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k-TOPOLOGIES ON E-COMPACTIFICATIONS

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Abstract: Given a cardinal k and an E-completely regular space X, for a topological space E, we define a k-topology on X. When X is a right topological semigroup, this process turns X into a topological semigroup. This is particularly true in the case of the right topological semigroup $(e, \delta_E X)$, the E-LMC compactification which is characterized by the fact that it is maximal w.r.t. the properties that it is right topological and that $\lambda_{e(x)}$ is continuous for each $x \in X$.

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1. Introduction

The idea that any compact Hausdorff space can be characterized as a space that is homeomorphic to some closed subspace of a topological product of the closed unit interval $\{x : 0 \leq x \leq 1\}$ in the real line, has been generalized by S. Mrowka and R. Engelking [2], [3]. Thus a space is E-completely regular (respectively, E-compact) if and only if it is homeomorphic to a subspace (respectively closed subspace) of some topological power of E, where E is a topological space. A special instance of this generalization is the class in which the space E is the real line. This class of spaces is the class of real compact spaces. It has been shown that each E-completely regular space has a maximum E-compact extension, which is called E-compactifications. For a properly chosen E, the maximal E-compactifications $\beta_E X$ is the space of all E-Z-ultra-filters with suitable topology, see [4].

We have the theory of compact right topological semigroups. It is known that

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any Hausdorff semitopological semigroup X (i.e. one which is both left and right topological) has a compactification (e, X) maximal with respect to the property that it is right topological and the requirement that $\lambda_{e(x)}$ be continuous for each $x \in X$.

The above two different concepts are combined to introduce the notion of E-LMC compactification of a topologized semigroup $X$and it has been proved (see [5]) that any Hausdorff semitopological semigroup X has an E-compactification $(e, \delta_E X)$for a topological space E maximal with respect to the property that it is right topological and the requirement that $\lambda_{e(x)}$ be continuous for each $x \in X$.

In the above background we have shown that for a given a cardinal k and an E-completely regular space X, for a topological space E, we define a k topology on X and when X is a right topological semigroup, this process turns X into a topological semigroup. This is particularly true in the case of the right topological semigroup $(e, \delta_E X)$, the E-LMC compactification, which is characterized by the fact that it is maximal w.r.t. the properties that it is right topological and that $\lambda_{e(x)}$ is continuous for each $x \in X$.

2. Preliminaries

Let E be a given topological space.

Definition 2.1. (see [2], [3]) 1) A space X is E-completely regular if X is homeomorphic to a subspace of a product of copies of E and X is called E-compact if X is homeomorphic to a closed subspace of a product of copies of E.

2) A subset U of X is called E-open if it is of the form $f^{-1}(V)$, where V is an open subset of some finite power E^n and $f \in C(X, E^n)$. A subset F of X is E-closed if and only if its complement is E-open, where

$$C(X, E^n) = \{f : X \longrightarrow E^n, f \text{ continuous}\}.$$

Theorem 2.2. (see [6]) *Let X and E be spaces. The following are equivalent.*

1. *X is E-completely regular.*
2. *For each closed subset A of X and each $p \in X \backslash A$, there is a positive integer n and $f \in C(X, E^n)$ such that $f(p)$ not in $cl_{E^n} f(A)$.*
3. *E-open subsets of X form a base for the open subsets of X.*
   In general we cannot replace $\cup\{C(X, E^n) : n \in \mathbb{N}\}$ by $C(X, E)$.

Convention. *1. It is a topological field.*

2. We take w copies of E and name them $\{E_i : i \in w\}$. Then by E^n, we mean $E_1 \times E_2 \times E_3 \times \cdots \times E_n$. If $n < m$, there is an obvious embedding of E^n in E^m, namely

$$(x_1, x_2, ..., x_n) \longleftrightarrow (x_1, x_2, ..., x_n, 0, 0, ..., 0).$$

This convention is needed for defining algebraic operations in our further developments. However this does not conflict with notation used in [6] *since any rearrangement of co-ordinates is a homeomorphism.*

3. We consider the class of all spaces X such that for each closed set $A \subset X$ and a point $x \in X \backslash A$, there is a positive integer n and $f \in C(X, E^n)$ such that $f(A) = 0$ and $f(x) \neq 0$ where 0 is the origin $(0, 0, ..., 0)$ in E^n, the additive identity. Such spaces are called E-spaces.

Definition 2.2.

$$C_E(x) = \cup\{C(X, E^n) : n \in \mathbb{N}\},$$

$$Z_{E^n}(f) = \{x \in X : f(x) = 0\},$$

where $f \in C(X, E^n)$ is called an E- zero set of f.

