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STOCHASTIC ANALYSIS OF TIME TO THREE VITAL ORGANS FAILURE AND TREATMENT OF A DIABETIC PERSON

K. Usha[1], Eswari Prem[2 §], R. Ramanarayanan[3]

[1]Department of Mathematics
Queen Mary's College for Women
Chennai, Tamilnadu, 600 005, INDIA

[2]Department of Mathematics
Governmental Arts College for Men
Krishnagiri, Tamilnadu, 635 001, INDIA
e-mail: premtols@yahoo.com

[3]Technical University
Chennai, Tamilnadu, 600 062, INDIA
e-mail: r.ramanarayanan@gmail.com

Abstract: A diabetic person is liable for damages of vital organs like heart, kidneys and legs, if he is not given proper treatment. The heart, two kidneys and legs are considered as three sub-systems of the main system. These three sub-systems are exposed to damages. The two kidneys have constant failure rate. In model 1, the organ heart has a general life time and the damages occur in the leg in accordance with Poisson process and in model 2 the heart has an exponential life time where as the leg is exposed to renewal damage processes. Considering treatment to the diabetic person, in this paper we present the expected time to failure of the organs, variance and the joint Laplace-Stieltjes transforms of the curing and damage times. Marginal distribution functions are calculated when there exists a cure time for every damage.

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§Correspondence author

1. Introduction

Diabetes is a worldwide problem and is a significant cause of morbidity and mortality. According to the study by WHO an estimated 135 million people worldwide had been diagnosed diabetes in 1995 and this number is expected to rise to at least 300 million by 2025. Diabetes is a metabolic disorder caused by insufficient insulin action, characterized by increased sugar level in the blood. Besides strong genetic predisposition to non-insulin dependent diabetic mellitus, its insulin secretion is also determined by interaction of some host factors and environmental factors as well. The reason for this escalation in diabetes prevalence are changes in life style, decreasing physical activity, increasing weight changes in eating habits, etc., which lead to diabetes during adulthood. Cardio vascular disease is cause of death of persons with diabetes. Diabetic nephropathy can cause renal failure. Diabetic neuropathy, by leading to loss of feeling, particularly in the feet, makes affected people very susceptible to infections and gangrene. For more details on such deficiency and disease one may refer [4], [1], [3], [5], [6], [8]. Using the mathematical approach given in [2], [7], [9] we study the case of a diabetic person.

Model

1. Consider the vital organs heart, kidneys and legs of the human body of the diabetic person.

2. Assume that heart has a life time U with distribution $F(x)$ whose density is $f(x)$ and two kidneys have constant failure rate. Renal failure occurs when the 2 kidneys are damaged and it works when at least one is in good condition. It is assumed that the probability that either of the 2 kidneys fails during the interval $(t, t+\Delta t)$ is $\lambda_1 \Delta t + 0(\Delta t)$. When the treatment is given to one kidney the probability that second fails during the interval is $(t, t+\Delta t)$ is $\lambda_2 \Delta t + 0(\Delta t)$.

3. The leg is also exposed to the damage process. The damages occur to it in accordance with a renewal process with inter-occurrence time distribution function $G(x)$ whose density is $g(x)$.

4. At time 0, the person is in normal condition, i.e. the organs heart, kidneys and legs are in good condition.

5. The person is hospitalized if either of the organs heart or kidney is in damaged state. The damaged part of kidney and the damaged part of the legs are also curable by taking medicine in the initial stage. The time to hospitalize the person is (T), $T = \min\{T_1, T_2\}$ where $T_i,\ i = 1, 2$ are the times that the damage occurs in the organs heart and kidney.

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6. Curing time (S) of heart is a random variable with distribution function $H(x)$, density $h(x)$. Curing times of kidney are exponential with parameter μ. Treatment times for the damages of legs are independent with distribution function $R(x)$ whose density is $r(x)$.

We present joint Laplace-Stieltjes transforms of time to hospital treatment and curing times when damages and organs are attended one by one in models (1) and (2).

2. Model 1: Heart Has General Life Time and Legs Are Exposed to Poisson Damage Process

We first obtain the distribution function of time to failure of the two kidneys, its expectation and variance.

1. At time 0 the two kidneys of the diabetic person are in normal condition. The time to damage of any one of them has exponential distribution with rate λ_1.

2. When one is in damaged state and under treatment, the curing time distribution is exponential with rate μ. The other may get damaged in an exponential time with rate λ_2.

3. The state of the human body is 0 when both the kidneys are good. The state is 1 when one kidney is in damaged state and the state is 2 when both kidneys are in damaged state.

To find the distribution function of the life and the illness times we need the following functions.

