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# LINEAR STABILITY ANALYSIS OF REACTION FRONTS PROPAGATION IN LIQUIDS WITH VIBRATIONS

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**Abstract:** Influence of vibrations on the onset of convective instability of reaction fronts in a liquid medium is studied. The model consists of a reaction-diffusion system coupled with the Navier-Stokes equations under the Boussinesq approximation. Linear stability analysis of the problem is fulfilled, and the convective instability boundary is found.

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**Key Words:** convective instability, reaction front propagation, vibrations

## 1. Introduction

Various instabilities can accompany and influence propagation of reaction fronts. Among them, thermo-diffusional, hydrodynamical and convective instabilities. The first one appears as a result of competition between heat production in the reaction zone and heat transfer from the reaction zone to the cold reactants. To study this type of instability, the density of the medium can be taken constant in order to remove the influence of hydrodynamics and to simplify the model. Stability

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conditions and nonlinear dynamics of front propagation in this case have been intensively studied [1], [3], [9], [14]. Hydrodynamic instability of reaction fronts, also called Darrieus-Landau instability, can occur if the density of the medium is variable. Usually it is considered as a given function of the temperature. The instability is caused by heat expansion of the gas or liquid in a neighborhood of the reaction zone [4], [7], [8], [17]. Convective instability appears due to natural convection. This instability can be distinguished from the hydrodynamic instability if we consider the Navier-Stokes equations under the Boussinesq approximation, i.e., we neglect the change of density everywhere except for the buoyancy term. The Boussinesq approximation was justified and used to study the front stability in [10], [11].

Influence of vibrations on convective instability of reaction fronts with a liquid reactant and a solid product was studied in [2]. This investigation was motivated by the preparation of microgravity experiments at the International Space Station where g-jitter, that is high frequency small amplitude vibrations should be taken into account. The convective instability boundary was found depending on different parameters.

This work is devoted to the influence of vibrations on reaction fronts when both, the reactant and the product of the reaction are in the liquid phase. This is the case of some polymerization fronts. Convective instability of reaction fronts in liquids without vibrations was studied in [5], [6], [12]. In order to study the influence of vibrations, we impose harmonic oscillations with some given frequency and amplitude. The time dependence of the instantaneous acceleration acting on the fluids is $g + b(t)$, where $g$ is the gravity acceleration and $b(t) = \lambda \sin(\mu t)$.

The paper is organized as follows. We present the model in Section 2. Jump conditions at the interface are derived in Section 3. In Section 4 we formulate the interface problem. Section 5 is devoted to linear stability analysis. We discuss the results and give conclusions in the last two sections.

## 2. Model

Reaction front propagation is described by the system of equations which includes heat equation, equation for the concentration and Navier-Stokes equations under the Boussinesq approximation. Harmonic oscillations of the gravity acceleration are applied in the vertical direction. The fluid is incompressible. The model studied in this work is as follows:

$$\frac{\partial T}{\partial t} + (v.\nabla)T = \kappa \Delta T + qW, \tag{2.1}$$

$$\frac{\partial \alpha}{\partial t} + (v.\nabla)\alpha = W, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{\rho}\nabla p + \nu \Delta v + g(1 + \lambda \sin(\sigma t))\beta(T - T_0)\gamma, \tag{2.3}$$

$$div(v) = 0, \tag{2.4}$$

with the conditions:

$$z \to +\infty : \quad T = T_i,\ \alpha = 0,\ v = 0,$$

$$z \to -\infty : \quad T = T_b,\ \alpha = 1,\ v = 0.$$

Here $T$ is the temperature, $\alpha$ the concentration of the reaction product, $v$ the velocity, $p$ the pressure, $\kappa$ the coefficient of thermal diffusivity, $q$ the adiabatic heat release, $\rho$ the density, $\nu$ the coefficient of kinematic viscosity, $\gamma$ the unit vector in the z-direction (upward), $\beta$ the coefficient of thermal expansion, $g$ the gravity acceleration, $T_0$ is the mean value of temperature, $T_i$ is the initial temperature and $T_b = T_i + q$ is the temperature of the reacted mixture. We assume that the chemical reaction is one-step zero order reaction. The reaction rate is considered in the following form:

$$W = k(T)\phi(\alpha), \qquad \phi(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha < 1 \\ 0 & \text{if} \quad \alpha = 1 \end{cases}.$$

The temperature dependence of the reaction rate is given by the Arrhenius law, $k(T) = k_0 exp(-E/R_0T)$, where $k_0$ is the pre-exponential factor, $R_0$ the universal gas constant, $E$ is the activation energy. In the sequel, we will assume that the activation energy is sufficiently large. The coefficient of mass diffusion is supposed to be small comparatively to the thermal diffusivity coefficient, so that the diffusion term in the equation for the concentration is neglected.

