International Electronic Journal of Pure and Applied Mathematics

Volume 1 No. 3 2010, 229-234

ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE VEKUA EQUATION IN THE L_p-SPACE

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Abstract: In this paper, we discuss on the existence and uniqueness of the solution of Vekua equation

$$\frac{\partial w}{\partial \overline{z}} = Aw + B\overline{w} + C, \tag{1}$$

where A, B, C are complex valued functions usually supposed to belong to L_p-space in a domain $D \subset C$ and also the uniqueness of the solution of the Cauchy-Riemann system

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = a(x,y)u + b(x,y)v + f(x,y), \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = c(x,y)u + d(x,y)v + g(x,y), \end{cases} \tag{2}$$

where a, b, c, d, f and g are analytic functions in the (simply connected) domain D with condition

$$\alpha_0 w(z, \overline{z}) = \varphi(z)$$

by fixed point theorem.

AMS Subject Classification: 35J60, 35A05
Key Words: Vekua equation, Cauchy-Riemann system, fixed point theorem, holomorphic function

Received: March 23, 2010

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1. Introduction

The theory of generalizations of analytic functions goes back to the early fifties (see, e.g., the survey by Tutshke [4] and reference therein). The interest to the developing of this area is connected first of all with different type applications of these functions called usually generalized analytic functions (see [2], [3], [5]). The most known constructions are those generalized analytic functions of Vekua type (see [5]) defined as a solution to elliptic system of differential equations generalizing the Cuachy-Riemann system, or pseudo-analytic functions of Bers type determined by ordinary differential equations in complex domains containing so called (F, G)-derivatives (see [1]). Generalized analytic functions are solutions the system (2) that is equivalent to the partial complex differential Vekua equation where $z = x + iy$, $\overline{z} = x - iy$, $w = u + iv$, $4A = a + d - ic - ib$, $4B = a - d + ic - ib$, $2F = f + ig$.

This paper is organized as follows: In Section 2 some notations and preliminaries are introduced. The uniqueness results of the solution Vekua equation and the Cuachy-Riemann system are verified in Section 3.

2. Preliminaries

Definition 2.1. Let the function $w = w(z, \overline{z}) \in W_{1,p}(D)$ and $k \in R$. Define the operator α_k,

$$\alpha_k w(z, \overline{z}) = w(z, k) = \varphi(z),$$

where $\varphi(z)$ is holomorphic function.

Example 2.1. With assumption $k = 1$ and arbitrary $w = w(z, \overline{z}) = e^{z+2\overline{z}} + z^2 + 3z\overline{z}$, we have

$$\alpha_1(e^{z+2\overline{z}} + z^2 + 3z\overline{z}) = e^{z+2} + z^2 + 3z.$$

Definition 2.2. Let $D \subseteq R^n$, $n \geq 2$ and $\int w(z, \overline{z})d\overline{z} = W(z, \overline{z})$. Define for every $a, b \in D$

$$\int_a^b w(z, \overline{z})d\overline{z} = \alpha_b W(z, \overline{z}) - \alpha_a W(z, \overline{z}).$$

Lemma 2.1. *Let:*

I. D be a simply connected domain;

II. $\varphi(z)$ be holomorphic function in the $D \subseteq R^n$ and $\alpha_0 w(z, \overline{z}) = \varphi(z)$;

III. $Aw + B\overline{w} + C$ be a continuous function of its variables in the domain D.

Then differential equation (1) with the initial condition $\alpha_0 w(z, \overline{z}) = \varphi(z)$ is equivalent with integral equation

$$w = \varphi(z) + \int_0^{\overline{z}} (Aw + B\overline{w} + C)d\overline{z}. \tag{3}$$

Proof. Let $w(z,\overline{z})$ be the solution of equation (1) with the initial condition

$$\alpha_0 w(z,\overline{z}) = \varphi(z).$$

Now by integrating of (1) with respect to $\overline{z}$, we have

$$w = \int (Aw + B\overline{w} + C)d\overline{z} + g(z), \tag{4}$$

where $g(z)$ is integration constant and holomorphic function.

Let

$$\int (Aw + B\overline{w} + C)d\overline{z} = k(z,\overline{z}).$$

We have

$$\alpha_0 w = \alpha_0 [\int (Aw + B\overline{w} + C)d\overline{z}] + \alpha_0 g(z) = \alpha_0 k(z,\overline{z}) + g(z). \tag{5}$$

By (4), (5) we have

$$w - \alpha_0 w = k(z,\overline{z}) - \alpha_0 k(z,\overline{z}) = \int_0^{\overline{z}} (Aw + B\overline{w} + C)d\overline{z}.$$

Therefore

$$w = \varphi(z) + \int_0^{\overline{z}} (Aw + B\overline{w} + C)d\overline{z}.$$

Conversely, let $w(z,\overline{z})$ be the solution of integral equation (3), therefore by differentiating (3) with respect to $\overline{z}$ we have

$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial \varphi(z)}{\partial \overline{z}} + \frac{\partial}{\partial \overline{z}} \int_0^{\overline{z}} (Aw + B\overline{w} + C)d\overline{z} = Aw + B\overline{w} + C,$$

and

$$\alpha_0 w(z,\overline{z}) = \varphi(z).$$

Remark 2.1. Let $\int^*$ denote the reverse operation to conjugated differentiation $\frac{\partial}{\partial \overline{z}}$, i.e.

$$\frac{\partial w}{\partial \overline{z}} = F \Leftrightarrow w = \int^* F = \int^* (A(z,\overline{z})w + B(z,\overline{z})\overline{w} + C(z,\overline{z})). \tag{6}$$

Here the right-hand side of (6) is a complex integral equation. But as analytic (in wider sense) equation has analytic solution (1) then it is

$$w = \int_0^{\overline{z}} (A(z,\overline{z})w + B(z,\overline{z})\overline{w} + C(z,\overline{z}))d\overline{z} + \varphi(z), \tag{7}$$

where $\varphi(z)$ is an arbitrary analytic function with a constant integral. □

Proposition 2.1. *Let domain D have a finite area and the function $Aw+B\overline{w}+C$ be continuous function in the domain D and satisfy a Lipschitz condition of the form*

$$|\ (Aw+B\overline{w}+C)-(A\widetilde{w}+B\widetilde{\overline{w}}+C)\ |\leq K\ |\ w-\widetilde{w}\ |\ .$$

Let K be positive real number. Then operator $\int^{*} Aw+B\overline{w}+C$ has fixed point.