$P_{0,0}(x) = P$ (at time x the person is in state 0, the person does not move to state 2 during $(0, x)|$ at time 0 the person is in state 0).

$P_{0,1}(x) = P$ (at time x the person is in state 1 and is under treatment, the person does not move to state 2 during $(0, x)|$ at time 0 the person is in state 0).

$P_{0,2}(x)dx = P$ (the person moves to state 2 during $(x, x + dx)|$ at time 0 he is in state 0), where $P_{0,2}$ is the failure density of the kidneys of the person.

We now calculate the $P_{\cdot,\cdot}(x)$ function.

$P_{0,0}(x)$ satisfies the following:

$$P_{0,0}(x) = e^{-\lambda_1 x} + \int_0^x \lambda_1 e^{-\lambda_1 x} P_{1,0}(x - u)du,$$

where $P_{1,0}(x) = P$ (at time x the two kidneys under consideration are good $|$ at time 0 one kidney is under treatment for damage).

$P_{1,0}(x)$ is given by

$$P_{1,0}(x) = \int_0^x \mu e^{-\mu u} e^{-\lambda_2 u} e^{-\lambda_1 (x-u)} du + \int_0^x \int_0^v \mu e^{-\mu u} e^{-\lambda_2 u} \lambda_1 e^{-\lambda_1 (v-u)} du P_{1,0}(x-v) dv.$$

The first term is the probability that the diabetic person's damaged organ is cured at u and no organ of diabetic person is damaged during $(0, x)$ and the second term is the probability that the damaged organ is cured at u, no organ is damaged during $(0, u)$, an organ fails at $v > u$ and at x all the organs are normal. Laplace transforms of the above equations give

$$P^*_{0,0}(s) = \frac{\lambda_2 + \mu + s}{[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1 \lambda_2]}, \tag{2.1}$$

where $*$ indicates Laplace transform.

Using a similar argument, we find

$$P_{0,1}(x) = \int_0^x \lambda_1 e^{-\lambda_1 u} e^{-\mu(x-u)} e^{-\lambda_2 (x-u)} du + \int_0^x \int_0^v \lambda_1 e^{\lambda_1 u} \mu e^{-\mu(v-u)} e^{-\lambda_2 (v-u)} du P_{0,1}(x-v) dv,$$

and

$$P^*_{0,1}(s) = \frac{\lambda_1}{[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1 \lambda_2]}. \tag{2.2}$$

The failure density $P_{0,2}(x)$ satisfies,

$$P_{0,2}(x) = \int_0^x \lambda_1 e^{-\lambda_1 x} P_{1,2}(x-u) du,$$

where $P_{1,2}(x)dx = P$(the person moves to state 2 during $(x, x+dx)|$ at time 0 damaged organ is given treatment).

$$P_{1,2}(x) = \lambda_2 e^{-\lambda_2 u} e^{-\mu x} + \int_0^x e^{\lambda_2 u} \mu e^{-\mu u} P_{0,2}(x-u) du.$$

By Laplace transformation,

$$P^*_{0,2}(s) = \frac{\lambda_1\lambda_2}{[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1\lambda_2]}. \tag{2.3}$$

Equations (2.1)-(2.3) can be inverted easily:

$$\begin{aligned} P_{0,0}(t) &= \frac{1}{2}e^{-at}[e^{bt} + e^{-bt}] \\ &+ \frac{1}{4b}[\lambda_2 - \lambda_1 + \mu]e^{-at}[e^{bt} - e^{-bt}], \end{aligned} \tag{2.4}$$

$$P_{0,1}(t) = \frac{\lambda_1}{2b}e^{-at}[e^{bt} - e^{-bt}], \tag{2.5}$$

$$P_{0,2}(t) = \frac{\lambda_1\lambda_2}{2b}e^{-at}[e^{bt} - e^{-bt}], \tag{2.6}$$

where

$$a = \frac{1}{2}(\lambda_1 + \lambda_2 + \mu); \quad b = \frac{1}{2}\sqrt{(\lambda_1 - \lambda_2)^2 + \mu^2 + 2\mu(\lambda_1 + \lambda_2)}.$$

