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SOME REMARKS ON THE PRINCIPAL EIGENVALUE OF A SECOND ORDER ELLIPTIC OPERATOR

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Abstract: In this paper we report some new bound and some remarks on the principal eigenvalue of a second order elliptic operator defined in a general domain.

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1. Introduction

In this paper we deal with an uniformly elliptic operator L defined on an open and connex subset $\Omega \subseteq R^n$. Let $a_{ij} : \Omega \to R$ be continuous mappings, let $a_j : \Omega \to R$ and $c : \Omega \to R$ be bounded given mappings.

The elliptic operator has the following form

$$L = M + c(x) = \sum_{1 \le i,j \le n} a_{ij}(x)\partial_{ij} + \sum_{1 \le j \le n} b_j(x)\partial_j + c(x) \tag{1.1}$$

and satisfies the condition of uniform ellipticity, i.e. there exist two positive constants C_0 and c_0 such that

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$$c_0|\xi|^2 \leq \sum_{1\leq i,j\leq n} a_{ij}(x)\xi_i\xi_j \leq C_0|\xi|^2 \tag{1.2}$$

for all $\xi \in R^n$.

Moreover we assume that $a_{ij} \in C(\Omega)$, $b_j, c \in L^\infty(\Omega)$ and that there exists $b > 0$ such that for all $x \in \Omega$

$$\sum_{1\leq j\leq n} b_j^2 \leq b^2 \quad \text{and} \quad |c(x)| \leq b. \tag{1.3}$$

Definition 1. (Classical) A complex number λ and a function $\psi : \Omega \to C$ are respectively an eigenvalue and an eigenfunction for the Dirichlet problem relatively to the operator $-L$ if one has

$$\begin{cases} (L+\lambda)\psi = 0 & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.4}$$

Remark 1. In the special case where L is a selfadjoint operator with respect to $L^2(\Omega)$, i.e. L has the following form

$$L = \sum_i \partial_i(\sum_j a_{ij}(x)\partial_j) + c(x), \tag{1.5}$$

then all the eigenvalues are real. On the other hand for a general uniformly elliptic operator L, as before introduced, the eigenvalue values are not necessarily real. Nevertheless it has been proved that, assuming that the coefficients of L and the boundary of Ω to be smooth, then there exists a principal eigenvalue $\lambda_1 \in R$ and a principal eigenfunction $\psi_1 : \Omega \to R$ such that the following properties are satisfyed:

a) the eigenfunction ψ_1 is simple;

b) if Ψ is a positive eigenfunction with eigenvalue λ, then one has $\lambda = \lambda_1$;

c) if λ is an eigenvalue, then one has $Re(\lambda) \geq \lambda_1$.

The proof of the existence of pair (λ_1, ψ_1) can be obtained by means of classical Krein-Rutman Theorem.

The aim of the paper of H. Berestycki, L. Niremberg and S.R. Varadhan is to extend the definition of principal eigenvalue to a general bounded, open and connected domain, without a regular boundary. To obtain such generalization we suggest as a start point the following remark.

Remark 2. In general the pair $\lambda_0 = -b$, $\phi_0(x) = 1$ is not a pair of the type eigenvalue and eigenfunction for the operator $-L$, but in any case for all operators L the following inequality

$$(L + \lambda_0)\phi_0 \leq 0 \quad \text{in} \quad \Omega \tag{1.6}$$

is satisfyed. It is immediate to verify that

$$L\phi_0 + \lambda_0\phi_0 = c(x) - b \le 0. \tag{1.7}$$

Hence for all operators L the subset $\Lambda(\Omega)$ of R of all λ, such that there exists ϕ positive in Ω such that $(L+\lambda)\phi \le 0$, is not empty.

Definition 2. (Generalized) We call principal eigenvalue and we denote by $\lambda_1(\Omega)$ the real number defined in the following way

$$\lambda_1(\Omega) = \sup \Lambda = \sup\{\lambda \in R |\ \text{there exists}\ \phi > 0 \quad \text{in} \quad \Omega : (L+\lambda)\phi \le 0\}. \tag{1.8}$$

Remark 3. It follows at once, from Definition 2, the following characterization of $\lambda_1(\Omega)$

$$\lambda_1(\Omega) = \sup_{\phi} \inf_{\Omega} (-\frac{L\phi}{\phi}). \tag{1.9}$$

Proposition 1. *If $\Omega_1 \subseteq \Omega_2$ are two open subset of R^n then $\lambda_1(\Omega_2) \le \lambda_1(\Omega_1)$.*

This result follows directly from Definition 2. Conversely,

Remark 4. If $\Omega_1 \subset \Omega_2$ then $\lambda_1(\Omega_2) < \lambda_1(\Omega_1)$. This result, which is not immediate, is a consequence of a theorem of BNV (see Theorem 2.4).

