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ON THE CYCLE INDICES OF FROBENIOUS GROUPS

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Abstract: There are several very useful formulas, which give the cycle indices of the binary operation of the sum, product, composition and power group of M and H in terms of cycle indices of M and H. One very useful binary operation on groups, which has not been exploited, is the semidirect product.

Suppose $G = M \rtimes H$, a semi direct product; the question is: how can we express the cycle index of G in terms of the cycle indices of M and H? This work partially answers this question by considering the cycle indices of some particularly semidirect product groups; namely – Frobenious groups.

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1. Introduction

A Frobenius group is a group G acting on a set X, transitively, in such a way that the stabilizer H of a point is nontrivial, but only the identity fixes two or more points. That means that $H \cap \left(xHx^{-1}\right) = \{1\}$, if $x \in G \setminus H$. Define $M^* = G \setminus \bigcup \left\{xHx^{-1} : x \in G\right\}$, the set of all elements in G having no fixed points. Then $M = M^* \cup \{1\}$ is a normal subgroup of G. Furthermore $G = M \rtimes H$.

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If M and H are permutation groups with cycle indices Z_M and Z_H respectively and if $*$ is some binary operation on permutation groups then a fundamental problem is the determination of a formula for Z_{M*H} in terms of Z_M and Z_H. To this end a number of results have already been obtained as discussed in [4], [5], [7].

Kamuti [11], gave a method for deriving the cycle index of Frobenious groups. In this work we give an alternative method of deriving the same and also express the cycle index of G in terms of cycle indices of M and H.

2. Preliminary Definitions and Results

Let X be a set and G be a group. We say that G **acts** on the left on X if for each $x \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that for each $x \in X$ and $g_1, g_2 \in G$:

(i) $(g_1 g_2)x = g_1(g_2 x)$.

(ii) For any $x \in X, 1x = x$ where 1 is the identity in G.

Similarly a group acts on a set on the right by writing g on the right.

If a finite group G acts on a set S with n elements, each $x \in G$ corresponds to a permutation σ of S, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of length 2, $\dots$, α_n cycles of length n, we say that σ and hence x has *cycle type* $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$

Theorem 2.1. (see [2]) *Let the cycle type of a permutation σ be $(j_1, j_2, \dots, j_n)$, then the cycle type of σ^k is $\left(j_{1/(1,k)}^{(1,k)}, j_{1/(2,k)}^{(2,k)}, \dots, j_{1/(n,k)}^{(n,k)}\right)$.*

Remark 2.2. If a finite group G acts on a set X, the permutation σ corresponding to $g \in G$ has cycles of lengths less than or equal to the order of g.

If a finite group G acts on a set X, $|X| = n$ and $g \in G$ has cycle type $(j_1, j_2, \dots, j_n)$, we define the *monomial* of g to be $\mathrm{mon}(g) = \prod_k t_k^{j_k}$, where $t_k,\ k = 1, 2, \dots, n$ are distinct commuting indeterminates. The *cycle index* of the action of G on X is the polynomial (say over the rational field $\mathbb{Q}$) in $t_1, t_2, \dots, t_n$ given by $Z(G) = |G|^{-1} \sum_{g \in G} mon(g)$. If G has conjugacy classes $K_1, K_2, \dots, K_m$ with $g_i \in K_i$, then $Z(G) = |G|^{-1} \sum_{i=1}^{m} |K_i| \,\mathrm{mon}(g)$.