In order to give a dimensionless formulation of the problem, we introduce the following variables:

$$x_1 = \frac{xc_1}{\kappa}, \quad y_1 = \frac{yc_1}{\kappa}, \quad z_1 = \frac{zc_1}{\kappa},$$

$$t_1 = \frac{tc_1^2}{\kappa}, \quad p_1 = \frac{p}{c_1^2\rho}, \quad c_1 = \frac{c}{\sqrt{2}}, \quad v_1 = \frac{v}{c_1}, \quad \theta = \frac{T - T_b}{q},$$

where $c$ denotes the stationary propagation front velocity, which can be calculated asymptotically for large activation energies [15]:

$$c^2 = \frac{2k_0\kappa R_0 T_b^2}{qE}\exp(-\frac{E}{R_0T_b}).$$

For convenience, we keep the same notation for all variables except for the temperature. We rewrite the system in the form:

$$\frac{\partial\theta}{\partial t} + (v.\nabla)\theta = \Delta\theta + Z\exp\left(\frac{\theta}{Z^{-1} + \delta\theta}\right)\phi(\alpha), \tag{2.5}$$

$$\frac{\partial\alpha}{\partial t} + (v.\nabla)\alpha = Z\exp(\left(\frac{\theta}{Z^{-1} + \delta\theta}\right)\phi(\alpha), \tag{2.6}$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\nabla p + P\Delta v + PR(1+\lambda\sin(\mu t))(\theta+\theta_0)\gamma, \tag{2.7}$$

$$div(v) = 0, \tag{2.8}$$

The conditions at infinity for the dimensionless model are:

$$z \to +\infty: \ \theta = -1, \ \alpha = 0, \ v = 0,$$

$$z \to -\infty: \ \theta = 0, \ \alpha = 1, \ v = 0,$$

where $P = \dfrac{\nu}{\kappa}$ is the Prandtl number, $R = g\beta q\kappa^2/(\nu c_1^3)$ is the Rayleigh number, $Z = qE/R_0T_b^2$ is the Zeldovich number, $\delta = R_0T_b/E$, $\theta_0 = (T_b - T_0)/q$ and $\mu = 2\kappa\sigma/c^2$.

## 3. Approximation of Narrow Reaction Zone

To study the problem analytically we use the Zeldovich-Frank-Kamenetskii approximation, called also the narrow zone method. For this purpose, we assume that the activation energy is large and the reaction zone is narrow [16]. We perform a formal asymptotic analysis with $\epsilon = Z^{-1}$ taken as a small parameter. The new independent variable is given by $z_1 = z - \zeta(x,y,t)$, where $\zeta(x,y,t)$ denotes the reaction zone location. We introduce the new functions $\theta_1, \alpha_1, v_1, p_1$:

$$\theta(x,y,z,t) = \theta_1(x,y,z_1,t), \qquad \alpha(x,y,z,t) = \alpha_1(x,y,z_1,t),$$

$$v(x,y,z,t) = v_1(x,y,z_1,t), \qquad p(x,y,z,t) = p_1(x,y,z_1,t).$$

We rewrite the equations (2.5)-(2.8) in the form:

$$\frac{\partial \theta}{\partial t} - \frac{\partial \theta}{\partial z_1}\frac{\partial \zeta}{\partial t} + (v.\tilde{\nabla})\theta = \tilde{\Delta}\theta + Z\exp\left(\frac{\theta}{Z^{-1}+\delta\theta}\right)\phi(\alpha), \tag{3.1}$$

