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A CERTAIN SUBCLASS OF MEROMORPHICALLY MULTIVALENT FUNCTIONS ASSOCIATED WITH A LINEAR OPERATOR

K. Suchithra[1], T. Thulasiram[2 §], R. Sattanathan[3], B. Adolf Stephen[4]

[1,2]Department of Mathematics
A.M. Jain College
Meenambakkam, Chennai, Tamilnadu, 600 114, INDIA
[1]e-mail: suchithravenkat@yahoo.co.in
[2]e-mail: ltthulasiram@gmail.com

[3]Department of Mathematics
D.G. Vasihnav College
Arumbakkam, Chennai, Tamilnadu, 600 106, INDIA
e-mail: rsattanathan@gmail.com

[4]Department of Mathematics
Madras Christian College
Tambaram, Chennai, Tamilnadu, 600059, INDIA
e-mail: adolfmcc2003@yahoo.co.in

Abstract: We introduce the class $\mathcal{M}^*(n, p, \lambda, \alpha)$ of meromorphically mutlivalent functions with negative coefficients by making use of a familiar analogue of Ruschewyeh derivative. The aim of the present paper is to obtain various properties and characteristics of this class. We also derive many interesting results for the Hadamard product of functions belonging to the class $\mathcal{M}^*(n, p, \lambda, \alpha)$.

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1. Introduction

Let Σ_p denote the class of functions of the form

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§Correspondence author

$$f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k \quad (a_k \geq 0, p \in \mathbb{N} = \{1, 2, \dots\}) \tag{1}$$

which are analytic and multivalent in the punctured open unit disc

$$\Delta^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \Delta \backslash \{0\}.$$

For functions $f(z)$ in the class Σ_p, we define an analogue of the familiar Ruscheweyh derivative [10] by

$$\begin{aligned} D^{n+p-1} f(z) &= \frac{1}{z^p (1-z)^{n+p}} * f(z) \ (n > -p,\ p \in \mathbb{N},\ f \in \Sigma_p) \\ &= \frac{1}{z^p} \left(\frac{z^{n+2p-1} f(z)}{(n+p-1)!} \right)^{n+p-1} \\ &= \frac{1}{z^p} - \sum_{k=p}^{\infty} \delta(n,k) a_k z^k, \end{aligned}$$

where here and in what follows n is assumed to be an integer $(> -p)$ and $f \in \Sigma_p$, $\delta(n,k) = \binom{k+n+2p-1}{n+p-1}$.

The linear operator D^{n+p-1} was studied by many researchers including Joshi and Srivastava [8] and Liu and Owa [7].

Further, various subclasses of Σ_p were studied by Aouf et al [1, 2, 3], Srivastava et al [12], Uralegaddi and Ganigi [15] and Chen et al [4]. Motivated by the works of Kamali et al [9], we aim at investigating various properties and characteristics of a new class $\mathcal{M}^*(n,p,\lambda,\alpha)$ of meromorphically multivalent functions, which is given as follows.

Definition 1. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$ if

$$\Re \left[\frac{(\lambda(p+1)-1) z (D^{n+p-1} f(z))' + \lambda z^2 (D^{n+p-1} f(z))''}{(D^{n+p-1} f(z))} \right] > \alpha \tag{2}$$

$$(0 \leq \alpha < p, p \in \mathbb{N})$$

for some suitably restricted real parameter λ.

We note that

(i) $\mathcal{M}^*(0,p,\lambda,\alpha) = \Sigma^*_{\lambda,p,\alpha}$ studied by Kamali et al [9].

(ii) $\mathcal{M}^*(0,p,0,\alpha) = \Sigma^*_p(\alpha) = \{f(z) \in \Sigma_p : \Re\left(\frac{-zf'(z)}{f(z)}\right) > \alpha, z \in \Delta, 0 \leq \alpha < p\}$, where $\Sigma^*_p(\alpha)$ is the class of meromorphically p-valent starlike functions of order α $(0 \leq \alpha < p)$.

(iii) $\mathcal{M}^*(0,1,\lambda,\alpha)$ is closely related to the class studied by Suchithra et al [14].

2. Coefficient Estimates and Closure Theorem

We first establish a necessary and sufficient condition for a function $f(z)$ given by (1) to be in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$ and obtain the coefficient estimates, by restricting the parameter λ as $\lambda > \frac{1}{p}$.

