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# EXISTENCE CRITERIA FOR SECOND ORDER SYSTEMS WITH “MAXIMA”

Dimitar Kolev[1], Nedyalka Markova[2 §]

[1]Department of Mathematics
University of Chemical Technology and Metallurgy
8, Kliment Ohridski Blvd., Sofia, 1756, BULGARIA
e-mails: mkolev@math.uctm.edu, kolev@uctm.edu

[2]Department of Mathematics, Physics and Chemistry
Technical University of Sofia, Branch of Sliven
Sliven, 8800, BULGARIA
e-mail: n_markova_54@abv.bg

**Abstract:** In this note we consider a system of two second order nonlinear ordinary differential equations with “maxima”. Of special interest here are the cases when the vectorial field in the right-hand side of the system under consideration satisfies different integral conditions. Some criteria for existence of positive solutions are established. To prove the existence we use Schauder Fixed Point Theorem (FPT). These differential equations with “maxima” are applicable in different fields as the mathematical simulation in theoretical physics, optimal control, chemistry, mechanics of materials, biology, ecology, etc.

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**Key Words:** maxima, second order ODEs, nonlinear system of ODEs, positive solutions, Schauder Fixed Point Theorem (FPT)

## 1. Introduction

In this note we consider the class of systems containing two second order nonlinear ordinary differential equations (ODEs) of the form

$$\begin{aligned} (a) \quad & x''(t) = a(t)f(y_s(t)), \\ (b) \quad & y''(t) = -b(t)g(x_s(t)), \end{aligned} \tag{1}$$

where $(x(t),\ y(t))$ is unknown vectorial function, and

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§Correspondence author

$$x_s(t) \equiv \max_{s\in[\sigma(t),\tau(t)]} x(s),$$
$$y_s(t) \equiv \max_{s\in[\sigma(t),\tau(t)]} y(s),$$
$$0 \le \sigma(t) \le \tau(t) \le t, \quad t \in [t_0, +\infty).$$

Here we use the notation $x'' \equiv \frac{d^2x}{dt^2}$, $y'' \equiv \frac{d^2y}{dt^2}$. Assume that $a(t)$, $b(t)$, $f(\xi)$, $g(\eta)$ are continuous functions w.r.t. their arguments, subject to some conditions given further. Both functions $\sigma(t)$ and $\tau(t)$ are continuous and monotone increasing in $\mathbb{R}_+$. The system (1) can be written in vectorial form as

$$z''(t) = \mathcal{F}(t, z_s(t)), \tag{2}$$

where $z_s(t) = (x_s(t),\ y_s(t))$, and

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad z''(t) = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}, \quad \mathcal{F}(t, z_s(t)) = \begin{pmatrix} a(t)f(y_s(t)) \\ -b(t)g(x_s(t)) \end{pmatrix}.$$

The initial data have also vectorial form

$$z(t_0) = z_0 \equiv \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad z'(t_0) = z_1 \equiv \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad x_i,\ y_i \in \mathbb{R}_+ \ \ (i = 1, 2), \tag{3}$$

which brings into line with the initial vectorial function $\eta_0(t)$ defined and at the same time sufficiently smooth in the closed interval $[\sigma(t_0), t_0]$,

$$\begin{aligned} &\eta_0(t) \equiv \begin{pmatrix} \eta_{0,1}(t) \\ \eta_{0,2}(t) \end{pmatrix}, \quad \eta_0'(t) \equiv \begin{pmatrix} \eta_{0,1}'(t) \\ \eta_{0,2}'(t) \end{pmatrix}, \\ &\eta_{0,i} \in C([\sigma(t_0), t_0]; \mathbb{R}_+) \ \ (i = 1, 2), \\ &z_0 = \eta_0(t_0), \quad z_1 = \eta_0'(t_0), \end{aligned} \tag{4}$$

where $0 \le \sigma(t_0) \le \tau(t_0) \le t_0$.

**Remark 1.** In the problem with "maxima" (1), (3), (4) we require both the initial function $\eta_0(t)$ and its derivative $\eta_0'(t)$ to be known in advance, and in general case being continuous in the initial point $t_0$.

**Remark 2.** We note that the eventual solution of (1) (in the case that there exists at least one) could be continuous in the point $t_0$ but not "compulsory" differentiable in the same point, since the initial function and the solution of (1), (3), (4) are "stitched together" in $t_0$ not always smoothly.