$$\frac{\partial \alpha}{\partial t} - \frac{\partial \alpha}{\partial z_1}\frac{\partial \zeta}{\partial t} + (v.\tilde{\nabla})\alpha = Z\exp\left(\frac{\theta}{Z^{-1}+\delta\theta}\right)\phi(\alpha), \tag{3.2}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z_1}\frac{\partial \zeta}{\partial t} + (v.\tilde{\nabla})v = -\tilde{\nabla}p + P\tilde{\Delta}v + Q(1+\lambda\sin(\mu t))(\theta+\theta_0)\gamma, \tag{3.3}$$

$$\frac{\partial v_x}{\partial x} - \frac{\partial v_x}{\partial z_1}\frac{\partial \zeta}{\partial x} + \frac{\partial v_y}{\partial y} - \frac{\partial v_y}{\partial z_1}\frac{\partial \zeta}{\partial y} + \frac{\partial v_z}{\partial z_1} = 0, \tag{3.4}$$

where

$$\tilde{\Delta} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_1^2} - 2\frac{\partial^2}{\partial x \partial z_1}\frac{\partial \zeta}{\partial x} - 2\frac{\partial^2}{\partial y \partial z_1}\frac{\partial \zeta}{\partial y} +$$

$$\frac{\partial^2}{\partial z_1^2}\left(\left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right) - \frac{\partial}{\partial z_1}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right),$$

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$$\tilde{\nabla} = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial z_1}\frac{\partial \zeta}{\partial x}, \frac{\partial}{\partial y} - \frac{\partial}{\partial z_1}\frac{\partial \zeta}{\partial y}, \frac{\partial}{\partial z_1} \right) \quad \text{and} \quad Q = PR.$$

We use matched asymptotic expansions and seek the outer solution of the problem (3.1)-(3.4) in the form of the expansion:

$$\theta = \theta_0 + \epsilon\theta_1 + ..., \quad \alpha = \alpha_0 + \epsilon\alpha_1 + ..., \quad v = v_0 + \epsilon v_1 + ..., \quad p = p_0 + \epsilon p_1 + ....$$

To obtain the jump conditions, we will consider the inner problem. The stretched coordinate is $\eta = z_1\epsilon^{-1}$, with $\epsilon = Z^{-1}$. We look for the inner solution in the following form:

$$\theta = \epsilon\tilde{\theta}_1 + ..., \quad \alpha = \tilde{\alpha}_0 + \epsilon\tilde{\alpha}_1 + ..., \tag{3.5}$$

$$v = \tilde{v}_0 + \epsilon\tilde{v}_1 + ..., \quad p = \tilde{p}_0 + \epsilon\tilde{p}_1 + ..., \quad \zeta = \zeta_0 + \varepsilon\zeta_1 + .... \tag{3.6}$$

Substituting these expansions in (3.1)-(3.4), we obtain:

**Order $\epsilon^{-2}$:**

$$P\left(1 + \left(\frac{\partial \zeta_0}{\partial x}\right)^2 + \left(\frac{\partial \zeta_0}{\partial y}\right)^2\right)\frac{\partial^2 \tilde{v}_0}{\partial \eta^2} = 0. \tag{3.7}$$

**Order $\epsilon^{-1}$:**

$$\left(1 + \left(\frac{\partial \zeta_0}{\partial x}\right)^2 + \left(\frac{\partial \zeta_0}{\partial y}\right)^2\right)\frac{\partial^2 \tilde{\theta}_1}{\partial \eta^2} + \exp\left(\frac{\tilde{\theta}_1}{1+\delta\tilde{\theta}_1}\right)\phi(\tilde{\alpha}_0) = 0, \tag{3.8}$$

$$-\frac{\partial \tilde{\alpha}_0}{\partial \eta}\frac{\partial \zeta_0}{\partial t} - \frac{\partial \tilde{\alpha}_0}{\partial \eta}\left(\tilde{v}_{0x}\frac{\partial \zeta_0}{\partial x} + \tilde{v}_{0y}\frac{\partial \zeta_0}{\partial y} - \tilde{v}_{0z}\right) = \exp\left(\frac{\tilde{\theta}_1}{1+\delta\tilde{\theta}_1}\right)\phi(\tilde{\alpha}_0), \tag{3.9}$$

