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INTEGRAL SOLUTIONS TO HEUN'S DIFFERENTIAL EQUATION VIA SOME RATIONAL TRANSFORMATION

A. Anjorin
Department of Mathematics
Lagos State University (LASU)
P.O. Box 1087, Apapa Lagos, NIGERIA
e-mail: anjomaths@yahoo.com

Abstract: The present work determines the integral form of solutions obtained from the transformation of Heun's equation to hypergeometric equation by rational substitution. All relevant solutions are provided.

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1. Introduction

The hypergeometric equation has three regular singular points. Heun's equation has four singular points. The problem of conversion from Heun's equation to hypergeometric equation has been treated in the works of K. Kuiken [10]. The purpose of this work is to derive some integrated forms of solutions to the Heun's equation via some rational transformation as stated in [10]. The steps taken will be the conversion of Heun's functions to the hypergeometric functions then taken the integration, and through a push and pull back process we arrive back to a new Heun's functions different from the original Heun's function.

Every homogenous linear second order differential equation with four regular singularities can be transformed into (see [12])

$$\frac{d^2u}{dt^2} + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-d}\right)\frac{du}{dt} + \frac{\alpha\beta t - q}{t(t-1)(t-d)}u = 0, \tag{1.1}$$

where $\{\alpha, \beta, \gamma, \delta, \epsilon, d, q\}$ $(d \neq 0, 1)$ are parameters, generally complex and arbitrary, linked by the Fuschian constraint $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. This equation has four

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regular singular points at $\{0, 1, a, \infty\}$, with the exponents of these singularities being respectively, $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$, and $\{\alpha, \beta\}$. The equation (1.1) is called Heun's equation (see [12]).

The hypergeometric equation (see [12])

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0, \tag{1.2}$$

has three regular singular points. In the works of [10], it has been shown that these two equations above can be transformed to one another via six rational polynomials $z = R(t)$, where $R(t) = t^2, 1-t^2, (t-1)^2, 2t-t^2, (2t-1)^2, 4t(1-t)$. The following parameter relations were deduced [10].

For the polynomial $R(t) = t^2$

- $\alpha + \beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = -1 + 2c$, $\delta = 1 + a + b - c$, $\epsilon = \delta$, $q = 0$ and $d = -1$.

For the polynomial $R(t) = 1 - t^2$

- $\alpha + \beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = 1 - 2c + 2a + 2b$, $\delta = c$, $\epsilon = \delta = c$, $q = 0$ and $d = -1$.

For the polynomial $R(t) = (t-1)^2$

- $\alpha + \beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = 1 + a + b - c$, $\delta = -1 + 2c$, $\epsilon = \gamma$, $q = 4ab$ and $d = 2$.

For the polynomial $R(t) = 2t - t^2$

- $\alpha + \beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = c$, $\delta = 1 - 2c + 2a + 2b$, $\epsilon = \delta = c$, $q = 4ab$ and $d = 2$.

For the polynomial $R(t) = (2t-1)^2$

- $\alpha + \beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = -1 + a + b - c$, $\delta = \gamma$, $\epsilon = \delta = -1 + 2c$, $q = 2ab$ and $d = 1/2$.

For the polynomial $R(t) = 4t(1-t)$

- $\alpha + \beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = c$, $\delta = \gamma$, $\epsilon = 1 - 2c + 2a + 2b$, $q = 2ab$ and $d = 1/2$.

Assuming $H(d, q; \alpha, \beta, \gamma, \delta, \epsilon; t)$ and ${}_2F_1(a, b; c; z = R(t))$ are representative forms of the solutions of (1.1) and (1.2) respectively, toghether with the parametres above relations can de established between these two forms via the polynomials data given above. We provide an answer to this in the resent paper. Indeed, we provide that the integral of the solution of **GHE** can be expressed in terms of another **GHE** solution.