Proof. Denote the right-hand side of (7) by $T(w)=\int_0^{\overline{z}}(Aw+B\overline{w}+C)d\overline{z}+\varphi(z)$. We obtain

$$\begin{aligned}
|\ T(w_1)-T(w_2)\ | &= |\int_0^{\overline{z}}(Aw_1+B\overline{w_1}+C)d\overline{z}+\varphi \\
&- \int_0^{\overline{z}}(Aw_2+B\overline{w_2}+C)d\overline{z}-\varphi\ | \\
&= |\int_0^{\overline{z}}((Aw_1+B\overline{w_1}+C)-(Aw_2-B\overline{w_2}+C))d\overline{z}\ | \\
&\leq K\int_0^{\overline{z}}|\ w_1-w_2\ |\ .\ |\ d\overline{z}\ | \\
&\leq K\ |\ z\ |\ .\ |\ w_1-w_2\ | \\
&\leq Khm.
\end{aligned}$$

Here $h=\sup_D|\ z\ |$ and $m=\sup_D|\ w_1-w_2\ |$. Iterating we get

$$\begin{aligned}
|\ T^2(w_1)-T^2(w_2)\ | &= |\ T(T(w_1))-T(T(w_2))\ | \\
&= |\int_0^{\overline{z}}[A(Tw_1-Tw_2)+B(\overline{Tw_1}-\overline{Tw_2})]d\overline{z}\ | \\
&= |\int_0^{\overline{z}}[(A^2+B^2)(w_1-w_2) \\
&+ (AB+BA)(\overline{w_1}-\overline{w_2})]d\overline{z}d\overline{z}\ | \\
&\leq \int_0^{\overline{z}}|\ [(A^2+B^2)(w_1-w_2) \\
&+ (AB+BA)(\overline{w_1}-\overline{w_2})]\ |\ .\ |\ d\overline{z}\ |\ .\ |\ d\overline{z}\ | \\
&\leq K^2m\frac{h^2}{2!}.
\end{aligned}$$

By induction we obtain that

$$|\ T^n(w_1)-T^n(w_2)\ |\leq K^n m\frac{h^n}{n!},$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 3 (2010)

and as we can choose n so that $q = \frac{(Kh)^n}{n!}$ becomes arbitrary small, we can obtain that

$$| T^n(w_1) - T^n(w_2) | \leq q \mid w_1 - w_2 |$$

with condition $0 < q < 1$, the operator T is a contraction operator. □

Corollary 2.1. *We consider the recursive sequence* $\{w_n\}$ *with relation*

$$w_{n+1} = T(w_n) = \int_0^{\overline{z}} (Aw_n + B\overline{w_n} + C)d\overline{z} + \varphi(z) \qquad n = 0, 1, 2, \dots .$$

The sequence $\{w_n\}$ *converges and since* T *has a fixed point therefore we have*

$$\exists w \in D \;\; T(w) = \lim_{n\to\infty} T(w_n) = \lim_{n\to\infty} w_n = w.$$

Therefore w *is the solution of the integral equation*

$$w = \int_0^{\overline{z}} (Aw + B\overline{w} + C)d\overline{z} + \varphi(z).$$

3. Main Results

Theorem 3.1. *Let the following conditions hold true:*

1. *The domain* D *has a finite area.*
2. *As a function of the variables* $z \in D, w, \overline{w}; Aw + B\overline{w} + C$ *is a continuous function of its variables.*
3. *The function* $Aw + B\overline{w} + C$ *satisfies a Lipschitz condition of the form*

   $$| (Aw + B\overline{w} + C) - (A\widetilde{w} + B\overline{\widetilde{w}} + C) | \leq K \mid w - \widetilde{w} |$$

   and K *is arbitrary positive number.*
4. *There exist* $w \in L_p(D)$ $1 < p < \infty$*, such that* $Aw + B\overline{w} + C \in L_p(D)$*.*

Then the Vekua equation

$$\frac{\partial w}{\partial \overline{z}} = Aw + B\overline{w} + C$$

has unique solution $w(z, \overline{z})$ *with condition* $\alpha_0 w(z, \overline{z}) = \varphi(z)$*.*

Proof. By the equivalence of the integral equation (3) with the differential equation (1), with initial condition $\alpha_0 w(z, \overline{z}) = \varphi(z)$, it is enough to prove that the integral equation has a unique solution.

The uniqueness solution of the integral equation (3) can be obtained from Corollary 2.1. □

Corollary 3.1. *The Vekua equation (1) is equivalent to Cauchy-Riemann differential equations (2), therefore the solution* $w \in W_{1,p}(D)$*,* $1 < p < \infty$*,* $mD < \infty$ *of the partial differential equation (1) is a solution of the partial differential equation (2) with condition* $\alpha_0 w(z, \overline{z}) = \varphi(z)$ *where* $\varphi(z)$ *is holomorphic function.*

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