$$-\frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_0}{\partial t} - \tilde{v}_{0x}\frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_0}{\partial x} - \tilde{v}_{0y}\frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_0}{\partial y} + \tilde{v}_{0z}\frac{\partial \tilde{v}_0}{\partial \eta}$$
$$= t_0\frac{\partial \tilde{p}_0}{\partial \eta} + P\left(A\frac{\partial^2 \tilde{v}_1}{\partial \eta^2} + t_3\frac{\partial^2 \tilde{v}_0}{\partial \eta^2} + F_0\frac{\partial \tilde{v}_0}{\partial \eta}\right), \tag{3.10}$$

$$-\frac{\partial \tilde{v}_{0x}}{\partial \eta}\frac{\partial \zeta_0}{\partial x} - \frac{\partial \tilde{v}_{0y}}{\partial \eta}\frac{\partial \zeta_0}{\partial y} + \frac{\partial \tilde{v}_{0z}}{\partial \eta} = 0. \tag{3.11}$$

**Order $\epsilon^0$:**

$$\frac{\partial \tilde{v}_0}{\partial t} - \frac{\partial \tilde{v}_1}{\partial \eta}\frac{\partial \zeta_0}{\partial t} - \frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_1}{\partial t} + \tilde{v}_{0x}\left(\frac{\partial \tilde{v}_0}{\partial x} - \frac{\partial \tilde{v}_1}{\partial \eta}\frac{\partial \zeta_0}{\partial x} - \frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_1}{\partial x}\right) + \tilde{v}_{1x}\frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_0}{\partial x} + \tilde{v}_{1y}\frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_0}{\partial y}$$
$$+\tilde{v}_{0y}\left(\frac{\partial \tilde{v}_0}{\partial y} - \frac{\partial \tilde{v}_1}{\partial \eta}\frac{\partial \zeta_0}{\partial y} - \frac{\partial \tilde{v}_0}{\partial \eta}\frac{\partial \zeta_1}{\partial y}\right) + \tilde{v}_{0z}\frac{\partial \tilde{v}_1}{\partial \eta} + \tilde{v}_{1z}\frac{\partial \tilde{v}_0}{\partial \eta} = -\nabla_0\tilde{p}_0 + t_1\frac{\partial \tilde{p}_0}{\partial \eta} + t_0\frac{\partial \tilde{p}_1}{\partial \eta}$$
$$+P\left(A\frac{\partial^2 \tilde{v}_2}{\partial \eta^2} + t_3\frac{\partial^2 \tilde{v}_1}{\partial \eta^2} + F_0\frac{\partial \tilde{v}_1}{\partial \eta} + t_4\frac{\partial^2 \tilde{v}_0}{\partial \eta^2} + F_1\frac{\partial \tilde{v}_0}{\partial \eta} + \Delta_1\tilde{v}_0\right) + Q(1+\lambda\sin(\mu t))\gamma\theta_0, \tag{3.12}$$

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$$\frac{\partial\tilde{v}_{0x}}{\partial x}-\frac{\partial\tilde{v}_{1x}}{\partial\eta}\frac{\partial\zeta_0}{\partial x}-\frac{\partial\tilde{v}_{0x}}{\partial\eta}\frac{\partial\zeta_1}{\partial x}+\frac{\partial\tilde{v}_{0y}}{\partial y}-\frac{\partial\tilde{v}_{1y}}{\partial\eta}\frac{\partial\zeta_0}{\partial y}-\frac{\partial\tilde{v}_{0y}}{\partial\eta}\frac{\partial\zeta_1}{\partial y}+\frac{\partial\tilde{v}_{1z}}{\partial\eta}=0. \quad (3.13)$$