We find the mean and variance for the failure density function from (2.3),

$$Mean = -\frac{d}{ds}P^*_{0,2}(s)|_{s=0} = \frac{(\lambda_1\lambda_2)(\lambda_1 + \lambda_2 + \mu)}{(\lambda_1\lambda_2)^2}.$$

$$Mean = \frac{\lambda_1 + \lambda_2 + \mu}{(\lambda_1\lambda_2)}.$$

Now,

$$E(Z^2) = \frac{d^2}{ds^2}P^*_{0,2}(s)|_{s=0},$$

$$\begin{aligned} \frac{d^2}{ds^2}P^*_{0,2}(s) = &\frac{[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1\lambda_2]^2(-2\lambda_1\lambda_2)}{[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1\lambda_2]^4} \\ &+ \frac{\lambda_1\lambda_2(2s + \lambda_1 + \lambda_2 + \mu)^2 2[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1\lambda_2]}{[s^2 + s(\lambda_1 + \lambda_2 + \mu) + \lambda_1\lambda_2]^4}, \end{aligned}$$

$$P''_{0,2}(s)|_{s=0} = \frac{-2\lambda_1\lambda_2 + 2(\lambda_1 + \lambda_2 + \mu)^2}{[\lambda_1\lambda_2]^2},$$

$$Var(Z) = E(z^2) - [E(Z)]^2 = \frac{-2\lambda_1\lambda_2 + 2(\lambda_1 + \lambda_2 + \mu)^2}{[\lambda_1\lambda_2]^2}$$

$$- \left(\frac{\lambda_1 + \lambda_2 + \mu}{\lambda_1 \lambda_2} \right)^2$$
$$= \frac{{\lambda_1}^2 + {\lambda_2}^2 + \mu^2 + 2\lambda_1\mu + 2\lambda_2\mu}{[\lambda_1\lambda_2]^2}.$$

Note $\frac{1}{\mu} = E$ (time to cure) if the curing time distribution is exponential with rate μ. If μ is large we get quickly cure – state 0. Otherwise it is slow and if μ is small curing is slow .

When $\mu = 0, E(X) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ and $Var(X) = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}$ the person will not survive.

Now we calculate the joint distribution function of hospitalization time and curing time. Let $\widehat{A}$ denote the sum of the curing times of the failed organs and S denote the curing time of heart.

When $T_1 < T_2$ kidneys are in normal condition but leg has k damages (gangrene), then the total curing time $\widehat{A}$ is given by $\widehat{A} = R_1 + R_2 + \cdots + R_k + S$ and when one kidney is also in damaged state we have $\widehat{A} = R_1 + R_2 + \cdots + R_k + S + S_1$. When $T_2 < T_1$ and the leg has k damages we get $\widehat{A} = R_1 + R_2 + \cdots + R_k + S_1 + S_2$, where S_1, S_2 have exponential distribution with parameter μ and R_i has the distribution function $R(x)$.

The joint distribution of T and $\widehat{A}$ is given by

$$\begin{aligned} P(T \leq x, \widehat{A} \leq y) &= P(T_1 \leq x, \widehat{R} \leq y, T_2 > T_1) \\ &\quad + P(T_2 \leq x, \widehat{A} \leq y, T_1 > T_2), \end{aligned}$$

$$\begin{aligned} P(T \leq x, \widehat{A} \leq y) &= \left\{ \left[\int_0^x f(u) P_{0,0}(u) \sum_{k=0}^{\infty} e^{-\beta u} \frac{(\beta u)^k}{k'!} du \right] \times \left[\int_0^y \int_0^u r_k(v) h(u-v) du dv \right] \right\} \\ &+ \left\{ \left[\int_0^x f(u) P_{0,1}(u) \sum_{k=0}^{\infty} e^{-\beta u} \frac{(\beta u)^k}{k'!} du \right] \times \left[\int_0^y \int_0^z \int_0^w r_k(v) \mu e^{-\mu(w-v)} h(z-w) dv dw dz \right] \right\} \\ &+ \left\{ \left[\int_0^x \bar{F}(u) P_{0,2}(u) \sum_{k=0}^{\infty} e^{-\beta u} \frac{(\beta u)^k}{k'!} du \right] \times \left[\int_0^y \int_0^w r_k(v) \mu^2 (w-v) e^{-\mu(w-v)} dv dw \right] \right\}, \end{aligned} \quad (2.7)$$

where $r_n(u)$ is the n-fold convolution of $r(u)$ with itself and $\bar{F}(u) = 1 - F(u)$. To write down (2.7), the three mutually exclusive and exhaustive cases are considered:

1. The functioning of heart deters even though the 2 kidneys are working.
2. The functioning of heart deters when one of the kidneys is under treatment.
3. The heart continues to work even though the two kidneys fail.

