\delta_3 > 0$ such that for $||x|| < \delta_3$ the inequality $V(x) < \delta_1$ holds.

Let $\rho = \min(\rho_1, \rho_2, \rho_3)$.

Choose $M_1 > 1$ and $\delta_2 > 0$:

$$M_1 > \frac{MC_2}{C_1}, \qquad M_1\delta_2 < \rho. \tag{11}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Choose the initial function $\varphi \in C([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$||\varphi(t_0)||_r < \delta. \tag{12}$$

From (11) and (12) we get the following inequalities

$$||\varphi(t_0 + s)|| \leq ||\varphi(t_0)||_r < \delta \leq \delta_2 < \frac{\rho}{M_1} < \rho, \quad s \in [-r, 0]. \tag{13}$$

According to (13) we obtain $\varphi(t_0 + s) \in S_\rho$ for $s \in [-r, 0]$.

We will prove that if inequality (12) is satisfied then

$$||x(t; t_0, \varphi)|| \leq M_1||\varphi(t_0)||_r, \quad t \geq t_0 - r, \tag{14}$$

where $x(t) = x(t; t_0, \varphi)$ is the solution of the initial value problem (1), (2) with the chosen above initial function $\varphi$.

From $M_1 > 1$ it follows that $||\varphi(t)|| \leq ||\varphi(t_0)||_r < M_1||\varphi(t_0)||_r$ for $t \in [t_0 - r, t_0]$.

Assume that (14) is not true for $t > t_0$. Therefore there exists a point

$$t^* = \sup\{t > t_0 : ||x(s)|| \leq M_1||\varphi(t_0)||_r \quad \text{for } s \in [t_0, t]\} < \infty.$$

Consider the following two cases:

*Case 1.* Let $t^* \neq \tau_k, \quad k = 1, 2, \ldots$. Therefore

$$||x(t^*)|| = M_1||\varphi(t_0)||_r, \qquad ||x(t)|| < M_1||\varphi(t_0)||_r, \quad t \in [t_0 - r, t^*). \tag{15}$$

From inequalities (11), (12), (15), and according to the choice of $\delta$ follows

$$||x(t)|| \leq M_1||\varphi(t_0)||_r < M_1\delta \leq M_1\delta_2 < \rho, \quad t \in [t_0 - r, t^*], \tag{16}$$

i.e. $x(t) \in S_\rho$ on $[t_0 - r, t^*]$.

From conditions $3(i)$ and $3(ii)$ of Theorem 1 it follows the validity of the condition 6 of Lemma 2 for $\Omega = S_\rho$.

Let $u^*(t) = u^*(t; t_0, u_0)$ be the maximal solution of (5) with initial condition $u_0 = \sup_{s\in[t_0-r,t_0]} V(\varphi(s))$.

From inequality (16) it follows that $x(t) \in S_\rho$ for $t \in [t_0 - r, t^*]$. Apply Lemma 2 for $\Omega = S_\rho$, $T = t^*$ and obtain

$$V(x(t)) \le |u^*(t)|, \quad t \in [t_0, t^*]. \tag{17}$$

From the choice of $u_0$ and the constants $\delta$ and $\delta_3$ follows that $u_0 < \delta_1$.

From condition $3(iii)$ of Theorem 1, inequalities (17), (10) we get

$$\begin{aligned} M_1||\varphi(t_0)||_r &= ||x(t^*)|| \le \frac{1}{C_1} V(x(t^*)) \le \frac{1}{C_1}|u^*(t^*)| \le \frac{M}{C_1}|u_0| \\ &= \frac{M}{C_1} \sup_{s\in[t_0-r,t_0]} V(\varphi(s)) \le \frac{MC_2}{C_1}||\varphi(t_0)||_r < M_1||\varphi(t_0)||_r. \end{aligned}$$

The obtained contradiction proves the validity of inequality (14).

*Case 2.* Let there exists a natural number $k$ such that $t^* = \tau_k$.

Note the following two cases are possible:

*Case 2.1.* Let $||x(t)|| \le M_1||\varphi(t_0)||_r$ for $t \in [t_0 - r, t^*]$ and $||x(\tau_k + 0)|| > M_1||\varphi(t_0)||_r$.

*Case 2.2.* Let $||x(t)|| \le M_1||\varphi(t_0)||_r$ for $t \in [t_0 - r, t^*]$, $||x(\tau_k)|| < M_1||\varphi(t_0)||_r$, $||x(\tau_k+0)|| = M_1||\varphi(t_0)||_r$ and $||x(s)|| > M_1||\varphi(t_0)||_r$ for $s \in (\tau_k, \tau_k+\epsilon]$, where $\epsilon > 0$ is small enough number.