**Order $\epsilon^1$:**

$$\begin{aligned}
&\frac{\partial\tilde{v}_1}{\partial t}-\left(\frac{\partial\tilde{v}_2}{\partial\eta}\frac{\partial\zeta_0}{\partial t}+\frac{\partial\tilde{v}_1}{\partial\eta}\frac{\partial\zeta_1}{\partial t}+\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_2}{\partial t}\right)\\
&\quad+\tilde{v}_{0x}\left(\frac{\partial\tilde{v}_1}{\partial x}-\frac{\partial\tilde{v}_1}{\partial\eta}\frac{\partial\zeta_1}{\partial x}-\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_2}{\partial x}-\frac{\partial\tilde{v}_2}{\partial\eta}\frac{\partial\zeta_0}{\partial x}\right)\\
&\quad+\tilde{v}_{1x}\left(\frac{\partial\tilde{v}_0}{\partial x}-\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_1}{\partial x}-\frac{\partial\tilde{v}_1}{\partial\eta}\frac{\partial\zeta_0}{\partial x}\right)\\
&\quad-\tilde{v}_{2x}\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_0}{\partial x}+\tilde{v}_{0y}\left(\frac{\partial\tilde{v}_1}{\partial y}-\frac{\partial\tilde{v}_1}{\partial\eta}\frac{\partial\zeta_1}{\partial y}-\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_2}{\partial y}-\frac{\partial\tilde{v}_2}{\partial\eta}\frac{\partial\zeta_0}{\partial y}\right)\\
&+\tilde{v}_{1y}\left(\frac{\partial\tilde{v}_0}{\partial y}-\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_1}{\partial y}-\frac{\partial\tilde{v}_1}{\partial\eta}\frac{\partial\zeta_0}{\partial y}\right)-\tilde{v}_{2y}\frac{\partial\tilde{v}_0}{\partial\eta}\frac{\partial\zeta_0}{\partial y}+\tilde{v}_{0z}\frac{\partial\tilde{v}_2}{\partial\eta}+\tilde{v}_{1z}\frac{\partial\tilde{v}_1}{\partial\eta}+\tilde{v}_{2z}\frac{\partial\tilde{v}_0}{\partial\eta}\\
&\quad=t_0\frac{\partial\tilde{p}_2}{\partial\eta}+t_1\frac{\partial\tilde{p}_1}{\partial\eta}-\nabla_0\tilde{p}_1+t_2\frac{\partial\tilde{p}_0}{\partial\eta}\\
&\quad+P\left(A\frac{\partial^2\tilde{v}_3}{\partial\eta^2}+t_3\frac{\partial^2\tilde{v}_2}{\partial\eta^2}+F_0\frac{\partial\tilde{v}_2}{\partial\eta}+F_1\frac{\partial\tilde{v}_1}{\partial\eta}+\Delta_1\tilde{v}_1+F_2\frac{\partial\tilde{v}_0}{\partial\eta}\right)\\
&\quad+Q(1+\lambda\sin(\mu t))\gamma\theta_1, \quad (3.14)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial\tilde{v}_{1x}}{\partial x}-\frac{\partial\tilde{v}_{2x}}{\partial\eta}\frac{\partial\zeta_0}{\partial x}-\frac{\partial\tilde{v}_{1x}}{\partial\eta}\frac{\partial\zeta_1}{\partial x}-\frac{\partial\tilde{v}_{0x}}{\partial\eta}\frac{\partial\zeta_2}{\partial x}+\frac{\partial\tilde{v}_{1y}}{\partial y}-\frac{\partial\tilde{v}_{2y}}{\partial\eta}\frac{\partial\zeta_0}{\partial y}-\frac{\partial\tilde{v}_{1y}}{\partial\eta}\frac{\partial\zeta_1}{\partial y}-\frac{\partial\tilde{v}_{0y}}{\partial\eta}\frac{\partial\zeta_2}{\partial y}\\
&\quad+\frac{\partial\tilde{v}_{2z}}{\partial\eta}=0. \quad (3.15)
\end{aligned}$$

**Order $\epsilon^2$:**

$$\begin{aligned}
&\frac{\partial\tilde{v}_{2x}}{\partial x}-\frac{\partial\tilde{v}_{3x}}{\partial\eta}\frac{\partial\zeta_0}{\partial x}-\frac{\partial\tilde{v}_{2x}}{\partial\eta}\frac{\partial\zeta_1}{\partial x}-\frac{\partial\tilde{v}_{1x}}{\partial\eta}\frac{\partial\zeta_2}{\partial x}-\frac{\partial\tilde{v}_{0x}}{\partial\eta}\frac{\partial\zeta_3}{\partial x}+\frac{\partial\tilde{v}_{2y}}{\partial y}\\
&\quad-\frac{\partial\tilde{v}_{3y}}{\partial\eta}\frac{\partial\zeta_0}{\partial y}-\frac{\partial\tilde{v}_{2y}}{\partial\eta}\frac{\partial\zeta_1}{\partial y}-\frac{\partial\tilde{v}_{1y}}{\partial\eta}\frac{\partial\zeta_2}{\partial y}-\frac{\partial\tilde{v}_{0y}}{\partial\eta}\frac{\partial\zeta_3}{\partial y}+\frac{\partial\tilde{v}_{3z}}{\partial\eta}=0. \quad (3.16)
\end{aligned}$$