Let us now introduce the joint Laplace-Stieltjes transform of T and $\widehat{A}$,

$$E(e^{-\xi T}e^{-\eta \widehat{A}}) = \int_0^\infty\int_0^\infty e^{-\xi x}e^{-\eta y}\frac{\partial^2}{\partial x \partial y}P(T \le x, \widehat{A} \le y)dxdy.$$

From (2.7) we get

$$\begin{aligned} E(e^{-\xi T}e^{-\eta \widehat{A}}) &= \int_0^\infty e^{-x[\xi+\beta(1-r^*(\eta))]}f(x)P_{0,0}(x)dxh^*(\eta) \\ &\quad + \int_0^\infty e^{-x[\xi+\beta(1-r^*(\eta))]}f(x)P_{0,1}(x)dxh^*(\eta)(\frac{\mu}{\mu+\eta}) \\ &\quad + \int_0^\infty e^{-x[\xi+\beta(1-r^*(\eta))]}\bar{F}(x)P_{0,2}(x)dx(\frac{\mu}{\mu+\eta})^2. \end{aligned}$$

After simplification,

$$\begin{aligned} E(e^{-\xi T}e^{-\eta \widehat{A}}) &= f^*(s+a-b)\left[h^*(\eta)\left(\frac{1}{2}+\frac{1}{4b}(\lambda_2-\lambda_1+\mu)+\frac{\lambda_1\mu}{2b(\mu+\eta)}\right)\right. \\ &\quad \left. - \frac{\lambda_1\lambda_2\mu^2}{2b(\mu+\eta)^2(s+a-b)}\right] \\ &\quad - f^*(s+a-b)\left[h^*(\eta)\left(\frac{-1}{2}+\frac{1}{4b}(\lambda_2-\lambda_1+\mu)+\frac{\lambda_1\mu}{2b(\mu+\eta)}\right)\right. \\ &\quad \left. - \frac{\lambda_1\lambda_2\mu^2}{2b(\mu+\eta)^2(s+a+b)}\right] \\ &\quad + \left[\frac{\lambda_1\lambda_2\mu^2}{2b(\mu+\eta)^2}\right]\left[\frac{1}{s+a-b}-\frac{1}{s+a+b}\right], \end{aligned} \tag{2.8}$$

where $s = \xi + \beta(1 - r^*(\eta))$. Equation (2.8) for $\xi = 0$ and $\eta = 0$ gives respectively $E(e^{-\eta\widehat{A}})$, and $E(e^{-\xi T})$. $E(T)$ and $E(\widehat{A})$ can be obtained from $E(e^{-\xi T})$ and $E(e^{-\eta\widehat{A}})$. They are

$$E(T) = (\frac{\lambda_1\lambda_2}{2b})\left\{\left[\frac{1-f^*(a-b)}{(a-b)^2}\right] - \left[\frac{1-f^*(a+b)}{(a+b)^2}\right]\right\}, \tag{2.9}$$

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$$\begin{aligned} E(\widehat{A}) &= (\frac{2}{\mu}) + \frac{\beta\lambda_1\lambda_2 E(R)}{2b}\left\{\left[\frac{1-f^*(a-b)}{(a-b)^2}\right] - \left[\frac{1-f^*(a+b)}{(a+b)^2}\right]\right\} \\ &\quad - \left[\frac{f^*(a-b)}{2b(a-b)}\right]\left[\lambda_1\lambda_2(\frac{2}{\mu} - E(S)) - \frac{(a-b)\lambda_1}{\mu}\right] \\ &\quad + \left[\frac{f^*(a+b)}{2b(a+b)}\right]\left[\lambda_1\lambda_2(\frac{2}{\mu} - E(S)) - \frac{(a+b)\lambda_1}{\mu}\right]. \end{aligned} \tag{2.10}$$

Numerical Example. We consider a special case for numerical results with in which the organ heart has constant failure rate λ. Then

$$f^*(t) = \frac{\lambda}{\lambda + t}, \tag{2.11}$$

where $t = a - b$ and $a + b = t + 2b$.

Substituting (2.11) in (2.9) and (2.10), then we get

$$E(T) = \lambda_1\lambda_2\left\{\frac{(\lambda + 2t + 2b)}{(t(\lambda+t)(t+2b)(\lambda+t+2b))}\right\}, \tag{2.12}$$

$$\begin{aligned} E(\widehat{A}) &= (\frac{2}{\mu}) + \left\{\frac{(\lambda_1\lambda_2)(\lambda + 2t + 2b)}{(t(\lambda+t)(t+2b)(\lambda+t+2b))}\right\} \\ &\quad \times \left\{\beta E(R) - (\frac{2}{\mu} - E(S))\lambda + \frac{\lambda}{\lambda_2\mu}t(t+2b)\right\}. \end{aligned} \tag{2.13}$$

Now we calculate the expected time to damage of the organs and time to cure by using the graph.