2. Main Results: Integral Solutions

In this section we shall apply the relations above in deriving the integral form of solutions via these polynomial transformations. Let $\mathcal{I} = \int_C$ be an integral operator defined over a compact interval C. Since $(a)_{n-1} = \frac{(a-1)_n}{a-1}$, we have

$$\mathcal{I}_2F_1(a,b;c;z=R(t)) = \frac{R^*(t)(c-1)}{(a-1)(b-1)}{}_2F_1(a-1,b-1;c-1;z=R(t)),$$

where $R^*(t)$ is a polynomial factor derived from the integrand and through a push and pull-back processes we have the following possible solutions;

1. For polynomial $R(t) = t^2$:

   (a) Using $c = (\gamma+1)/2$, we obtain

$$\begin{aligned}&\mathcal{I}H(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)\\ &= \frac{2(\gamma-1)t^3}{3(\alpha-2)(\beta-2)}{}_2F_1(\frac{\beta-2}{2},\frac{\alpha-2}{2};\frac{\gamma-1}{2};R(t)=t^2)|_C\\ &= \frac{2(\gamma-1)t^3}{3(\alpha-2)(\beta-2)}H(-1,0;\alpha-2,\beta-2,\gamma-2,\\ &\quad \frac{\alpha+\beta-\gamma-1}{2},\frac{\alpha+\beta-\gamma-1}{2};t))|_C.\end{aligned} \tag{2.1}$$

   (b) Using $c = 1-\delta+a+b$, we get

$$\begin{aligned}&\mathcal{I}H(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)\\ &= \frac{4(\alpha+\beta-2\delta)t^3}{3(\alpha-2)(\beta-2)}{}_2F_1(\frac{\beta-2}{2},\frac{\alpha-2}{2};\alpha+\beta-2\delta;R(t)=t^2)|_C\\ &= \frac{4(\alpha+\beta-2\delta)t^3}{3(\alpha-2)(\beta-2)}\times H(-1,0;\alpha-2,\beta-2,2(\alpha+\beta-2\delta)-1,\\ &\quad \frac{4\delta-(\alpha+\beta)-2}{2},\frac{4\delta-(\alpha+\beta)-2}{2};t))|_C.\end{aligned} \tag{2.2}$$

2. For polynomial $R(t) = 1-t^2$:

   (a) Using $c = \delta$, we have

$$\begin{aligned}&\mathcal{I}H(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)\\ &= \frac{4(\delta-1)(3t-t^3)}{3(\alpha-2)(\beta-2)}{}_2F_1(\frac{\beta-2}{2},\frac{\alpha-2}{2};\delta-1;R(t)=1-t^2)|_C\\ &= \frac{4(\delta-1)(3t-t^3)}{3(\alpha-2)(\beta-2)}\\ &\quad \times H(-1,0;\alpha-2,\beta-2,\alpha+\beta-2\delta-1,\delta-1,\delta-1;t))|_C.\end{aligned} \tag{2.3}$$

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(b) Using $c = \epsilon$, we have

$$\begin{aligned}
&\mathcal{I}H(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)\\
&\quad= \frac{4(\epsilon-1)(3t-t^3)}{3(\alpha-2)(\beta-2)}\,{}_2F_1\Big(\frac{\beta-2}{2},\frac{\alpha-2}{2};\epsilon-1;R(t)=1-t^2\Big)|_C\\
&\quad= \frac{4(\epsilon-1)(3t-t^3)}{3(\alpha-2)(\beta-2)}\\
&\qquad\times H(-1,0;\alpha-2,\beta-2,\alpha+\beta-2\epsilon-1,\epsilon-1,\epsilon-1;t))|_C. \qquad (2.4)
\end{aligned}$$

(c) Using $c = (1-\gamma+2a+2b)/2$, we arrive at

$$\begin{aligned}
&\mathcal{I}H(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)\\
&= \frac{2(\alpha+\beta-\gamma-1)(3t-t^3)}{3(\alpha-2)(\beta-2)}\,{}_2F_1\Big(\frac{\beta-2}{2},\frac{\alpha-2}{2};\frac{\alpha+\beta-\gamma-1}{2};R(t)=1-t^2\Big)|_C\\
&= \frac{2(\alpha+\beta-\gamma-1)(3t-t^3)}{3(\alpha-2)(\beta-2)}\\
&\quad\times H\Big(-1,0;\alpha-2,\beta-2,\gamma-2,\frac{\alpha+\beta-\gamma-1}{2},\frac{\alpha+\beta-\gamma-1}{2};t\Big)|_C. \qquad (2.5)
\end{aligned}$$