The Proposition 2 below shows that (1.9) has been obtained in classical context and that Definition 2 is a good generalization of standard one.

Proposition 2. *If $\phi \in C^2(\Omega)$, with $\phi > 0$ in $\overline{\Omega}$ and λ is an eigenvalue for the Dirichlet problem of operator $-L$ on Ω then*

$$Re(\lambda) \ge \inf_{\Omega} (-\frac{L\phi}{\phi}). \tag{1.10}$$

Proof. We argue by contradiction. Let $\lambda < \inf_{\Omega}(-\frac{L\phi}{\phi})$, we obtain that λ is not an eigenvalue. We prove that if u solves the problem of Dirichlet

$$\begin{cases} (L+\lambda)u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

$u = 0$ in Ω. Setting $v = \frac{u}{\phi}$, we have $u = v\phi$ and then the boundary value problem becomes

$$\sum_1^n \phi a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_1^n (\phi b_i + 2\sum_{j=1}^n a_{ij}) \frac{\partial \phi}{\partial x_j} \frac{\partial v}{\partial x_i} + (L\phi + \lambda\phi) v = 0 \tag{1.11}$$

and

$$v = 0 \quad \text{on} \quad \partial\Omega. \tag{1.12}$$

The maximum principle is valid since, for the absurd assumption, one has $L\phi+\lambda\phi \leq 0$, hence $v = 0$ which implies $u = 0$.

If λ is not a real number, we multiply the previous calculus by $\overline{v}$ and we take the real parts obtaining the inequality

$$Re(\lambda) \geq \inf_{\Omega}(-\frac{L\phi}{\phi}). \tag{1.10}$$

2. A New Upper Bound for the Principal Eigenvalue $\lambda_1(\Omega)$

We recall that in the special case where L is the Laplacian Δ and Ω is the open ball $B_r = \{x \in R^n : |x| < r\}$, if $\lambda_1(r) = \lambda_1(B_r)$ denotes the corresponding first eigenvalue, then the following formula

$$\lambda_1(r) = \frac{\lambda_1(1)}{r^2} \tag{2.1}$$

holds.

Indeed, for $r = 1$, let $u(x)$ be an eigenfunction for the corresponding eigenvalue $\lambda_1(1)$, i.e.

$$\begin{cases} -\Delta u(x) = \lambda_1(1)u(x) & |x| < 1 \quad \text{in} \quad \Omega, \\ u(x) = 0 & |x| = 1. \end{cases} \tag{2.2}$$

Now note that the mapping $u_r(x) = u(rx)$ satisfies

$$-\Delta u_r(x) = r^{-2}\lambda_1(1)u_r(x) \quad |x| < r, \tag{2.3}$$

and by definition it must be

$$-\Delta u_r(x) = \lambda_1(r)u_r(x) \quad |x| < r. \tag{2.4}$$

Equating the right hand side of (2.3) and (2.4) we get formula (2.1).

The following lemma is reported in [2].

Lemma 1. (BNV) *Assume that the operator L satisfies* (1.1), (1.2) *and* (1.3) *and is defined on the open set $\Omega \subseteq R^n$ containing the ball $B = B(0, R) = \{x : |x| < R\}$ with $R \leq 1$. Then the following bound*

$$\lambda_1(\Omega) \leq \frac{C}{R^2} \tag{2.5}$$

holds, where the constant C depends only upon n, c_0, C_0 and b.

Now we give the following generalization of the previous lemma.

Lemma 2. *Assume that the operator L satisfies* (1.1), (1.2) *and* (1.3) *and is defined on the open set $\Omega \subseteq R^n$ containing the ball $B(a,R) = \{x : |x-a| < R\}$ where $a \in \Omega$ and with $R \leq 1$. Then the following bound*

$$\lambda_1(\Omega) \leq \frac{C}{R^2} \tag{2.6}$$

holds where the constant C depends only upon n, c_0, C_0 and b.