Theorem 2. *A function $f(z)$ given by (1) is in $\mathcal{M}^*(n,p,\lambda,\alpha)$ if and only if*

$$\sum_{k=p}^{\infty} \delta(n,k)\{k[\lambda(p+k)-1]-\alpha\}a_k \leq p-\alpha, \tag{3}$$

where $0 \leq \alpha < p$, $\lambda > \frac{1}{p}$, $p \in \mathbb{N}$.

Proof. Suppose $f(z)$ is in $\mathcal{M}^*(n,p,\lambda,\alpha)$. Then from (1) and (2),

$$\Re\left[\frac{(\lambda(p+1)-1)z(D^{n+p-1}f(z))' + \lambda z^2(D^{n+p-1}f(z))''}{(D^{n+p-1}f(z))}\right] > \alpha$$

$$(0 \leq \alpha < p, \lambda > \tfrac{1}{p})$$

or equivalently,

$$\Re\left[\frac{p - \sum_{k=p}^{\infty}\delta(n,k)[k(\lambda(p+k)-1)]a_k z^{k+p}}{1-\sum_{k=p}^{\infty}\delta(n,k)a_k z^{k+p}}\right] > \alpha.$$

Choosing z to be real and letting $z \to 1^-$ through real values, we obtain

$$\frac{p - \sum_{k=p}^{\infty}\delta(n,k)k(\lambda(p+k)-1)a_k}{1-\sum_{k=p}^{\infty}\delta(n,k)a_k} \geq \alpha$$

which implies

$$\sum_{k=p}^{\infty}\delta(n,k)\{k[\lambda(p+k)-1]-\alpha\}a_k \leq p-\alpha, \quad \text{proving (3).}$$

Conversely, assume that (3) is true.

Then if we let $z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}$, we find that

$$\left|\frac{(\lambda(p+1)-1)z(D^{n+p-1}f(z))' + \lambda z^2(D^{n+p-1}f(z))''}{(D^{n+p-1}f(z))} - p\right|$$
$$\leq \frac{\sum_{k=p}^{\infty}\{k[\lambda(k+p)-1]-p\}\delta(n,k)a_k|z|^{k+p}}{1-\sum_{k=p}^{\infty}\delta(n,k)a_k|z|^{k+p}}$$
$$\leq \frac{(p-\alpha)+\sum_{k=p}^{\infty}\alpha a_k\delta(n,k)|z|^{k+p} - \sum_{k=p}^{\infty}pa_k\delta(n,k)|z|^{k+p}}{1-\sum_{k=p}^{\infty}\delta(n,k)a_k|z|^{k+p}}$$

$$= (p-\alpha);\ (0 \le \alpha < p,\ \lambda > \frac{1}{p},\ p \in \mathbb{N}).$$

Hence, by maximum modulus theorem, we have $f(z) \in \mathcal{M}^*(n,p,\lambda,\alpha)$.

This completes the proof of Theorem 2. □

Corollary 3. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$. Then*

$$a_k \le \frac{p-\alpha}{\delta(n,k)\{k[\lambda(p+k)-1]-\alpha\}}, \qquad \text{for } k \ge p,\ p \in \mathbb{N}. \tag{4}$$

The equality in (4) is attained for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} - \frac{p-\alpha}{\delta(n,k)\{k[\lambda(p+k)-1]-\alpha\}} z^k, \quad k \ge p,\ p \in \mathbb{N}. \tag{5}$$

Theorem 4. *The class $\mathcal{M}^*(n,p,\lambda,\alpha)$ is closed under convex linear combinations.*

Proof. Let each of the functions $f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k$ and $g(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} b_k z^k$ be in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$.

It is sufficient to show that the function defined by $h(z) = (1-\gamma)f(z) + \gamma g(z)$, $0 \le \gamma \le 1$ is also in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$.