An element $g \in G$ generates the group G and say g is a generator for G if $G =< g >$ i.e $G = \{g^n : n \in \mathbb{Z}\}$. Let G act on the set X and $x \in X$, then the *orbit* of x is given by $Orb_G(x) = \{gx \ : g \in G\}$. The action of a group G on the set X is said to be *transitive* if for each pair $x, y \in X$ there exists $g \in G$ such that $gx = y$;

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in other words if the action of G on X has only one orbit. The *stabilizer* of x in G is the set $\mathrm{Stab}_G(x) = \{g \in G\, gx = x\}$. The stabilizer forms a subgroup of G called the *isotropy* group of x in G denoted by G_x

Theorem 2.3. (see [13, p. 76]) *Let G be a finite group acting transitively on a set X. Let $x \in X$ and $H = Stab_G(x)$. Then the action of G on X is equivalent to the action by multiplication on the set of cosets of H in G.*

If G is a group with subgroups H and K, then G is said to be the semidirect product of K by H denoted by $G = K \rtimes H$ if:

(1) $H < G$ and $K \triangleleft G$;

(2) $HK = G$;

(3) $H \cap K = \{1\}$.

If G act on the set X and $g \in G$, then $\mathrm{Fix}(g) = \{x \ : \ gx = x\}$.

Theorem 2.4. (see [8]) *Let G be a finite transitive permutation group acting on the cosets of it's subgroup H. If $g \in G$ and $[G : H] = n$ then* $\dfrac{\varphi(g)}{n} = \dfrac{|C^g \cap H|}{|C^g|}$

Theorem 2.5. (see [2]) *Let g be a permutation with cycle type α_1, α_2, α_3, $\dots$, α_n, then:*

(i) *The number $\varphi(g^l)$ of 1-cycles in g^l is $\Sigma_{i/l} i\alpha_i$;*

(ii) $\alpha_l = \dfrac{1}{l} \sum\limits_{i/l} \varphi\left(g^{l/i}\right) \mu(i)$.

3. Main Results: The Cycle Index of Frobenious Groups

If G is a Frobenious group, then the action of G on X is equivalent to the action of G on $S = G/H$, the set of left cosets of G, the action being by left multiplication. Furthermore $|S| = |G/H| = |M|$.

Now, the cycle index of G is derived as follows. Let $x \in G$, then either $x \in M$ or else x is in a conjugate of H. So it's enough to determine $\mathrm{mon}(x)$ in Case I $x \in M$ and in Case II $x \in H$.

Case I. If $x \neq 1$ is in M, then $\varphi(x) = 0$. Since M consists of 1 and all elements of G with no fixed points, $|C^* \cap H| = 0$.

Now if $x \neq 1$ is in M, then $\alpha_l = 0$ if $l \neq |x|$ where α_l is the number of cycles of length l in x and if $l = |x|$, then $\alpha_l = \dfrac{1}{l} \sum\limits_{i/l} \varphi(x^{l/i})\mu(i)$. But $\varphi(x^{l/i}) = 0$, unless

$i = 1$, in which case $\varphi(x^l) = \varphi(1) = |M|$. Therefore since $l = |x|$, then $\alpha_l = \dfrac{|M|}{|x|}$ and $\mathrm{mon}(x) = t_{|x|}^{|M|/|x|}$. If $x = 1$, then $\varphi(x) = |M|$ and $\mathrm{mon}(x) = t_1^{|M|}$. Thus elements of M contribute $\dfrac{1}{|G|}\sum\left\{t_{|x|}^{|M|/|x|} \ : \ x \in M\right\}$ to the cycle index of G.

Case II. If $x = 1$, $\varphi(x) = \varphi(1) = |M|$ and $\mathrm{mon}(x) = t_1^{|M|}$. If $x \neq 1$, then $\varphi(x) = 1$ (from the definition of a Frobenious group). Let $|C^* \cap H| = a$ then $|C^*| = a|M|$. If $l \neq |x|$ then

$$\alpha_l = \frac{1}{l}\sum_{i/l}\varphi(x^{l/i})\mu(i) = \frac{1}{l}\sum_{i/l}\mu(i) = 0,$$

since $\varphi(x^{l/i}) = 1$, for $i \neq 1$ and also $\sum_{i/l}\mu(i) = 0$ if $l \neq 1$ from Theorem 2.5.