Where

$$\Delta_1=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \quad \nabla_0=\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big),$$

$$F_i=-2\left(\frac{\partial\zeta_i}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\zeta_i}{\partial y}\frac{\partial}{\partial y}\right)-\left(\frac{\partial^2\zeta_i}{\partial x^2}+\frac{\partial^2\zeta_i}{\partial y^2}\right)I, \quad i=0,1,2,$$

$$t_0=\left(\frac{\partial\zeta_0}{\partial x},\frac{\partial\zeta_0}{\partial y},-1\right), \quad t_i=\left(\frac{\partial\zeta_i}{\partial x},\frac{\partial\zeta_i}{\partial y},-1\right) \quad i=1,2, \quad t_3=2\left(\frac{\partial\zeta_1}{\partial x}\frac{\partial\zeta_0}{\partial x}+\frac{\partial\zeta_0}{\partial y}\frac{\partial\zeta_1}{\partial y}\right),$$

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$$t_4 = \left(\frac{\partial \zeta_1}{\partial x}\right)^2 + \left(\frac{\partial \zeta_1}{\partial y}\right)^2 + \frac{\partial \zeta_0}{\partial x}\frac{\partial \zeta_2}{\partial x} + \frac{\partial \zeta_0}{\partial y}\frac{\partial \zeta_2}{\partial y}, \quad A = \left(1 + \left(\frac{\partial \zeta_0}{\partial x}\right)^2 + \left(\frac{\partial \zeta_0}{\partial y}\right)^2\right).$$

Here $I$ is the identity operator. The matching conditions are:

$$\tilde{v}_0 \sim v_0|_{z_1=\pm 0}, \tag{3.17}$$

$$\tilde{v}_1 \sim \left(\left.\frac{\partial v_0}{\partial z_1}\right|_{z_1=\pm 0}\right)\eta + v_1|_{z_1=\pm 0}, \tag{3.18}$$

$$\tilde{v}_2 \sim \frac{1}{2}\left(\left.\frac{\partial^2 v_0}{\partial z_1^2}\right|_{z_1=\pm 0}\right)\eta^2 + \left(\left.\frac{\partial v_1}{\partial z_1}\right|_{z_1=\pm 0}\right)\eta + v_2\,|_{z_1=\pm 0}\ , \tag{3.19}$$

and

$$\tilde{v}_3 \sim \frac{1}{6}\left(\left.\frac{\partial^3 v_0}{\partial z_1^3}\right|_{z_1=\pm 0}\right)\eta^3 + \frac{1}{2}\left(\left.\frac{\partial^2 v_1}{\partial z_1^2}\right|_{z_1=\pm 0}\right)\eta^2 + \left(\left.\frac{\partial v_2}{\partial z_1}\right|_{z_1=\pm 0}\right)\eta + v_3\,|_{z_1=\pm 0}\ , \tag{3.20}$$

when $\eta \to +\infty$:

$$\tilde{\theta}_1 \approx \theta_1\,|_{z_1=+0} + \left(\left.\frac{\partial \theta_0}{\partial z_1}\right|_{z_1=+0}\right)\eta, \quad \tilde{\alpha}_0 \to 0, \tag{3.21}$$

when $\eta \to -\infty$:

$$\tilde{\theta}_1 \sim \theta_1\,|_{z_1=-0} \quad \tilde{\alpha}_0 \to 1. \tag{3.22}$$