3. For polynomial $R(t) = 2t - t^2$:

(a) Using $c = (\delta+1)/2$, we obtain

$$\begin{aligned}
&\mathcal{I}H(2,\alpha\beta;\ \alpha,\beta,\gamma,\delta,\epsilon;t)\\
&\quad= \frac{2(\delta-1)t^2(3-t^2)}{3(\alpha-2)(\beta-2)}\,{}_2F_1\Big(\frac{\beta-2}{2},\frac{\alpha-2}{2};\frac{\delta-1}{2};R(t)=2t-t^2\Big)|_C\\
&\quad= \frac{2(\delta-1)t^2(3-t^2)}{3(\alpha-2)(\beta-2)}H\Big(2,(\alpha-2)(\beta-2);\alpha-2,\beta-2,\frac{\alpha+\beta-\delta-1}{2},\delta-2,\\
&\qquad\quad \frac{\alpha+\beta-\delta-1}{2};t\Big)|_C. \qquad (2.6)
\end{aligned}$$

(b) Using $c = 1+a+b-\gamma$, we get

$$\begin{aligned}
&\mathcal{I}H(2,\alpha\beta;\ \beta,\alpha,\gamma,\delta,\epsilon;t)\\
&\quad= \frac{2(\alpha+\beta-2\gamma)t^2(3-t^2)}{3(\alpha-2)(\beta-2)}\,{}_2F_1\Big(\frac{\beta-2}{2},\frac{\alpha-2}{2};\gamma+1;R(t)=2t-t^2\Big)|_C\\
&\quad= \frac{2(\alpha+\beta-2\gamma)t^2(3-t^2)}{3(\alpha-2)(\beta-2)}H\Big(2,(\alpha-2)(\beta-2);\alpha-2,\beta-2,\frac{\gamma-2}{2},\\
&\qquad\quad \alpha+\beta-\gamma-1,\frac{\gamma-2}{2};t)\Big)|_C. \qquad (2.7)
\end{aligned}$$

4. For polynomial $R(t) = (t-1)^2$

(a) Using $c = (1 - \delta + 2a + 2b)/2$, we get

$$\begin{aligned} &\mathcal{I}H(2, \alpha\beta, \alpha, \beta, \gamma, \delta, \epsilon; t) \\ &= \frac{2(\alpha+\beta-\delta-1)(t-1)^3}{3(\alpha-2)(\beta-2)} {}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \frac{\alpha+\beta-\delta-1}{2}; R(t) = (t-1)^2)|_C \\ &= \frac{2(\alpha+\beta-\delta-1)(t-1)^3}{3(\alpha-2)(\beta-2)} \\ &\quad \times H(2, (\alpha-2)(\beta-2); \alpha-2, \beta-2, \frac{\alpha+\beta-\delta-1}{2}, \frac{\alpha+\beta-\delta-1}{2}, \\ &\quad \frac{\alpha+\beta-\delta-1}{2}; t))|_C. \end{aligned} \tag{2.8}$$

(b) Using $c = \gamma$, we have

$$\begin{aligned} &\mathcal{I}H(2, \alpha\beta; \alpha, \beta, \gamma, \delta, \epsilon; t) \\ &= \frac{2(\gamma-1)(t-1)^3}{3(\alpha-2)(\beta-2)} {}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \gamma-1; R(t) = (t-1)^2)|_C \\ &= \frac{2(\gamma-1)}{(\alpha-2)(\beta-2)} H(2, (\alpha-2)(\beta-2); \alpha-2, \beta-2, \gamma-1, \\ &\quad \alpha+\beta-2\gamma-1, \alpha+\beta-2\gamma-1; t))|_C. \end{aligned} \tag{2.9}$$

(c) By changing γ to ϵ in (2.9), similar relation can be obtained.