For $f, g \in C_E(X)$, define $(f + g)(x) = f(x) + g(x)$, for $x \in X$.

If $f \in C(X, E^n), g \in C(X, E^m)$ and if $n > m$, then since E^m is embedded in E^n as described above, $g(x)$ can be taken as a member of E^n, i.e. $g \in C(X, E^n)$. So$f(x) + g(x)$ makes sense. Likewise $(f \circ g)(x) = f(x)g(x)forx \in X$.

3. *E*-LMC Compactification

Let X represent a Hausdorff semitopological semigroup. Let K represent the disjoint union $\amalg E^n = \cup(E^n X\{n\})$.

Definition 3.1. A function $f \in C_E(X)$ is defined to be in E-LMC if:

1. $\{f.\lambda_x : x \in X\} \subseteq C_E(X)$.

2. $\sigma(C_E(X), C_E(X)^*)$-closure of $\{f \circ \rho_x : x \in X\} \subseteq C_E(X)$.

3. For each $y \in X$, $\sigma(C_E(X), C_E(X)^*)$-closure of $\{f \circ \lambda_y \circ \rho_x : x \in X\} \subseteq C_E(X)$. Put $\lambda_1 = P_1 = i$, the identity map. Then $f \in E$-LMC if for each $y \in X \cup \{1\}$, $\sigma(C_E(X), C_E(X)^*)$-closure of $\{f \circ \lambda_y \circ \lambda_x : x \in X \cup \{1\}\} \subseteq C_E(X)$, where $C_E(X)^*$ denote the topological dual of $C_E(X)$ and $\sigma(C_E(X), C_E(X)^*)$ denotes the weak topology induced on $C_E(X)$ by $C_E(X)^*$ and

$$\rho_x : X \longrightarrow X \text{ is given by } \rho_x(y) = yx,$$

$$\lambda_x : X \longrightarrow X \text{ is given by } \lambda_x(y) = xy.$$

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Theorem 3.2. *Let T be an E-compact Hausdorff right topological semigroup and $\phi : X \longrightarrow T$ be a continuous homomorphism such that $\lambda_{\phi(x)}$ is continuous for each $x \in X$. If $f \in C_E(T)$, then $f \circ \phi \in E - LMC(X)$.*

Proof. Let $t \in X \cup \{1\}$ and

$$g \in \sigma(C_E(X), C_E(X)^*) - cl\{f \cdot \lambda_y \cdot \rho_x : x \in X \cup \{1\}\}.$$

It suffices to show that there is some $a \in T$ such that for each $x \in X$, $f \cdot \lambda_{\phi(tx)}(a) = g(x)$, for then, if $t = 1$, $g = f \cdot \lambda_{\phi(a)} = f \cdot \rho_a \cdot \phi$ and if $t \in X$ $g = f \cdot \lambda_{\phi(t)} \cdot \rho_a \cdot \phi$ In either case, g is the composition of continuous functions.

Let $(x_\alpha)_{\alpha \in I}$ be an E-Z-net such that $(f \cdot \phi \cdot \lambda_t \cdot \rho_{X_a})_{\alpha \in I}$ converges to g. By taking a E-Z-subnet if necessary, we may assume $(\phi(X_\alpha))_{\alpha \in I}$ converges to some $a \in T$. Then for each $t \in X$, $\alpha \in I$,

$$f \cdot \lambda_{\phi_{(yt)}}(\phi(x_\alpha)) = \Pi_t(f \cdot \phi \cdot \lambda_y \cdot \rho_{x_\alpha}),$$

$$f \cdot \lambda_{\phi_{(yt)}}(a) = g(t) \text{ as required.}$$ □

Definition 3.3.

1. Define $e : X \longrightarrow \prod_{f \in E - LMC} E_f$, where $E_f = E$ for every $f \in E - LMC$, where $e(x)(f) = f(x)$.
2. $\delta_E X = cle[X]$.

Then $\delta_E X$ is E-compact and Hausdorff and $e[X]$ is dense in $\delta_E X$.

Theorem 3.4. *Let $f \in E - LMC$, $x \in X$. Then $f \circ \lambda_x \in E - LMC$.*

Proof. Let $t \in X \cup \{1\}$. Then

$$\{(f \cdot \lambda_x) \cdot \lambda_t \cdot \rho_y : y \in X \cup \{1\}\} = \{f \cdot \lambda_{xt} \cdot \rho_y : y \in X \cup \{1\}\}.$$ □

Definition 3.5. For $f \in E - LMC$, $\nu \in \delta_E X$, define $h_{\nu,f} : X \longrightarrow K$ by $h_{\nu,f}(x) = \nu(f \circ \lambda(x)$.