In both Case 2.1. and Case 2.2 the inclusion $x(t) \in S_\rho$ is valid for $t \in [t_0 - r, t^*]$ and as in Case 1 we prove that the conditions of Lemma 2 are satisfied for $\Omega = S_\rho$ and $t \in [t_0, t^*]$, we could apply it and obtain again a contradiction. □

In the case when the used Lyapunov like function depends on the time $t$ we could obtain sufficient conditions for Lipshitz stability.

**Theorem 2.** *Let the following conditions be fulfilled:*

1. *The conditions 1, 2, 4 and 5 of Theorem 1 are satisfied.*
2. *There exists a number $\rho_3 > 0$ and a function $V : [-r,\infty) \times S_{\rho_3} \to \mathbb{R}_+$, $V \in \Lambda$ such that:*

   *(i) for any number $t \in \mathbb{R}_+ :\ t \ne \tau_k,\ k = 1, 2, \dots$ and any function $\psi \in PC([t - r, t], S_{\rho_3})$ such that $V(t, \psi(t)) \ge V(t, \psi(t + s))$ for $s \in [-r, 0)$ the inequality*

   $$D_{(1),(2)}V(t, \psi(t)) \le g(t, V(\psi(t)))$$

*holds.*

*(ii)* $V(\tau_k, I_k(x)) \leq \xi_k(V(\tau_k, x))$, *for* $x \in S_{\rho_3}$, $k = 1, 2 \dots .$

*(iii) there exist constants* $C_1, C-2 > 0$ *such that* $C_1||x|| \leq V(t,x) \leq C_2||x||$ *for* $(t,x) \in [-r,\infty) \times S_{\rho_3}$.

3. *The zero solution of (5) is Lipschitz stable.*

*Then the zero solution of (1), (2) is Lipschitz stable.*

*Proof.* The proof is similar to the proof of claim (A) of Theorem 1, where $\delta_1$ and $\delta_3$ depends on the initial point $t_0$. □

As a partial case of Theorem 1 we obtain the following sufficient conditions:

**Theorem 3.** *Let the following conditions be fulfilled:*

1. *The conditions 1, 2, 4, and 5 of Theorem 1 are satisfied.*
2. *There exists a function* $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, $g(t,0) \equiv 0$ *and a constant* $\rho_3 > 0$ *such that for any function* $\psi \in PC([-r,\infty), \mathbb{R}^n)$*: if there exists* $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k = 1, 2, \dots$ *such that* $\psi(t) \in S_{\rho_3}$ *and* $||\psi(t+s)|| < ||\psi(t)||$, $s \in [-r, 0)$, *then the inequality*

$$||\psi(t) + \epsilon f(t, \psi(t), \max_{s\in[-r,0]} \psi(t+s)|| \leq ||\psi(t)|| + \epsilon g(t, ||\psi(t)||) + h(\epsilon)$$

*holds, where* $\lim_{\epsilon\to 0} \frac{h(\epsilon)}{\epsilon} = 0$.

*Then:*

(A) *If the zero solution of (5) is uniformly Lipschitz stable, then the zero solution of (1), (2) is uniformly Lipschitz stable;*

(B) *If the zero solution of (5) is uniformly eventually stable, then the zero solution of (1), (2) is uniformly eventually stable;*

(C) *If the zero solution of (5) is uniformly eventually Lipschitz stable, then the zero solution of (1), (2) is uniformly eventually Lipschitz stable.*

*Proof.* Consider the function $V(x) = ||x||$, $x \in \mathbb{R}^n$. Note that $V \in \Lambda$.

From condition 2 of Theorem 3 it follows the validity of condition $3(i)$ of Theorem 1 for the chosen above function $V$.

Let $t \in \mathbb{R}_+$, $t \neq \tau_k$, $k = 1, 2, \dots$. Let the function $\psi \in PC([t-r,t], S_{\rho_3})$ be such that $||\psi(t)|| > ||\psi(t+s)||$ for $s \in [-r, 0)$. Then according to condition 2 of Theorem 3 we get

$$D_{(1)}V(\psi(t)) = \limsup_{\epsilon\to 0} \frac{1}{\epsilon}\Big[V\Big(\psi(t) + \epsilon f(t, \psi(t), \max_{s\in[-r,0]} \psi(t+s))\Big) - V(\psi(t))\Big]$$

$$\begin{aligned} &= \limsup_{\epsilon\to 0} \frac{1}{\epsilon}\Big[||\psi(t)+\epsilon f(t,\psi(t),\max_{s\in[-r,0]}\psi(t+s))||-||\psi(t)||\Big] \\ &\le g(t,||\psi(t)||)+\limsup_{\epsilon\to 0}\frac{1}{\epsilon}h(\epsilon) \\ &= g(t,V(\psi(t))). \end{aligned}$$

Therefore the condition $3(ii)$ of Theorem 1 is satisfied.

We apply Theorem 1 and prove Theorem 3. □

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