The study of differential equations with “maxima” starts many years ago with the pioneer works of A. Magomedov [12], [13], where the first ideas for linear differential equations with “maxima” have been proposed in the connection with the theory of optimal control to different physical systems. In most cases we use “maxima” in the right-hand side when the control corresponds to the maximal deviation of the regulated quantity that could be for instance temperature, heat, current density, pressure and so on. The study of these equations continues in several directions: existence and uniqueness, oscillation property, stability, blow-up, bifurcation theory, etc. In the theory of automatic control of various mechanical and technological systems it often occurs that the low of regulation depends on maximum values of some regulated state parameters over certain time intervals whose boundaries evolve all along (see, e.g. [2], [14]). This is especially the case for stabilization systems where the regulated quantity usually represents the deviation of some state parameters from the given value. The mathematical models for such systems naturally include differential equations with “maxima”. The oscillation properties of the solutions of the ordinary differential equations with “maxima” were studied by Bainov and his group of associates (see, e.g. [1]-[6]). However the qualitative theory for ODEs with “maxima” is still at its initial stage. Only several works devoted to similar topics are published so far (see, e.g. [2], [4]-[6]).

The present note was motivated by the fact that most applications of the considered problem require existence of solutions. A similar system without “maxima” was studied in [11]. Unfortunately the theory given there could not applied directly for the problem (1), (3), (4) due to the special properties of the arguments $\max_{s\in[\sigma(t),\tau(t)]} x(s)$ and $\max_{s\in[\sigma(t),\tau(t)]} y(s)$.

## 2. Preliminary Notes

Let $t \in [t_0, +\infty) \equiv J \subseteq \mathbb{R}_+ \equiv [0, +\infty)$. Introduce the following conditions:

**H1.** $a, b$ are not identical zeros, and $a, b \in C(J,\ \mathbb{R}_+)$.

**H2.** Define the monotone increasing in $\mathbb{R}$ functions $f, g \in C(\mathbb{R},\ \mathbb{R})$, that satisfy the inequalities

$$\xi f(\xi) > 0, \qquad \eta g(\eta) > 0 \qquad \text{for} \qquad \xi, \eta \in \mathbb{R} \setminus \{0\}.$$

**H3.** The pairs of functions $\sigma$ and $\tau$ are nondecreasing and continuous $\sigma, \tau \in C(J,\ \mathbb{R}_+)$, moreover

$$\lim_{t\to+\infty} \sigma(t) = +\infty, \quad \text{provided that} \quad \sigma(t) \le \tau(t) \le t.$$

**H4.** $\eta_0(t)$ is a continuous vectorial function defined in the interval $[\sigma(t_0), t_0]$ and such that one of its Dini's derivatives is finite and continuous on the same interval.

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We remind some familiar notations for Dini's derivatives of some function $G(t)$. Denote by $D^+G(t)$, $D_+G(t)$, $D^-G(t)$ and $D_-G(t)$ its right-hand upper, right-hand lower, left-hand upper and left-hand lower Dini's derivatives at the point $t$, i.e.

$$D^+G(t) \equiv \lim\sup_{h\to 0+} \frac{G(t+h)-G(t)}{h}, \quad D_+G(t) \equiv \lim\inf_{h\to 0+} \frac{G(t+h)-G(t)}{h},$$
$$D^-G(t) \equiv \lim\sup_{h\to 0-} \frac{G(t+h)-G(t)}{h}, \quad D_-G(t) \equiv \lim\inf_{h\to 0-} \frac{G(t+h)-G(t)}{h}.$$

The Dini's derivatives for the vectorial function $\eta_0(t)$ mean Dini derivatives of its components, that is, $D^+\eta_{0,i}(t)$, $D_+\eta_{0,i}(t)$, $D^-\eta_{0,i}(t)$, $D_-\eta_{0,i}(t)$ $(i = 1, 2)$. Some properties of Dini's derivatives one could find in [10], [16], [18] and the references given therein.

We make use of the following statements due to Dini's theory (see, e.g. [16], Chapter 1, pp. 12-13):

**Theorem 1.** *Let the function $G(t)$ be continuous in an open interval $\Delta \subset \mathbb{R}$. Assume that one of its Dini's derivatives is finite and continuous at $t_0 \in \Delta$. Then $\frac{dG}{dt}(t_0)$ exists.*

The following statement is actually an immediate consequence of Theorem 1.