λ_1=1, λ_2=1, μ=1, β=1, $E(S) = 0.1$, $E(R)$ =0.1		
λ	E(T)	$E(\widehat{A})$
1	0.8000	1.3600
2	0.4545	1.2273
3	0.3158	1.1789
4	0.2415	1.1552
5	0.1951	1.1415
6	0.1636	1.1327
7	0.1408	1.1268
8	0.1236	1.1225
9	0.1101	1.1193
10	0.0992	1.1168

Table 2.1

λ_1=0.1, λ_2=0.1, μ=1, β=2, $E(S) = 0.1$, $E(R)$ =0.1		
λ	E(T)	$E(\widehat{A})$
1	0.9955	0.4072
2	0.4992	0.3027
3	0.3331	0.2680
4	0.2499	0.2508
5	0.1999	0.2406
6	0.1666	0.2337
7	0.1428	0.2289
8	0.1250	0.2252
9	0.1111	0.2224
10	0.1000	0.2202

Table 2.2

From Tables 2.1 to 2.6 we observe the behavior of mean time to damages $E(T)$ and mean time to curing $E(\widehat{A})$ for fixed values of λ_1, λ_2, μ, β, $E(S)$ and $E(R)$. If

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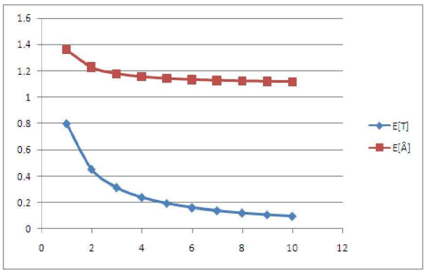

Figure 2.1

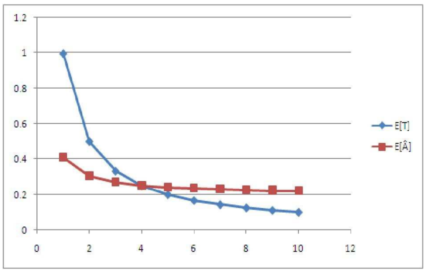

Figure 2.2

λ_1=5, λ_2=4, μ=1, β=2, $E(S)$=0.1, $E(R)$ =0.1		
λ	E(T)	$E(\widehat{A})$
1	0.3548	3.1710
2	0.2727	3.7455
3	0.2203	4.0932
4	0.1842	4.3211
5	0.1579	4.4789
6	0.1379	4.5931
7	0.1223	4.6784
8	0.1098	4.7439
9	0.0995	4.7953
10	0.0909	4.8364

Table 2.3

λ_1=0.1, λ_2=0.1, μ=5, β=2, $E(S)$=0.1, $E(R)$ =0.1		
λ	E(T)	$E(\widehat{A})$
1	0.4286	11.0714
2	0.3077	15.6615
3	0.2394	18.1901
4	0.1957	19.7696
5	0.1652	20.8374
6	0.1429	21.6000
7	0.1257	22.1671
8	0.1122	22.6020
9	0.1013	22.9441
10	0.0923	23.2185

Table 2.4

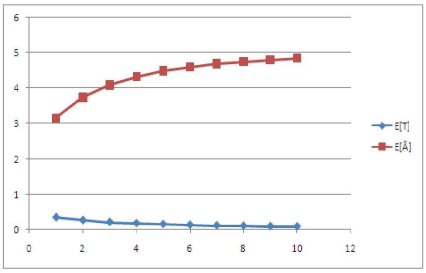

Figure 2.3

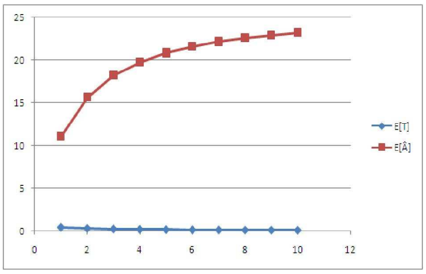

Figure 2.4

the parameter λ increases, both $E(T)$ and $E(\widehat{A})$ decrease in Tables 2.1, 2.2, 2.5 and 2.6. If the parameter λ increases, $E(T)$ decreases and $E(\widehat{A})$ increases in Tables 2.3 and 2.4.