5. For polynomial $R(t) = (2t-1)^2$

(a) Using $c = (\epsilon+1)/2 = (\delta+1)/2$

$$\begin{aligned} &\mathcal{I}H(1/2, \alpha\beta/2; \alpha, \beta, \gamma, \delta, \epsilon; t) \\ &= \frac{2(\epsilon-1)(2t-1)^3}{6(\alpha-2)(\beta-2)} {}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \frac{\epsilon-1}{2}; R(t) = (2t-1)^2)|_C \\ &= \frac{2(\epsilon-1)(2t-1)^3}{6(\alpha-2)(\beta-2)} H(1/2, \frac{(\alpha-2)(\beta-2)}{2}; \alpha-2, \beta-2, \frac{\alpha+\beta-\epsilon-5}{2}, \\ &\quad \frac{\alpha+\beta-\epsilon-5}{2}, \epsilon-2; t)|_C. \end{aligned} \tag{2.10}$$

By changing ϵ to δ a similar expression can be obtained.

(b) Using $c = -1 + a + b - \gamma$, we obtain

$$\begin{aligned} &\mathcal{I}H(1/2, \alpha\beta/2; \alpha, \beta, \gamma, \delta, \epsilon; t) \\ &= \frac{2(\alpha+\beta-2(\gamma+2))(2t-1)^3}{6(\alpha-2)(\beta-2)} \end{aligned}$$

$$\times\, {}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \frac{\alpha+\beta-2\gamma-4}{2}; R(t)=(2t-1)^2)|_C$$
$$= \frac{2(\alpha+\beta-2(\gamma+2))(2t-1)^3}{6(\alpha-2)(\beta-2)} H(1/2, \frac{(\alpha+2)(\beta+2)}{2}; \alpha-2,$$
$$\beta-2, \gamma-1, \gamma-1, \alpha+\beta-2\gamma-5; t)|_C. \quad (2.11)$$

6. For polynomial $R(t)=4t(1-t)$:

(a) Using $c=\gamma$, we have

$$\mathcal{I}H(1/2, \alpha\beta/2; \beta, \alpha, \gamma, \delta, \epsilon, ; t)$$
$$= \frac{4(\gamma-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} {}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \gamma-1; R(t)=4t(1-t))|_C$$
$$= \frac{4(\gamma-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} H(1/2, \frac{(\alpha-2)(\beta-2)}{2}; \alpha-2, \beta-2,$$
$$\gamma-1, \gamma-1, \alpha+\beta-2\gamma-1; t)|_C. \quad (2.12)$$

(b)

$$\mathcal{I}H(1/2, \alpha\beta/2; \beta, \alpha, \gamma, \delta, \epsilon, ; t)$$
$$= \frac{4(\delta-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} {}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \delta-1; R(t)=4t(1-t))|_C$$
$$= \frac{4(\delta-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} H(1/2, \frac{(\alpha-2)(\beta-2)}{2}; \alpha-2, \beta-2,$$
$$\delta-1, \delta-1, \alpha+\beta-2\delta-1; t)|_C. \quad (2.13)$$

(c) Using $c=(1-\epsilon+2a+2b)/2$

$$\mathcal{I}H(1/2, \alpha\beta/2; \alpha, \beta, \gamma, \delta, \epsilon, ; t)$$
$$= \frac{2(\alpha+\beta-\epsilon-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)}$$
$${}_2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \frac{\alpha+\beta-\epsilon-1}{2}; R(t)=4t(1-t))|_C$$
$$= \frac{2(\alpha+\beta-\epsilon-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} H(1/2, \frac{(\beta-2)(\alpha-2)}{2}; \alpha-2, \beta-2,$$
$$\frac{\alpha+\beta-\epsilon-1}{2}, \frac{\alpha+\beta-\epsilon-1}{2}, \epsilon-2; t)|_C. \quad (2.14)$$

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3. Concluding Remarks and Suggestions

In this paper, we have shown that the parameter relations obtained in the works of K. Kuiken [10] lead to some integral forms of solutions to the general Heun's equation. The multiple choise of close form solutions arises from the papameter relations. For example, consider the quadratic equation arising from the relations $\alpha+\beta = 2(a+b)$ and $\alpha\beta = 4ab$ leads to the parameter choise $a = \beta/2$ and $b = \alpha/2$ or $a = \alpha/2$ and $b = \beta/2$. The first leads to all the relations above while the later repeates all the relations described above by changing α to β. This method has being extended in the works [8] and the work of Robert Maier [11], pp. 15.

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