From (3.7), we obtain:

$$\frac{\partial^2 \tilde{v}_0}{\partial \eta^2} = 0,$$

thus $\tilde{v}_0(\eta)$ is a linear function of $\eta$ and from the fact that the velocity is bounded, it will be identically constant. Using (3.17) we have :

$$v_0\,|_{z_1=+0} = v_0\,|_{z_1=-0},\ , \tag{3.23}$$

$$\frac{\partial \tilde{v}_0}{\partial \eta} = 0. \tag{3.24}$$

Consequently, the first term in the expression of the velocity $v_0$ is continuous across the front. In view of (3.24), the equations (3.10) and (3.13) take the form:

$$PA\frac{\partial^2 \tilde{v}_1}{\partial \eta^2} + t_0\frac{\partial \tilde{p}_0}{\partial \eta} = 0, \tag{3.25}$$

and

$$\frac{\partial \tilde{v}_{0x}}{\partial x} - \frac{\partial \tilde{v}_{1x}}{\partial \eta}\frac{\partial \zeta_0}{\partial x} + \frac{\partial \tilde{v}_{0y}}{\partial y} - \frac{\partial \tilde{v}_{1y}}{\partial \eta}\frac{\partial \zeta_0}{\partial y} + \frac{\partial \tilde{v}_{1z}}{\partial \eta} = 0. \tag{3.26}$$

By derivation of the previous equation, we have :

$$\frac{\partial^2 \tilde{v}_{1x}}{\partial \eta^2}\frac{\partial \zeta_0}{\partial x} + \frac{\partial^2 \tilde{v}_{1y}}{\partial \eta^2}\frac{\partial \zeta_0}{\partial y} - \frac{\partial^2 \tilde{v}_{1z}}{\partial \eta^2} = 0. \tag{3.27}$$

Recall that (3.25) is a vector equation with three components. We multiply the first component by $\frac{\partial \zeta_0}{\partial x}$, the second by $\frac{\partial \zeta_0}{\partial y}$ and the third by $-1$, we have:

$$PA\left(\frac{\partial^2 \tilde{v}_{1x}}{\partial \eta^2}\frac{\partial \zeta_0}{\partial x} + \frac{\partial^2 \tilde{v}_{1y}}{\partial \eta^2}\frac{\partial \zeta_0}{\partial y} - \frac{\partial^2 \tilde{v}_{1z}}{\partial \eta^2}\right) + A\frac{\partial \tilde{p}_0}{\partial \eta} = 0.$$

We consider now the relation (3.27). It follow that:

$$\frac{\partial \tilde{p}_0}{\partial \eta} = 0, \tag{3.28}$$

from the equation (3.25), we have:

$$\frac{\partial^2 \tilde{v}_1}{\partial \eta^2} = 0, \tag{3.29}$$

using equation (3.18) and that the velocity is bounded, we obtain the following continuity equalities:

$$\left.\frac{\partial v_0}{\partial z_1}\right|_{z_1=+0} = \left.\frac{\partial v_0}{\partial z_1}\right|_{z_1=-0}, \tag{3.30}$$

$$v_1\,|_{z_1=+0} = v_1\,|_{z_1=-0}.$$

Differentiating (3.15) twice, (3.12) once with respect to $\eta$, using (3.24), (3.28) and (3.29), we obtain:

$$t_0\frac{\partial^2 \tilde{p}_1}{\partial \eta^2} + PA\frac{\partial^3 \tilde{v}_2}{\partial \eta^3} = 0. \tag{3.31}$$

We also have:

$$\frac{\partial^3 \tilde{v}_{2x}}{\partial \eta^3}\frac{\partial \zeta_0}{\partial x} + \frac{\partial^3 \tilde{v}_{2y}}{\partial \eta^3}\frac{\partial \zeta_0}{\partial y} - \frac{\partial^3 \tilde{v}_{2z}}{\partial \eta^3} = 0. \tag{3.32}$$

As above, multiplying the components of (3.31) respectively by $\dfrac{\partial \zeta_0}{\partial x}$, $\dfrac{\partial \zeta_0}{\partial y}$ and $-1$, we have:

$$PA\left(\frac{\partial^3 \tilde{v}_{2x}}{\partial \eta^3}\frac{\partial \zeta_0}{\partial x} + \frac{\partial^3 \tilde{v}_{2y}}{\partial \eta^3}\frac{\partial \zeta_0}{\partial y} - \frac{\partial^3 \tilde{v}_{2z}}{\partial \eta^3}\right) + A\frac{\partial^2 \tilde{p}_1}{\partial \eta^2} = 0.$$