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λ_1=0.1, λ_2=0.1, μ=1, β=0.1, $E(S)$=0.1, $E(R)$ =0.1		
λ	E(T)	$E(\widehat{A})$
1	0.9955	0.2181
2	0.4992	0.2078
3	0.3331	0.2048
4	0.2499	0.2034
5	0.1999	0.2026
6	0.1666	0.2021
7	0.1428	0.2017
8	0.1250	0.2015
9	0.1111	0.2013
10	0.1000	0.2012

Table 2.5

λ_1=2, λ_2=1, μ=0.1, β=0.2, $E(S)$=0.2, $E(R)$ =0.2		
λ	E(T)	$E(\widehat{A})$
1	0.6721	6.8531
2	0.4180	3.6298
3	0.3005	2.3431
4	0.2336	1.6988
5	0.1906	1.3300
6	0.1608	1.0990
7	0.1389	0.9448
8	0.1222	0.8366
9	0.1091	0.7578
10	0.0985	0.6987

Table 2.6

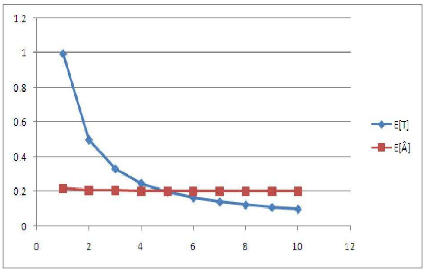

Figure 2.5

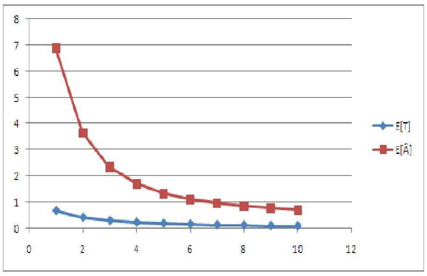

Figure 2.6

From equation (2.8), $E(T\widehat{A})$, is seen to be

$$
\begin{aligned}
E(T\widehat{A}) \;=\; & \left\{ \left[\frac{\lambda_1\lambda_2}{b(a-b)^2}\right] \left[\frac{1}{\mu} - \frac{\beta r^{*'}(0)}{a-b}\right] \right. \\
& \left. \times \left[1 - f^*(a-b) + (a-b) f^{*'}(a-b)\right] \right\} \\
& + \left\{ \left[\frac{\lambda_1\lambda_2}{b(a+b)^2}\right] \left[\frac{\beta r^{*'}(0)}{a+b} - \frac{1}{\mu}\right] \right. \\
& \left. \times \left[1 - f^*(a+b) + (a+b) f^{*'}(a+b)\right] \right\} \\
& + f^{*'}(a-b)\left(h^{*'}(0)(a+b) - \frac{\lambda_1}{\mu}\right)\frac{1}{2b} \\
& + f^{*'}(a+b)\left(\frac{\lambda_1}{\mu} - h^{*'}(0)(a-b)\right)\frac{1}{2b}. \qquad (2.14)
\end{aligned}
$$

where $'$ indicates differentiation. Using (2.9)-(2.11) we can get the co-variance.

3. Model 2: Heart Has an Exponential Life Time whereas the Leg Is Exposed to Renewal Damage Processes

Considering the failure time of the organ heart has exponential distribution with parameter β and considering the total curing time $\widehat{A}$ as the sum of the curing times required for the treatment we can set up the equation of $P(T \leq x, \widehat{A} \leq y)$ using the arguments given already. We get,

$$\begin{aligned} P(T \leq x, \widehat{R} \leq y) &= \Bigg[\int_0^x \beta e^{-\beta u} P_{0,0}(u) \sum_{k=0}^{\infty} [G_k(u) - G_{k+1}(u)] du \\ &\quad \times \int_0^y \int_0^u r_k(v) h(u-v) dv du\Bigg] \\ &\quad + \left\{\left[\int_0^x \beta e^{-\beta u} P_{0,1}(u) \sum_{k=0}^{\infty} [G_k(u) - G_{k+1}(u)] du\right]\right. \\ &\quad \left. \times \left[\int_0^y \int_0^z \int_0^w r_k(v) \mu e^{-\mu(w-v)} h(z-w) dv dw dz\right]\right\} \\ &\quad + \left\{\left[\int_0^x e^{-\beta u} P_{0,2}(u) \sum_{k=0}^{\infty} [G_k(u) - G_{k+1}(u)] du\right]\right. \\ &\quad \left. \times \left[\int_0^y \int_0^w r_k(v) \mu^2 (w-v) e^{-\mu(w-v)} dv dw\right]\right\}, \end{aligned} \tag{3.1}$$

where $G_k(u)$ is the k-fold Stieltjes convolution of $G(u)$ with itself and $P_{.,.}(U)$ are given by (2.4)-(2.6). The double Laplace-Stieltjes transform of (3.1) is seen as,