By Theorem 2.4 ν is defined at $f \circ \lambda_x$.

Theorem 3.6. *Let $f \in E - LMC$, $\nu \in \delta_E X$. Then $h_{\nu,f} \in E - LMC$.*

Proof. Let $t \in X \cup \{1\}$. We show that for each $x \in X \cup \{1\}$,$h_{\nu,f} \cdot \lambda_t \cdot \rho_x \in cl\{f \cdot \lambda_t \cdot \rho_s : s \in X \cup \{1\}\}$ so that $cl\{h_{\nu,f} \cdot \lambda_t \cdot \rho_x : x \in X \cup \{1\}\} \subseteq \{clf \cdot \lambda_t \cdot \rho_s : s \in X \cup \{1\}\} \subseteq C_E(X)$

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Let $x \in X \cup \{1\}$, F a finite subset of X, for each $y \in F$, U_Y be a neighbourhood of $h_{\nu,f} \cdot \lambda_t \cdot \rho_x(y)$ in K^X. Then, $U = \cap \Pi_{y^{-1}}(U_y)$ is a basic neighbourhood of $h_{\nu,f} \cdot \lambda_t \cdot \rho_x$ (here, $\Pi_y : K^X$ to K). Let $V = \bigcap_{y \in F} \Pi_{f \cdot \lambda_{tyx}}(U_y)$ (here $\Pi_{f \cdot \lambda_{tyx}} : \prod_{g \in E-LMC} E_g \longrightarrow K$, where $E_g = E$ for every $g \in E$-LMC).

Given $y \in F$, $h_{\nu,f} \cdot \lambda_t \cdot \rho_x(y) = h_{\nu,f}(tyx) = \nu(f \cdot \lambda_{tyx})$ so that V is a neighbourhood of ν. Pick $s \in X$ such that $e(s) \in V$. Then given $y \in F$,

$$f \cdot \lambda_t \cdot \rho_{xs}(y) = f(tyxs) = f \cdot \lambda_{tyx}(s) = e(s)(f \cdot \lambda_{tyx}) \in U_y.$$

Thus $f \cdot \lambda_t \cdot \rho_{xs} \in U$ as required. □

Definition 3.7. For $\mu, \nu \in \delta_E X$, define $\mu\nu \in K^{E-LMC}$ by $\mu\nu(f) = \mu(h_{\nu,f})$.

Theorem 3.8. *With the operation just defined, $\delta_E X$ is a right topological semigroup and for each $s \in X$, $\lambda_{e(s)}$ is continuous.*

Proof. We first show that for $\mu, \nu \in \delta_E X; \mu\nu \in \delta_E X$. Let F be a finite subset of E-LMC and for each $f \in F$, let F be a finite subset of E-LMC and for each $f \in F$, let U_f be a neighbourhood of $\mu\nu(f)$. Then $U = \cap \Pi_f^{-1}(U_f)$ is basic neighbourhood of $\mu\nu$. Now, given $f \in F, \nu(f.\lambda_s) = h_{\nu,f}(s) = e(s)(h_{\nu,f}) \in U_f$ so that $\cap \Pi^{-1}_{h\nu,f}(U_f)$ is a neighbourhood of ν. Pick $s \in X$ such that $e(s) \in \cap \Pi^{-1}_{h_{\nu,f}}(U_f)$. Then, given $f \in F, e(st)(f) = f(st) = (f \cdot \lambda_s)(t) = e(t)(f \cdot \lambda_s) \in U_f$ so that $e(st) \in U$ as required.

To see that the operation is associative, let $\mu, \nu, \eta \in \delta_E X$ and $f \in E$-LMC. Then, $(\mu\nu)\eta(f) = (\mu\nu)(h_{\mu,f}) = \mu(h_{\nu,h_{\eta,f}})$ and $\mu(\nu\eta)(f) = \mu(h_{\nu\eta,f})$, so it suffices to show $h_{\nu,h_{\eta,f}} = h_{\nu\eta,f}$.