If $l = |x|$, we have

$$\alpha_l = \frac{1}{l}\sum_{i/l}\varphi(x^{l/i})\mu(i) = \frac{1}{l}\left[|M| + \sum_{i/l}\mu(i), \ i \neq 1\right],$$

since $x \neq 1$ and $i \neq 1$ implies $l \neq 1$ and $\sum_{i/l}\mu(l) = \mu(1) + \sum_{i/l}\mu(i), \ i \neq 1$.

But $\mu(1) = 1$ and $\sum_{i/l}\mu(l) = 0, \ l \neq 1$. Therefore $\sum_{i/l}\mu(i) = -1, \ i \neq 1$ and thus $\alpha_i = \dfrac{1}{l}\left[|M| - 1\right]$. So $\alpha_{|x|} = \dfrac{|M|-1}{|x|}$ and $\mathrm{mon}(x) = t_1 t_{|x|}^{(|M|-1)/|x|}$. We conclude that since H has distinct conjugates which intersect trivially i.e at 1, the contribution of element of $G/M = \bigcup\{H^x \backslash I \ : \ x \in G\}$ to Z_G is

$$\begin{aligned}
&|G|^{-1}\left[|M|\sum\{\mathrm{mon}(x) \ : \ x \neq 1 \in H\}\right] \\
&\quad = |G|^{-1}\left[|M|\sum\{\mathrm{mon}(x) \ : \ x \in H\} - |M|t_1^{|M|}\right] \\
&\quad = |G|^{-1}\left[|M|\sum\left\{t_1 t_{|x|}^{(|M|-1)/|x|} \ : \ x \in H\right\} - |M|t_1^{|M|}\right] \\
&\quad = \frac{|M|}{|G|}\sum\left\{t_1 t_{|x|}^{(|M|-1)/|x|} \ : \ x \in H\right\} - \frac{|M|}{|G|}t_1^{|M|}.
\end{aligned}$$

But $\dfrac{|G|}{|H|} = |M|$ which implies $\dfrac{|M|}{|G|} = \dfrac{1}{|H|} = |H|^{-1}$. This gives

$$|H|^{-1}\sum\left\{t_1 t_{|x|}^{(|M|-1)/|x|} \ : \ x \in H\right\} - |H|^{-1}t_1^{|M|}.$$

Combining Case I and Cases II gives

$$Z_{G,M=S} = |G|^{-1} \sum \left\{ t_{|x|}^{|M|/|x|} \ : \ x \in M \right\} + |H|^{-1} \sum \left\{ t_1 t_{|x|}^{(|M|-1)/|x|} \ : \ x \in H \right\} - |H|^{-1} t_1^{|M|}.$$

If our aim is to express the cycle index of Frobenius group G in terms of the cycle index of M and H then we have

$$Z_{G,M=S} = |G|^{-1} \sum \left\{ t_{|x|}^{|M|/|x|} \ : \ x \in M \right\} + |H|^{-1} \sum \left\{ t_1 t_{|x|}^{(|M|-1)/|x|} \ : \ x \in H \right\} - |H|^{-1} t_1^{|M|}.$$

But

$$|G|^{-1} = \frac{1}{|H||M|} \text{ and } Z_{M,S} = \frac{1}{|M|} \sum \left\{ t_{|x|}^{|M|/|x|} \ : \ x \in M \right\}.$$

Therefore

$$Z_{G,S} = \frac{1}{|H||M|} \sum \left\{ t_{|x|}^{|M|/|x|} \ : \ x \in M \right\} + |H|^{-1} \sum \left\{ t_1 t_{|x|}^{(|M|-1)/|x|} \ : \ x \in H \right\} - |H|^{-1} t_1^{|M|},$$

and

$$Z_{G,S} = |H|^{-1} Z_{M,S} + Z_{H,S} - |H|^{-1} Z_{1,S}.$$

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