$$\begin{aligned} E(e^{-\xi T} e^{-\eta \widehat{A}}) &= \left\{\frac{[1 - g^*(\xi + \beta + a - b)]}{(\xi + \beta + a - b)} \times [1 - g^*(\xi + \beta + a - b) r^*(\eta)]\right\} \\ &\quad \times \left\{\beta h^*(\eta) \left[\frac{1}{2} + \frac{1}{4b}(\lambda_2 - \lambda_1 + \mu) + \frac{\lambda_1 \mu}{2b(\mu + \eta)}\right] + \frac{\mu^2 \lambda_1 \lambda_2}{2b(\mu + \eta)^2}\right\} \\ &\quad + \left\{\frac{[1 - g^*(\xi + \beta + a + b)]}{(\xi + \beta + a + b)} \times [1 - g^*(\xi + \beta + a + b) r^*(\eta)]\right\} \\ &\quad \times \left\{\beta h^*(\eta) \left[\frac{1}{2} - \frac{1}{4b}(\lambda_2 - \lambda_1 + \mu) - \frac{\lambda_1 \mu}{2b(\mu + \eta)}\right]\right. \end{aligned}$$

$$+ \frac{\mu^2\lambda_1\lambda_2}{2b(\mu+\eta)^2}\Bigg\}. \tag{3.2}$$

$E(e^{\xi T})$ and $E(e^{-\eta \widehat{A}})$ can be calculated from (3.2). The moments of T, $\widehat{A}$ are given by,

$$E(T) = (\frac{1}{2b})\left\{\frac{(a+b)\beta+\lambda_1\lambda_2}{(\beta+a-b)^2} - \frac{\beta(a-b)+\lambda_1\lambda_2}{(\beta+a+b)^2}\right\}, \tag{3.3}$$

$$\begin{aligned} E(\widehat{A}) &= \left[\frac{1}{2b(\beta+a-b)}\right]\left\{\frac{g^*(\beta+a-b)}{1-g^*(\beta+a-b)} \times E(R)[\beta(a+b)+\lambda_1\lambda_2] \right. \\ &\left. + \beta E(S)(a-b) + \frac{\beta\lambda_1}{\mu} + \frac{2\lambda_1\lambda_2}{\mu}\right\} \\ &- \left[\frac{1}{2b(\beta+a+b)}\right]\left\{\frac{g^*(\beta+a+b)}{1-g^*(\beta+a+b)} \times E(R)[\beta(a-b)+\lambda_1\lambda_2] \right. \\ &\left. + \beta E(S)(a-b) + \frac{\beta\lambda_1}{\mu} + \frac{2\lambda_1\lambda_2}{\mu}\right\}. \end{aligned} \tag{3.4}$$

Numerical Example. We consider a special case for numerical results with in which the organ heart has constant failure rate α. Then

$$\begin{aligned} g^*(\theta) &= \frac{\alpha}{\alpha+\theta}, \qquad \text{where} \quad \theta = \beta+a-b \\ \text{and} \quad \beta+a+b &= \theta+2b. \end{aligned} \tag{3.5}$$

Substituting (3.5) in (3.3) and (3.4), we get

$$E(T) = \frac{1}{\theta^2(\theta+2b)^2}\left\{\beta[\theta^2+2(a+b)(\theta+b)]+2\lambda_1\lambda_2(1+b)\right\},$$

$$\begin{aligned} E(\widehat{A}) &= \frac{E(R)}{\theta^2(\theta+2b)}\{\alpha\beta[\theta(\theta+2a+2b)+2b(a+b)]+4\alpha\lambda_1\lambda_2(\theta+b)\} \\ &+ \frac{1}{\theta(\theta+2b)}\left\{\beta(E(S)(a-b)+\frac{\lambda_1}{\mu})+2\lambda_1\lambda_2\frac{1}{\mu}\right\}. \end{aligned}$$

Now we calculate the expected time to damage of the organs and time to cure by using the graph.

From Tables 3.1 to 3.4 we observe the behavior of mean time to damages $E(T)$ and mean time to curing $E(\widehat{A})$ for fixed values of λ_1, λ_2, μ, $E(S)$ and $E(R)$. If the parameter α increases, $E(T)$ is a constant and $E(\widehat{A})$ increases in Table 3.1. If the parameter β increases, both $E(T)$ and $E(\widehat{A})$ decrease in Table 3.4 and $E(T)$ decreases and $E(\widehat{A})$ increases in Tables 3.2 and 3.3.