Since

$$\begin{aligned} h(z) &= (1-\gamma)\left[\frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k\right] + \gamma\left[\frac{1}{z^p} - \sum_{k=p}^{\infty} b_k z^k\right] \\ &= \frac{1}{z^p} - \sum_{k=p}^{\infty}[(1-\gamma)a_k + \gamma b_k] z^k, \quad 0 \le \gamma \le 1. \end{aligned}$$

With the aid of Theorem 2, we have

$$\begin{aligned} &\sum_{k=p}^{\infty} \delta(n,k)\{k(\lambda(p+k)-1)-\alpha\}[(1-\gamma)a_k + \gamma b_k] \\ &\quad = (1-\gamma)\sum_{k=p}^{\infty} \delta(n,k)\{k[\lambda(p+k)-1]-\alpha\} a_k \\ &\qquad + \gamma \sum_{k=p}^{\infty}\{k[\lambda(p+k)-1]-\alpha\}\delta(n,k) b_k \\ &\quad \le (1-\gamma)(p-\alpha) + \gamma(p-\alpha) \\ &\quad = p-\alpha;\quad (0 \le \alpha < p;\ p \in \mathbb{N}), \end{aligned}$$

which shows that $h(z) \in \mathcal{M}^*(n,p,\lambda,\alpha)$.

Hence Theorem 4 is proved. □

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3. Distortion Theorem

In this section, we prove the following growth and distortion theorem for the class $\mathcal{M}^*(n,p,\lambda,\alpha)$.

Theorem 5. *If a function $f(z)$ defined by (1) is in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$, then*

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!}\left[\frac{p-\alpha}{\delta(n,p)[p(2\lambda p-1)-\alpha]}\right] r^{2p}\right\} r^{-(p+m)} \leq |f^{(m)}(z)|$$
$$\leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!}\left[\frac{p-\alpha}{\delta(n,p)[p(2\lambda p-1)-\alpha]}\right] r^{2p}\right\} r^{-(p+m)}, \tag{6}$$

$(0<|z|=r<1; 0\leq\alpha<p, \lambda>\frac{1}{p}, p\in\mathbb{N}, p>m)$.

The result is sharp for the function given by

$$f(z) = \frac{1}{z^p} - \frac{p-\alpha}{\delta(n,p)[p(2\lambda p-1)-\alpha]} z^k, \ k\geq p, \ p\in\mathbb{N}. \tag{7}$$

Proof. In view of Theorem 2, we have

$$\sum_{k=p}^{\infty} \delta(n,k)[k(\lambda(p+k)-1)-\alpha]a_k \leq p-\alpha.$$

$$\Rightarrow \delta(n,p)\left[\frac{p(2\lambda p-1)-\alpha}{p!}\right]\sum_{k=p}^{\infty} k!a_k \leq \sum_{k=p}^{\infty}\delta(n,k)\frac{[k(\lambda(p+k)-1)\alpha]}{k!}k!a_k$$
$$\leq p-\alpha$$

which gives

$$\sum_{k=p}^{\infty} k!a_k \leq \frac{p!(p-\alpha)}{\delta(n,p)[p(2\lambda p-1)-\alpha]}. \tag{8}$$

Now, by differentiating both sides of (1) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m \frac{[p+m-1]!}{(p-1)!} z^{-(p+m)} - \sum_{k=p}^{\infty}\frac{k!}{(k-m)!}a_k z^{k-m}, \quad p\in\mathbb{N}, \ p>m, \tag{9}$$

and Theorem 5 follows easily from (8) and (9).

Further it is easy to see that the equality in (6) are attained for the function $f(z)$ given by (7). □

4. Radii of Meromorphically Multivalent Starlikeness and Convexity

In this section, we determine the radii of meromorphically multivalent starlikeness of order δ $(0 \leq \delta < p)$ and meromorphically multivalent convexity of order δ $(0 \leq \delta < p)$ for functions in the classs of $\mathcal{M}^*(n, p, \lambda, \alpha)$.

Theorem 6. *Let the function $f(z)$ defined by* (1) *be in the class $\mathcal{M}^*(n, p, \lambda, \alpha)$ then:*

(i) $f(z)$ is meromorphically multivalent starlike of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_1$. That is,

$$\Re\left\{\frac{-zf'(z)}{f(z)}\right\} > \delta, \ (|z| < r_1;\ 0 \leq \delta < p;\ p \in \mathbb{N}),$$

where

$$r_1 = \inf_{k \geq p}\left\{\left(\frac{p-\delta}{k+\delta}\right)\frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]}{p-\alpha}\right\}^{1/k+p}. \tag{10}$$

(ii) $f(z)$ is meromorphically multivalent convex of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_2$. That is,

$$\Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \delta, \ (|z| < r_2;\ 0 \leq \delta < p;\ p \in \mathbb{N}),$$

where

$$r_2 = \inf_{k \geq p}\left\{\left(\frac{p(p-\delta)}{k(k+\delta)}\right)\frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]}{p-\alpha}\right\}^{1/k+p}. \tag{11}$$

Each of these results are sharp for the function $f(z)$ given by (5).