**Corollary 1.** *For a function $G(t)$ that is continuous in an open interval $\Delta \subset \mathbb{R}$ assume that one of its Dini's derivatives is finite and continuous on $\Delta$. Then $\frac{dG}{dt}(t)$ exists and is continuous on $\Delta$.*

Introduce the following definitions.

**Definition 1.** The solution $(x(t),\ y(t))$ of the problem (1), (3), (4) is said to be proper if it is defined in some intervals $[T_x, +\infty)$, $[T_y, +\infty)$, respectively, and there is a number $T > t_0$ such that $\sup\{|x(t)| : \ t \geq T \geq T_x\} > 0$, $\sup\{|y(t)| : \ t \geq T \geq T_y\} > 0$.

**Definition 2.** The solution $(x(t),\ y(t))$ of (1), (3), (4) is said to be eventually positive if there is a number $\tilde{T} > t_0$ such that

$$x(t) > 0 \quad \text{and} \quad y(t) > 0$$

are defined for $t \geq \tilde{T}$.

The eventually negative solution can be defined analogously by reversing the above inequalities.

We employ the following notations

$$\alpha(t) \equiv \int_t^{+\infty} a(s)ds, \qquad \beta(t) \equiv \int_t^{+\infty} b(s)ds, \qquad t \geq t_0, \tag{5}$$

being used further for simplicity.

In order to prove existence of a solution to the problem (1) we take advantage of Schauder FPT. We refer the reader to [7], [9], [18]. The Schauder FPT is an extension of the Brouwer FPT to topological vector spaces, which may be of infinite dimension. It asserts that if $K$ is a convex subset of a topological vector space $V$ and $T$ is a continuous mapping of $K$ into itself so that $T(K)$ is contained in a compact subset of $K$, then $T$ has a fixed point $z \in K$, that is, $Tz = z$. In other words we can state that if $E$ be a normed space, $A \subset E$ convex and non-empty, and $C \subset A$ compact, then every continuous map $T : A \to C$ has at least one fixed point.

The theorem was conjectured and proven for special cases, such as Banach spaces, by J. Schauder [15] in 1930. His conjecture for the general case was published in the Scottish book (see, e.g. [18]). In 1934, Tychonoff proved the theorem for the case when $K$ is a compact convex subset of a locally convex space (see, e.g. [17]). This version is known as the Schauder-Tychonoff FPT. B.V. Singbal proved the theorem for the more general case where $K$ may be non-compact. The proof can be found in the appendix of Bonsall's book [7]. The proof of the full result (without the assumption of local convexity) was published by R. Cauty in 2001 (see, e.g. [8]).

## 3. Main Results

The following theorem that we intend to prove contains certain conditions that guarantee nonexistence of eventually positive solutions for (1), (3), (4).

**Theorem 2.** *Assume that the conditions H1-H4 hold and* $\alpha(t_0) = +\infty$, $\beta(t_0) = +\infty$ *(see (5)). Then the problem (1), (3), (4) does not possess eventually positive solutions.*

*Proof.* Note that the Cauchy problem for (1), (3), (4) is correctly posed by reason of H4 and Corollary 1. Suppose that $(x(t), y(t))$ is an eventually positive solution of (1). Then there exists a number $T_1 > t_0$ such that $x(t) > 0$, $y(t) > 0$ and also $x_s(t) > 0$ for $t \geq T_1$. Note that from (1)(b) we have $y''(t) < 0$ for $t \geq T_1$ which implies that $y'(t)$ is strictly monotone decreasing function in $[T_1, +\infty)$. Then we have two cases:

$$\text{(i)} \qquad y'(t) > 0 \text{ for } t \geq T_1,$$

$$\text{(ii)} \qquad y'(t) < 0 \text{ for } t \geq T_1.$$

We shall consider consecutively these cases.

*Case (i).* Note that $y(t) > 0$ is a strictly monotone increasing function for $t \geq T_1$ hence $y_s(t) \geq y(\sigma(t))$ and from the first equation of the system (1) get $x''(t) > 0$ for