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μ=0.5,β=0.1,λ_1=0.1, λ_2 =0.1,$E(S)$=0.1,$E(R)=0.1$		
α	E(T)	$E(\widehat{A})$
1	11.0751	1.0065
2	11.0751	1.3448
3	11.0751	1.6830
4	11.0751	2.0213
5	11.0751	2.3595
6	11.0751	2.6978
7	11.0751	3.0360
8	11.0751	3.3743
9	11.0751	3.7125
10	11.0751	4.0508

Table 3.1

μ=0.1, α=0.1,λ_1=0.1, λ_2=0.1,$E(S)$=0.1,$E(A)$=0.1		
β	E(T)	$E(\widehat{A})$
1	0.9941	0.2214
2	0.4984	0.3414
3	0.3327	0.6724
4	0.2497	1.2492
5	0.1998	2.1257
6	0.1666	3.3596
7	0.1428	5.0099
8	0.1250	7.1360
9	0.1111	9.7976
10	0.1000	13.0546

Table 3.2

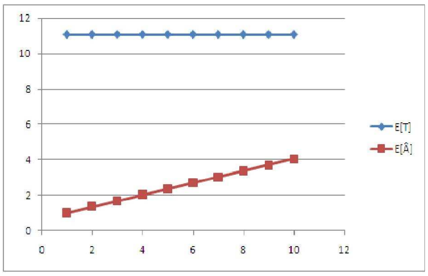

Figure 3.1

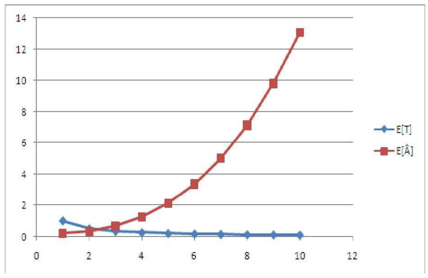

Figure 3.2

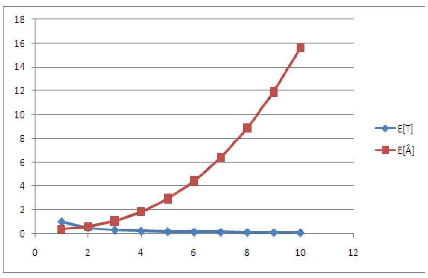

Figure 3.3

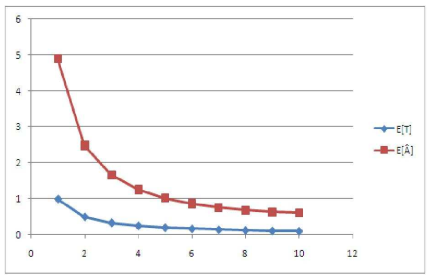

Figure 3.4

From the equation (3.2), the covariance is

$E(T\widehat{A})$

μ=0.8,α=0.1,λ_1 =0.2, λ_2=0.2,$E(S)$=0.1,$E(A) = 0.1$		
β	E(T)	$E(\widehat{A})$
1	0.9816	0.3738
2	0.4949	0.5933
3	0.3313	1.0686
4	0.2490	1.8302
5	0.1994	2.9307
6	0.1663	4.4276
7	0.1426	6.3796
8	0.1248	8.8462
9	0.1110	11.8868
10	0.0999	15.5612

Table 3.3

μ=0.02,α=0.01, λ_1 =0.1, λ_2=0.1,$E(S)$=0.01,$E(A) =$ 0.01		
β	E(T)	$E(\widehat{A})$
1	0.9910	4.8787
2	0.4978	2.4733
3	0.3325	1.6581
4	0.2469	1.2509
5	0.1998	1.0100
6	0.1665	0.8545
7	0.1428	0.7497
8	0.1249	0.6788
9	0.1111	0.6324
10	0.1000	0.6054

Table 3.4

$$
\begin{aligned}
= & \left\{\frac{r^{*'}(0)[\beta(a+b)+\lambda_1\lambda_2]}{2b(\beta+a-b)[1-g^*(\beta+a-b)]}\right\} \times \left\{\frac{g^{*'}(\beta+a-b)}{1-g^*(\beta+a-b)} - \frac{g^*(\beta+a-b)}{\beta+a-b}\right\} \\
& - \left[\frac{1}{2b}(\beta+a-b)^2][\beta h^{*'}(0)(a+b) - \beta\frac{\lambda_1}{\mu} - \frac{2\lambda_1\lambda_2}{\mu}]\right] \\
& - \left\{\frac{r^{*'}(0)[\beta(a-b)+\lambda_1\lambda_2]}{2b(\beta+a-b)[1-g^*(\beta+a+b)]}\right\} \times \left\{\frac{g^{*'}(\beta+a+b)}{1-g^*(\beta+a+b)} - \frac{g^*(\beta+a+b)}{\beta+a+b}\right\} \\
& + \left[[\frac{1}{2b}(\beta+a+b)^2][\beta h^{*'}(0)(a-b) - \beta\frac{\lambda_1}{\mu} - \frac{2\lambda_1\lambda_2}{\mu}]\right],
\end{aligned}
$$

where $'$ indicates differentiation.

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