Proof. (i) From the definition of $f(z)$ given by (1), we get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq \frac{\sum_{k=p}^{\infty}(k+p)a_k|z|^{k+p}}{2(p-\delta)-\sum_{k=p}^{\infty}(k-p+2\delta)a_k|z|^{k+p}}.$$

Thus, we have the desired inequality:

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq 1 \quad (0 \leq \delta < p, p \in \mathbb{N})$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\delta}{p-\delta}\right)a_k|z|^{k+p} \leq 1. \tag{12}$$

Hence, by Theorem 2, (12) will be true if

$$\left(\frac{k+\delta}{p-\delta}\right)|z|^{k+p} \leq \frac{\delta(n,k)\{k(\lambda(p+k)-1)-\alpha\}}{p-\alpha}, \quad \left(\lambda > \frac{1}{p}, k \geq p, p \in \mathbb{N}\right). \tag{13}$$

The inequality (13) lead us immediately to the disc $|z| < r_1$, where r_1 is given by (10).

(ii) In order to prove the second assertion of Theorem 6, we find from the definition of $f(z)$ given by (1) that

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}+p}{1+\dfrac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq \frac{\sum_{k=p}^{\infty} k(k+p)a_k|z|^{k+p}}{2p(p-\delta)-\sum_{k=p}^{\infty}(k^2-kp+2k\delta)a_k|z|^{k+p}}.$$

Thus we have

$$\left|\frac{1+\dfrac{zf''(z)}{f'(z)}+p}{1+\dfrac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq 1 \quad (0 \leq \delta < p,\ p \in \mathbb{N}),$$

if

$$\sum_{k=p}^{\infty} \frac{k(k+\delta)}{p(p-\delta)} a_k|z|^{k+p} \leq 1. \tag{14}$$

Hence, by Theorem 2, (14) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p} \leq \frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]}{p-\alpha} \quad \left(k \geq p, p \in \mathbb{N}, \lambda > \frac{1}{p}\right). \tag{15}$$

This inequality (15) readily yields the disc $|z| < r_2$ where r_2 is given by (11). Clearly the results are sharp for the function $f(z)$ defined by (5), which completes the proof of Theorem 6.

□

5. Convolution Properties

Let

$$f_j(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geq 0, j = 1, 2, \ldots, p \in \mathbb{N}). \tag{16}$$

Then the convolution (or Hadamard product) $(f_1 * f_2)(z)$ of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_{k,1} a_{k,2} z^k.$$

Theorem 7. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by* (16) *be in the class $\mathcal{M}^*(n, p, \lambda, \alpha)$. Then $(f_1 * f_2)(z) \in \mathcal{M}^*(n, p, \lambda, \gamma)$, where*

$$\gamma = p - \frac{2p(p-\alpha)^2(\lambda p - 1)}{\delta(n,p)[p(2\lambda p - 1) - \alpha]^2 - (p-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = \frac{1}{z^p} - \frac{(p-\alpha)}{\delta(n,p)\{p(2\lambda p - 1) - \alpha\}} z^p \qquad (j = 1, 2, p \in \mathbb{N}). \tag{17}$$

Proof. Since $f_j(z) \in \mathcal{M}^*(n, p, \lambda, \alpha)$, we have

$$\sum_{k=p}^{\infty} \frac{\delta(n,k)[k(\lambda(p+k) - 1) - \alpha]}{p - \alpha} a_{k,j} \leq 1.$$

Using Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{\delta(n,k)[k(\lambda(p+k) - 1) - \alpha]}{p - \alpha} \sqrt{a_{k,1} a_{k,2}} \leq 1.$$