$t \geq T_1$. Moreover, for any fixed $t \geq T_1$, $y_s(t) = y(\tau(t)) \geq y(\sigma(t))$ therefore obtain

$$\begin{aligned} x'(t) =& x'(T_2) + \int_{T_2}^{t} a(r) f(y_s(r)) dr \\ =& x'(T_2) + \int_{T_2}^{t} a(r) f(y(\tau(r))) dr \geq x'(T_2) + \int_{T_2}^{t} a(r) f(y(\sigma(r)) dr \\ \geq& x'(T_2) + f(y(\sigma(T_2)) \int_{T_2}^{t} a(r) dr \longrightarrow +\infty \quad \text{as} \quad t \to +\infty, \end{aligned} \tag{6}$$

where $T_2$ is a positive number such that $T_1 \leq \sigma(T_2) \leq \tau(T_2) \leq T_2 \leq t$. Then we have from (6) that $x'(t) > 0$, therefore, $x(t)$ is a strictly monotone increasing and at the same time unbounded function. Hence $x_s(t) = x(\tau(t)) \geq x(\sigma(t))$, for $t \geq T_1$. Thus from the second equation (b) of system (1) we get

$$\begin{aligned} y'(t) =& y'(T_2) - \int_{T_2}^{t} b(r) g(x_s(r)) dr \\ =& y'(T_2) - \int_{T_2}^{t} b(r) g(x(\tau(r))) dr \leq y'(T_2) - \int_{T_2}^{t} b(r) g(x(\sigma(r))) dr \\ \leq& y'(T_2) - g(x(\sigma(T_2))) \int_{T_2}^{t} b(r) dr \longrightarrow -\infty \quad \text{as} \quad t \to +\infty, \end{aligned}$$

which contradicts the statement that $y'(t) > 0$ for all $t \geq T_1$.

*Case (ii).* If $y'(t) < 0$ for all $t \geq T_1$ then from $y''(t) < 0$ for $t \geq T_1$ infer that $y'(t)$ is a strictly monotone decreasing function and hence there exists a constant $c > 0$ such that

$$y'(t) \leq -c \quad \text{for} \quad t \geq T_1. \tag{7}$$

Then integrating both sides of (7) we obtain that

$$y(t) \leq y(T_1) - \int_{T_1}^{t} c dr \leq 0 \quad \text{for} \quad t \geq \frac{y(T_1)}{c} + T_1,$$

whence infer that $y(t) \leq 0$ and $y(t) \to -\infty$ as $t \to +\infty$. This contradicts the fact that $y(t) > 0$. Therefore, taking into account all the above we conclude that there exists no eventually positive solution of the problem (1), (3), (4). □

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**Theorem 3.** *Let $t_1$ be some positive number such that $t_1 \geq t_0$ and following conditions be satisfied:*

*1). H1-H4.*

*2). The functions $\alpha, \beta$ defined by (5) take real and finite values at $t = t_1 \geq t_0$,*

$$\alpha(t_1) < +\infty, \quad \beta(t_1) < +\infty.$$

*3). There exist positive numbers $c_0 > 0$ and $c > 0$ such that*

$$f(c) \int_{t_1}^{+\infty} \int_{s}^{+\infty} a(u) du ds < +\infty$$

$$g(c_0) \int_{t_1}^{+\infty} \int_{s}^{+\infty} b(u) du ds < +\infty.$$

*Then the problem (1), (3), (4) has an eventually positive solution $(x(t),\ y(t))$.*

*Proof.* In this proof we use the same note as it was in the proof of Theorem 2 for the correctness of Cauchy problem (1), (3), (4). Infer by the same argument that our problem has correct initial data. We have from 2) that there is a positive number $T$ such that $t_1 \leq \sigma(T) \leq T$, and the following pair of inequalities hold

$$\int_{T}^{+\infty} \int_{s}^{+\infty} a(u) f(c) du ds < \frac{c_0}{2} \tag{8}$$

and

$$\int_{T}^{+\infty} \int_{s}^{+\infty} b(u) g(c_0) du ds < \frac{c}{2}\ . \tag{9}$$

Further we follow the method used in [11]. Denote by $BC$ the space of all bounded and continuous vectorial functions $z : \mathbb{R}_+ \to \mathbb{R}^2$, $z(t) = (x(t),\ y(t)) \in \mathbb{R}^2$ with a norm defined by

$$\|z\| \equiv \|(x, y)\| \equiv \max\left\{\sup_{t\geq T} |x(t)|,\ \sup_{t\geq T} |y(t)|\right\} < +\infty.$$