Proof. Let us consider the ball $B(a,r) = \{x : |x-a| < r\}$, where $2r = R$ and the function $\sigma = 1/4(r^2 - |x-a|^2)^2$. The partial derivative with respect to x_i is given by

$$\sigma_i = -(r^2 - |x-a|^2)(x_i - a_i) \tag{2.7}$$

and the derivative od second order with respect to x_i and x_j is given

$$\sigma_{ij} = -(r^2 - |x-a|^2)\delta_{ij} + 2(x_i - a_i)(x_j - a_j), \tag{2.8}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Computing $\frac{-L\sigma}{4\sigma}$ we get

$$\begin{aligned} \frac{-L\sigma}{4\sigma} &= (r^2 - |x-a|^2)^{-1} \sum_i a_{ii} - (r^2 - |x-a|^2)^{-2} 2(x_i - a_i)(x_j - a_j) \\ &\quad - (r^2 - |x-a|^2)^{-1} b_i (x_i - a_i) \leq (r^2 - |x-a|^2)^{-1}(nc_0 + br) \\ &\quad - (r^2 - |x-a|^2)^{-2}(2c_0|x-a|^2) + b. \end{aligned} \tag{2.9}$$

Now if in the previous line the second term is greater than the first we get

$$\frac{-L\sigma}{4\sigma} \leq b, \tag{2.10}$$

this happen when we consider the set of points satisfying the following definition

$$\leq (r^2 - |x-a|^2)^{-1}(nc_0 + br) \leq (r^2 - |x-a|^2)^{-2}(2c_0|x-a|^2) \tag{2.11}$$

or

$$nC_0 + br) \leq (r^2 - |x-a|^2)^{-1}(2c_0|x-a|^2) \tag{2.12}$$

or

$$2c_0|x-a|^2 \geq nC_0 r^2 - nC_0|x-a|^2 + br^3 - br|x-a|^2 \tag{2.13}$$

or, equivalently the following condition

$$(2c_0 + nC_0 + br)|x-a|^2 \geq r^2(nC_0 + br). \tag{2.14}$$

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On the other hand in the complementary set $\{x : |x - a| < qr\}$ where $q^2 = \frac{nC_0+br}{2c_0+nC_0+br} < 1$ which is a subset of $B(a,r)$ we deduce

$$\frac{-L\sigma}{4\sigma} \leq \frac{(nC_0 + br)(2c_0 + nC_0 + br)}{2c_0 r^2} + b. \tag{2.15}$$

Example. We give the following example of application of our lemma to obtain a sharper estimates. Let $n = 2$ and $\Omega = \{(x_1, x_2) : -\frac{1}{2} < x_1 < \frac{3}{2}\}$. Then $B(0, \frac{1}{2}) \subset \Omega$ and $B(\frac{1}{2}, 1) \subset \Omega$. From Lemma BNV we get the bound

$$\lambda_1(\Omega) \leq \frac{C}{\frac{1}{4}} = 4C. \tag{2.16}$$

On the other hand, from our lemma we get the sharper bound

$$\lambda_1(\Omega) \leq \frac{C}{1} = C. \tag{2.17}$$

Theorem. *The operator L admits a principal eigenvalue with a positive eigenfunction.*

Proof. Since Ω is a nonempty open subset of R^n, then there exist $a \in \Omega$ and $R > 0$ such that the ball $B(a, 2R)$ is contained in Ω. Hence $\overline{B(a,R)}$ is contained in Ω. Fix $\delta > 0$, let K be a closed subset of Ω such that: $U = \Omega - K$ has measure $|U| < \delta$, $\overline{B(a,R)}$ is included in Ω and $x_0 \in K$. Let Ω_j be a sequence of expanding subdomains of Ω satisfying the following conditions

$$K \subset \Omega_j \subset \overline{\Omega_j} \subset \Omega_{j+1} ... \subset \Omega, \quad \cup\Omega_j = \Omega. \tag{2.20}$$

Since the coefficients a_{ij} are continuous on the closure of Ω_j, by standard theory there exist a principal eigenvalue μ_j and an unique eigenfunction $\phi_j > 0$ on Ω such that

$$L\phi_j + \mu_j \phi_j = 0 \quad \text{in} \quad \Omega_j, \tag{2.21}$$

$$\phi_j = 0 \quad \text{on} \quad \partial\Omega_j, \quad \phi_j(x_0) = 1. \tag{2.22}$$

Since $\phi_{j+1} > 0$ on the closure of Ω_j, the maximum principle implies

$$\mu_j > \mu_{j+1}. \tag{2.23}$$

Arguing by contradiction, we deduce that sequence μ_j must be convergent, set $\mu = \lim_{j\to+\infty} \mu_j$.