We have to find largest γ such that

$$\sum_{k=p}^{\infty} \frac{\delta(n,k)[k(\lambda(p+k) - 1) - \gamma]}{p - \gamma} a_{k,1} a_{k,2} \leq 1. \tag{18}$$

(18) is satisfied if

$$\frac{\delta(n,k)[k(\lambda(p+k) - 1) - \gamma]}{p - \gamma} a_{k,1} a_{k,2} \leq \frac{\delta(n,k)[k(\lambda(p+k) - 1) - \alpha]}{p - \alpha} \sqrt{a_{k,1} a_{k,2}}$$

or if

$$\sqrt{a_{k,1} a_{k,2}} \leq \left(\frac{p - \gamma}{p - \alpha}\right) \left[\frac{k[\lambda(p+k) - 1] - \alpha}{k[\lambda(p+k) - 1] - \gamma}\right].$$

But

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{p-\alpha}{k(\lambda(p+k)-1)-\alpha}.$$

Thus (18) will be satisfied if

$$\frac{p-\alpha}{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]} \leq \left(\frac{p-\gamma}{p-\alpha}\right)\left[\frac{k[\lambda(p+k)-1]-\alpha}{k[\lambda(p+k)-1]-\gamma}\right]$$

or if

$$\gamma \leq p - \frac{(p-\alpha)^2[k(\lambda(p+k)-1)-p]}{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]^2-(p-\alpha)^2}, (k \geq p).$$

The right hand side of this inequality is an increasing function of k.

Setting $k = p$, we get

$$\gamma \leq p - \frac{(p-\alpha)^2[p(2\lambda p-1)-p]}{\delta(n,p)[p(2\lambda p-1)-\alpha]^2-(p-\alpha)^2}.$$

Thus $p - \frac{2p(p-\alpha)^2(\lambda p-1)}{\delta(n,p)(p(2\lambda p-1)-\alpha)^2-(p-\alpha)^2}$ is the largest γ such that (18) is true, which implies that $(f_1 * f_2)(z) \in \mathcal{M}^*(n,p,\lambda,\gamma)$ there by completing the proof of Theorem 7. □

Theorem 8. *Let $f_1(z)$ be in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$. Suppose if $f_2(z) \in \mathcal{M}^*(n,p,\lambda,\beta)$, then $(f_1 * f_2)(z) \in \mathcal{M}^*(n,p,\lambda,\xi)$ where*

$$\xi = p - \frac{2p(p-\alpha)^2(p-\beta)^2(\lambda p-1)}{\delta(n,p)[p(2\lambda p-1)-\alpha]^2[p(2\lambda p-1)-\beta^2]-(p-\alpha)^2(p-\beta)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ given by

$$f_1(z) = \frac{1}{z^p} - \frac{p-\alpha}{\delta(n,p)[p(2\lambda p-1)-\alpha]}z^p$$

and

$$f_2(z) = \frac{1}{z^p} - \frac{p-\beta}{\delta(n,p)[p(2\lambda p-1)-\beta]}z^p.$$

Proof. If $f_1(z) \in \mathcal{M}^*(n,p,\lambda,\alpha)$ and $f_2(z) \in \mathcal{M}^*(n,p,\lambda,\beta)$, then

$$\sum_{k=p}^{\infty} \frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]}{p-\alpha}a_{k,1} \leq 1$$

and

$$\sum_{k=p}^{\infty} \frac{\delta(n,k)[k(\lambda(p+k)-1)-\beta]}{p-\beta}a_{k,2} \leq 1.$$

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By Cauchy-Schwarz inequality

$$\sum_{k=p}^{\infty}\delta(n,k)\left[\frac{[k(\lambda(p+k)-1)-\alpha]}{p-\alpha}\right]\left[\frac{[k(\lambda(p+k)-1)-\beta]}{p-\beta}\right]\sqrt{a_{k,1}a_{k,2}}\leq 1.$$

We have to find a largest ξ so that

$$\sum_{k=p}^{\infty}\frac{\delta(n,k)[k(\lambda(p+k)-1)-\xi]}{p-\xi}a_{k,1}a_{k,2}\leq 1. \tag{19}$$