Define the set

$$\tilde{D} \equiv \left\{\tilde{z}(t) \equiv \Big(x(t), y(t)\Big) : \quad (x, y) \in C\Big([\sigma(T), +\infty);\ \mathbb{R} \times \mathbb{R}\Big)\right\} \subset BC, \tag{10}$$

that contains all continuous and bounded vectorial functions $\tilde{z} = \tilde{z}(t)$ being defined by

$$\tilde{z} \equiv \begin{cases} \Big(x(t), y(t)\Big), & x(t) = \dfrac{c_0}{2}, \qquad y(t) = \dfrac{c}{2}, \qquad \sigma(T) \leq t < T, \\ \Big(x(t), y(t)\Big), & \dfrac{c_0}{2} \leq x(t) \leq c_0, \quad \dfrac{c}{2} \leq y(t) \leq c, \qquad t \geq T. \end{cases} \tag{11}$$

Therefore for each function $\tilde{z} \in \tilde{D}$ we have

$$\min\left\{\frac{c_0}{2}, \frac{c}{2}\right\} \le \|\tilde{z}\| \le \max\{c_0, c\}.$$

Next we define an operator $S: \ \tilde{D} \to C\Big([\sigma(T), +\infty); \ \mathbb{R} \times \mathbb{R}\Big)$ by the equality

$$Sx(t) = \begin{cases} \dfrac{c_0}{2}\ , & \sigma(T) \le t < T, \\ \dfrac{c_0}{2} + \displaystyle\int_t^{+\infty}\int_s^{+\infty} a(u) f(y_r(u)) du ds\ , & t \ge T, \end{cases} \tag{12}$$

and

$$Sy(t) = \begin{cases} \dfrac{c}{2}\ , & \sigma(T) \le t < T, \\ \dfrac{c}{2} + \displaystyle\int_T^{t}\int_s^{+\infty} b(u) g(x_r(u)) du ds\ , & t \ge T, \end{cases} \tag{13}$$

where $x_r(u) \equiv \max\limits_{r \in [\sigma(u), \tau(u)]} x(r)$, $y_r(u) \equiv \max\limits_{r \in [\sigma(u), \tau(u)]} y(r)$. Here we have in mind that $\sigma(T) < T \le \sigma(u) \le \tau(u) \le u < +\infty$. Let $z$ be a vectorial function $z = (x, y) \in \tilde{D}$. Then we have from (8) and (9) that

$$\frac{c_0}{2} \le Sx(t) \le \frac{c_0}{2} + \int_t^{+\infty}\int_s^{+\infty} a(u) f(c) du ds \le \frac{c_0}{2} + \frac{c_0}{2} = c_0, \quad t \ge T,$$

and

$$\frac{c}{2} \le Sy(t) \le \frac{c}{2} + \int_T^{t}\int_s^{+\infty} b(u) g(c_0) du ds \le \frac{c}{2} + \frac{c}{2} = c, \quad t \ge T,$$

respectively. Therefore,

$$\min\left\{\frac{c_0}{2}, \frac{c}{2}\right\} \le \|Sz\| \le \max\{c_0, c\}$$

and infer that $S\tilde{D} \subseteq \tilde{D}$ that leads to the conclusion that the conditions of the Schauder FPT are satisfied and hence a fixed point $z = (x, y)$ exists, that is, $Sz = z$. Moreover, it turns out that the components of a proper positive solution of the problem (1), (3), (4) can be obtained by the system of integral equations

$$x(t) = \frac{c_0}{2} + \int_t^{+\infty}\int_s^{+\infty} a(u) f(y_r(u)) du ds, \qquad t \ge T,$$

$$y(t) = \frac{c}{2} + \int_T^t \int_s^{+\infty} b(u)g(x_r(u))duds, \qquad t \geq T.$$

As a final note we point at the case when the number $T$ can be too close to $t_0$ in the sense that $t_0 \leq t_1 \leq T \leq t \leq \tilde{t}$, where $\sigma(\tilde{t}) = t_0$, then by the condition H4 we have the same result as it was above. □

**Theorem 4.** *Let $t_1$ be some positive number such that $t_1 \geq t_0$ and conditions 1) and 2) in Theorem 3 be satisfied. Then if*

$$\begin{aligned} &(a) \quad \int_{t_1}^{+\infty} a(r)f(cy_s(r))dr < +\infty, \\ &(b) \quad \int_{t_1}^{+\infty} b(r)g(c_0 x_s(r))dr < +\infty, \end{aligned} \tag{14}$$

*where $c, c_0$ are some positive numbers, the problem (1), (3), (4) has an eventually positive solution $(x(t),\ y(t))$.*