Let $s \in X$. Then $h_{\nu,h_{\eta,f}}(s) = (h_{\eta,f} \cdot \lambda_s)$ and $h_{\nu\eta,f}(s) = \nu\eta(f \cdot \lambda_s) = \nu(h_{\eta,f} \cdot \lambda_s)$ so it suffices to show that

$$h_{\eta,f} \cdot \lambda_s = h_{\eta,f} \cdot \lambda_s.$$

So let $t \in X$. Then,

$$(h_{\eta,f} \cdot \lambda_s)(t) = h_{\eta,f}(st) = \eta(f \cdot \lambda_{st}) = \eta(f \cdot \lambda_s \cdot \lambda_t) = h_{\eta,f} \cdot \lambda_s(t).$$

To see that $\delta_E X$ is right topological, let $\nu \in \delta_E X$, $f \in E$-LMC and U open in K. Then $\Pi_f^{-1}(U) \cap \delta_E X$ is a sub basic open set in $\delta_E X$. $\Pi^{-1}_{h_{\nu,f}}(U) \cap \delta_E X = \rho_\nu^{-1}[\Pi_f^{-1}(U) \cap \delta_E X]$. □

Theorem 3.9. *The function $e : X \longrightarrow \delta_E X$ is a continuous homomorphism.*

Remark 3.10. $e : X \longrightarrow \delta_E X$ is a homeomorphism of X into $\delta_E X$ if and only if for every closed subset A of X and each $x \in X \setminus A$, there exists $f \in E - LMC$ such that $f(A) = 0$ and $f(x) \neq 0$.

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4. κ-Topologies on E-Compactifications

Definition 4.1. Let κ be a cardinal number. We define κ-topology as a topology for which the intersection of any family of open sets with no more than κ members is again open.

Definition 4.2. Let X be E-completely regular space. Define $V = \cap U_i$ where $\{U_i : i \in I\}$ is any family of E-open sets in X with $card I \leq \kappa$. Then a base of open sets for a κ-topology on X is provided by sets of the form V. This κ-topology on X is denoted by $\kappa_E X$ and its members are called κ_E-open sets.

Remark 3.3.

1. Suppose that Y is an E-completely regular space with any κ-topology and $f : Y \longrightarrow X$ is continuous, where X is E-completely regular. Then $f : Y \longrightarrow (X, \tau)$ is continuous where τ is the κ_E-topology on X.
2. Every E-open (E-closed) set in X is κ_E-open (κ_E-closed).
3. If κ is finite, then κ-topology is just the weak topology on X induced by $C_E(X)$, the original topology of X.

Result 4.4. *Let X be an E-completely regular Hausdorff space. Let κ be an infinite cardinal. Then every point has a base of E-open neighbourhoods in $\kappa_E X$ that are E-closed in X (and are E-closed in $\kappa_E X$).*

Proof. Let $G = \cap U_i$ be any basic κ_E-open neighbourhood of X with each U_i, E-open in X and $card\ I \leq \kappa$. For each i, choose inductively a sequence (W_i^n) of E-open neighbourhoods of x with $W_i^0 = U_i^0 \subset U_i$ and $clW_i^{n+1} \subseteq W_i^n$ for $n \geq 0$. Then, $\cap W_i^n = \cap clW_i^n$ is κ_E-open neighbourhood of x (since $card\ I \times \mathbb{N} \leq \kappa$) contained in G and is E-closed in X. □

Result 4.5. *For $E - LMC$ compactification $\delta_E X$ of X, κ, infinite and for $X \subseteq \wedge(\delta_E X)$, where X is dense in $\delta_E X$ and $card\ X \leq \kappa$, multiplication is continuous from $\delta_E X \times \kappa - \delta_E X$ to $\delta_E X$, where $\Lambda(\delta_E X)$ is the center of $\delta_E X$.*

Proof. Take $a, b \in \delta_E X$. Let U be an E-open neighbourhood of ab. Let U_0 be an E-open neighbourhood of ab with $clU_0 \subset U$. Using the fact that $\delta_E X$ is right topological, find an E-open neighbourhood V of a with $Vb \subset U_0$. Since $X \subseteq \wedge(\delta_E X)$ for each $x \in X \cap V$, we can find an E-open neighbourhood W_x of b with $xW_x \subseteq U_0$. Then $G = \cap\{W_x : x \in X \cap V\}$ is a κ_E-open neighbourhood of b. Since X is dense in $\delta_E X$, $X \cap V$ is dense in V. Therefore for $v \in V$ and $g \in G$, $vg \in cl(X \cap V).g \subseteq clU_0 \subseteq U$, using the fact that $\delta_E X$ is right topological. Thus $VG \subseteq U$ as required. □

Result 4.6. *Under the hypotheses above, $\kappa - \delta_E X$ is a topological semigroup.*

Proof. Let $a, b \in \delta_E X$ and E be a κ_E-open neighbourhood ab, say $E = \cap_{i\in I} U_i$ with each U_i, E-open in $\delta_E X$ and *card* $I \leq \kappa$. For each $i \in I$, use the result above to find an E-open neighbourhood V_i of a and κ_E-open neighbourhood G_i of b with $V_i G_i \subseteq U_i$.

Then $F = \cap V_i, G = \cap G_i$ are κ_E-open neighbourhood of a, b respectively with $FG \subseteq E$. □

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