Harnack inequality says that if $u > 0$ solve the equation

$$Lu = 0, \tag{2.24}$$

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then for any closed subset K of Ω there is a constant $C_1 = C_1(c_0, C_0, b, K)$ such that

$$\max_{x\in K} u(x) \leq C_1 \min_{x\in K} u(x). \tag{2.25}$$

Here we assume, for all $j > 0$, $b \geq |c(x) + \mu_j|$ and the domain $\Omega = \Omega_1$. Applying previous inequality to the case where $u = \phi_j > 0$ solves the equation

$$(L + \mu_j)\phi_j = 0, \tag{2.26}$$

then we get

$$\max_{x\in K} \phi_j \leq C_1 \min_{x\in k} \phi_j \leq C_1\phi_j(x_0) = C_1. \tag{2.27}$$

Inequality $\max_{x\in K} \phi_j \leq C_1$ implies $\phi_j \leq C_1$ and

$$(M - c^-)v_j \geq -b\phi_j - \mu_j\phi_j. \tag{2.28}$$

Since the domain Ω_j contains the closure of the ball $B(a, R)$, applying our Lemma we deduce that the any eigenvalue μ_j satisfies the following estimates

$$\mu_j \leq \frac{C}{R^2}. \tag{2.29}$$

Let $U_j = \Omega_j - K$, we consider the maximum of ϕ_j on the closure of U_j. Alexandroff inequality enables us to write

$$\max \phi_j - C_1 \leq (b + \frac{C}{R^2})B \max \phi_j \delta^{1/n}. \tag{2.30}$$

Now taking δ such that the following inequality

$$(b + \frac{C}{R^2})B\delta^{1/n} \leq \frac{1}{2} \tag{2.31}$$

holds, then we obtain

$$\max \phi_j - C_1 \leq \frac{1}{2} \max \phi_j \tag{2.32}$$

and finally

$$\max \phi_j \leq 2C_1. \tag{2.33}$$

Define for any $j \geq 1$

$$\psi_j : \Omega \to R \tag{2.34}$$

by

$$\begin{cases} \psi_j = \phi_j & \text{on} \quad \Omega_j, \\ \psi_j = 0 & \text{on} \quad \Omega - \Omega_j. \end{cases} \tag{2.35}$$

Standard interior $W^{2,p}(\Omega)$ estimates yield that for any $k \geq j + 1$

$$\|\psi_k\|_{W^{2,p}} \leq C(j). \tag{2.37}$$

Since $W^{2,p}(\Omega)$ is a reflexive Banach space, any ball is weakly compact, thus there exists a subsequence ψ_{j_k} and ψ_j converging to a positive function ψ in Ω such that $\psi_{j_k} \to \psi$ weakly in $W^{2,p}$ and strongly in $W^{1,+\infty}$ in every compact subset of Ω.

Clearly ψ satisfies

$$\begin{cases} L\psi + \mu\psi = 0 \quad \text{in} \quad \Omega, \\ \psi(x_0) = 1, \end{cases} \tag{2.38}$$

and

$$\psi \leq 2C_1 \quad \text{on} \quad \Omega. \tag{2.40}$$

In view of Definition 2 (generalized) we remark that equation (2.38) implies that $\mu \in \Lambda$ and hence $\Lambda \neq \emptyset$.

Since $\Omega_1 \subset \Omega$ then Λ is bounded from above and thus $\lambda_1 = \sup \Lambda$ exists as real number.

Now necessarily the following inequality $\mu \leq \lambda_1$ holds. On the other hand we have $\mu \geq \lambda_1$, letting $j \to +\infty$ in the inequality $\mu_j > \lambda_1$, in view of inclusion $\overline{\Omega_j} \subset \Omega$.

Finally we get $\lambda_1 = \mu$. □

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