*Proof.* For the correctness of the initial data here we use the same argument as in the proof of Theorem 3. The condition (14) of the theorem leads to the existence of a positive number $T \geq t_1$ and such that the following estimates hold:

$$\begin{aligned} &(a) \quad \int_{T}^{+\infty} a(r)f(cy_s(r))dr < \frac{c_0}{2}, \\ &(b) \quad \int_{T}^{+\infty} b(r)g(c_0 x_s(r))dr < \frac{c}{2}. \end{aligned} \tag{15}$$

Here $x_s(r)$ and $y_s(r)$ are defined analogously to those in (14) being taken $T$ instead $t_1$. Denote by $B$ the Banach space of all continuous vectorial functions $z(t) = (x(t), y(t))$ with the norm

$$\|z\| = \|(x, y)\| \equiv \max\left\{\sup_{t\geq T}\left|\frac{x(t)}{t}\right|,\ \sup_{t\geq T}\left|\frac{y(t)}{t}\right|\right\} < +\infty.$$

Define the set

$$\Omega \equiv \left\{z(t) = \Big(x(t), y(t)\Big) \in C\Big([\sigma(T), +\infty); \mathbb{R} \times \mathbb{R}\Big)\right\}, \tag{16}$$

that contains all vectorial functions $z = z(t)$ defined by

$z(t) \equiv$

$$\begin{cases} x(t) = \dfrac{c_0 t}{2}, & y(t) = \dfrac{ct}{2}, \quad \sigma(T) \le t < T, \\ \Big(x(t),\ y(t)\Big),\ \dfrac{c_0 t}{2} \le x(t) \le c_0 t, & \Big(x(t),\ y(t)\Big),\ \dfrac{ct}{2} \le y(t) \le ct,\ t \ge T. \end{cases} \tag{17}$$

Next define the operator $S : \Omega \to C\Big([\sigma(T), +\infty); \mathbb{R} \times \mathbb{R}\Big)$ by the formulae

$$Sx(t) = \begin{cases} \dfrac{c_0 t}{2}\ , & \sigma(T) \le t < T, \\ \dfrac{c_0 t}{2} + \displaystyle\int_T^t \int_T^r a(u) f(y_s(u)) du dr\ , & t \ge T, \end{cases} \tag{18}$$

and

$$Sy(t) = \begin{cases} \dfrac{ct}{2}\ , & \sigma(T) \le t < T, \\ \dfrac{ct}{2} + \displaystyle\int_T^t \int_r^{+\infty} b(u) g(x_s(u)) du dr\ , & t \ge T. \end{cases} \tag{19}$$

Let $z = (x, y)$ be an arbitrary point in $\Omega$, and then having in mind (15) and the definition of $S$ by (18) and (19) obtain the inequalities

$$\begin{aligned} &(a) \quad \frac{c_0 t}{2} \le Sx(t) \le \frac{c_0 t}{2} + t \int_T^{+\infty} a(r) f(c y_s(r)) dr \le \frac{c_0 t}{2} + \frac{c_0 t}{2} = c_0 t, \quad t \ge T, \\ &(b) \quad \frac{ct}{2} \le Sy(t) \le \frac{ct}{2} + t \int_T^{+\infty} b(r) g(c_0 x_s(r)) dr \le \frac{ct}{2} + \frac{ct}{2} = ct, \quad t \ge T. \end{aligned} \tag{20}$$

Therefore we conclude from (20), (a) and (b) that $S\Omega \subseteq \Omega$. Further we prove by the same arguments as in the proof of Theorem 3 (see, e.g. [11]), that the operator $S$ has a fixed point $z = (x, y)$, that is, $S(x, y) = (x, y)$. One can show that the solution of (1) could be obtained by solving the system

$$\begin{aligned} x(t) &= \frac{c_0 t}{2} + \int_T^t \int_T^r a(u) f(y_s(u)) du dr\ , \quad t \ge T, \\ y(t) &= \frac{ct}{2} + \int_T^t \int_r^{+\infty} b(u) g(x_s(u)) du dr\ , \quad t \ge T. \end{aligned}$$

In the case when the number $T$ is taken too close to $t_0$ we use the same conclusion as in the proof of Theorem 3. □

As a final note we touch on some open questions resulting from our investigation. First one is when the functions $\alpha(t), \beta(t)$ satisfy different requirements as either $\alpha < +\infty, \beta = +\infty$ or $\alpha = +\infty, \beta < +\infty$. These cases for a system without "maxima" could be seen in [11]. Other open questions are the existence of periodic solutions, oscillating properties and blowing-up. Bifurcation theory for these equations with "maxima" also would be of great interest.

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