(19) is satisfied if,

$$\sqrt{a_{k,1}a_{k,2}}\leq\left[\frac{p-\xi}{(p-\alpha)(p-\beta)}\right]\left[\frac{(k(\lambda(p+k)-1)-\alpha)(k(\lambda(p+k)-1)-\beta)}{(k(\lambda(p+k)-1)-\xi)}\right].$$

But

$$\sqrt{a_{k,1}a_{k,2}}\leq\frac{(p-\alpha)(p-\beta)}{\delta(n,k)[k(\lambda(p+k)-1)-\alpha][k(\lambda(p+k)-1)-\beta]}.$$

Thus (19) will be satisfied if

$$\frac{(p-\alpha)(p-\beta)}{\delta(n,k)[k(\lambda(p+k)-1)-\alpha][k(\lambda(p+k)-1)-\beta]}\leq \left[\frac{p-\xi}{(p-\alpha)(p-\beta)}\right]\left[\frac{(k(\lambda(p+k)-1)-\alpha)(k(\lambda(p+k)-1)-\beta)}{(k(\lambda(p+k)-1)-\xi)}\right]$$

or if,

$$\xi\leq p-\frac{(p-\alpha)^2(p-\beta)^2[k(\lambda(p+k)-1)-p]}{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]^2[k(\lambda(p+k)-1)-\beta^2]-(p-\alpha)^2(p-\beta)^2}.$$

We see that the right hand side of this inequality is an increasing function of k. Setting $k=p$, we get

$$\xi\leq p-\frac{2p(p-\alpha)^2(p-\beta)^2(\lambda p-1)}{\delta(n,p)[p(2\lambda p-1)-\alpha]^2[p(2\lambda p-1)-\beta^2]-(p-\alpha)^2(p-\beta)^2},$$

which is the largest ξ such that (19) is true, which implies $(f_1*f_2)(z)\in\mathcal{M}^*(n,p,\lambda,\xi)$. □

Theorem 9. *Let the function $f_j(z)$ $(j=1,2)$ defined by (16) be in the class $\mathcal{M}^*(n,p,\lambda,\alpha)$. Then the function $h(z)$ defined by*

$$h(z)=\frac{1}{z^p}-\sum_{k=p}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^k\in\mathcal{M}^*(n,p,\lambda,\zeta),$$

where

$$\zeta=p\left[1-\frac{2(p-\alpha)^2(2\lambda p-2)}{\delta(n,p)[p(2\lambda p-1)-\alpha]^2-2(p-\alpha)^2}\right].$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ given by (17).

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Proof. Since

$$\sum_{k=p}^{\infty}\left\{\frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]}{(p-\alpha)}\right\}^2 a_{k,j}^2$$

$$\leq \left[\sum_{k=p}^{\infty}\frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]}{(p-\alpha)}a_{k,j}\right]^2 \leq 1,$$

for $f_j(z) \in \mathcal{M}^*(n,p,\lambda,\alpha)$ $(j=1,2)$, we have,

$$\sum_{k=p}^{\infty}\frac{[\delta(n,k)[k(\lambda(p+k)-1)-\alpha]]^2}{2(p-\alpha)^2}(a_{k,1}^2+a_{k,2}^2) \leq 1.$$

We have to find largest ζ so that

$$\sum_{k=p}^{\infty}\frac{\delta(n,k)[k(\lambda(p+k)-1)-\zeta]}{p-\zeta}(a_{k,1}^2+a_{k,2}^2) \leq 1. \tag{20}$$

(20) is satisfied if

$$\frac{k(\lambda(p+k)-1)-\zeta}{p-\zeta} \leq \frac{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]^2}{2(p-\alpha)^2}$$

or if

$$\zeta \leq p - \frac{2(p-\alpha)^2[k(\lambda(p+k)-1)-p]}{\delta(n,k)[k(\lambda(p+k)-1)-\alpha]^2-2(p-\alpha)^2}.$$

We see that the right side of this inequality is an increasing function of k.

Setting $k=p$, we get

$$\zeta \leq p\left[1-\frac{2(p-\alpha)^2(2\lambda p-2)}{\delta(n,k)[p(2\lambda p-1)-\alpha]^2-2(p-\alpha)^2}\right],$$

which is the largest ζ such that (20) is true, which implies that $h(z) \in \mathcal{M}^*(n,p,\lambda,\zeta)$.

The proof is complete. □

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