From this equation and (3.32), we can write:

$$\frac{\partial^2 \tilde{p}_1}{\partial \eta^2} = 0. \tag{3.33}$$

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From (3.31), we have:

$$\frac{\partial^3 \tilde{v}_2}{\partial \eta^3} = 0. \tag{3.34}$$

Since the velocity is bounded and taking into account (3.19), we obtain the following jump conditions:

$$\begin{aligned} \frac{\partial^2 v_0}{\partial z_1^2}\bigg|_{z_1=+0} &= \frac{\partial^2 v_0}{\partial z_1^2}\bigg|_{z_1=-0}, \\ \frac{\partial v_1}{\partial z_1}\bigg|_{z_1=+0} &= \frac{\partial v_1}{\partial z_1}\bigg|_{z_1=-0}, \\ v_2\,|_{z_1=+0} &= v_2\,|_{z_1=-0}. \end{aligned} \tag{3.35}$$

Taking the third derivative of (3.16) and the second derivative of (3.14) with respect to $\eta$ and considering (3.24), (3.28), (3.33) and (3.34), we have :

$$t_0 \frac{\partial^3 \tilde{p}_2}{\partial \eta^3} + PA \frac{\partial^4 \tilde{v}_3}{\partial \eta^4} + Q(1+\lambda \sin(\mu t))\gamma \frac{\partial^2 \tilde{\theta}_1}{\partial \eta^2} = 0, \tag{3.36}$$

and

$$\frac{\partial^4 \tilde{v}_{3x}}{\partial \eta^4}\frac{\partial \zeta_0}{\partial x} + \frac{\partial^4 \tilde{v}_{3y}}{\partial \eta^4}\frac{\partial \zeta_0}{\partial y} - \frac{\partial^4 \tilde{v}_{3z}}{\partial \eta^4} = 0. \tag{3.37}$$

We proceed as before and get:

$$PA\left(\frac{\partial^4 \tilde{v}_{3x}}{\partial \eta^4}\frac{\partial \zeta_0}{\partial x} + \frac{\partial^4 \tilde{v}_{3y}}{\partial \eta^4}\frac{\partial \zeta_0}{\partial y} - \frac{\partial^4 \tilde{v}_{3z}}{\partial \eta^4}\right) + A\frac{\partial^3 \tilde{p}_2}{\partial \eta^3} - Q(1+\lambda \sin(\mu t))\frac{\partial^2 \tilde{\theta}_1}{\partial \eta^2} = 0.$$

Then, from the equation (3.36) and (3.37), we obtain:

$$A\frac{\partial^3 \tilde{p}_2}{\partial \eta^3} - Q(1+\lambda \sin(\mu t))\frac{\partial^2 \tilde{\theta}_1}{\partial \eta^2} = 0,$$

and

$$\gamma_0 \frac{\partial^2 \tilde{\theta}_1}{\partial \eta^2} = \frac{\partial^4 \tilde{v}_3}{\partial \eta^4}, \tag{3.38}$$

with

$$\gamma_0 = \left(-\frac{\partial \zeta_0}{\partial x}\frac{R}{A^2}, -\frac{\partial \zeta_0}{\partial y}\frac{R}{A^2}, \frac{R}{A^2} - \frac{R}{A}\right)(1+\lambda \sin(\mu t)).$$

Integrating (3.38) with respect to $\eta$ and using (3.20), (3.21) and (3.22), we have:

$$\frac{\partial^3 v_0}{\partial z_1^3}\bigg|_{z_1=-0} - \frac{\partial^3 v_0}{\partial z_1^3}\bigg|_{z_1=+0} = -\gamma_0 \frac{\partial \theta_0}{\partial z_1}\bigg|_{z_1=+0}. \tag{3.39}$$

The equations (3.23), (3.30), (3.35) and (3.39) give the jump conditions for the velocity at the front.

From equations (3.9), (3.11) and (3.24), we conclude that $\tilde{\alpha}_0$ is a monotonic function satisfying $0 < \tilde{\alpha}_0 < 1$. Since the reaction is of the order zero, we have $\phi(\tilde{\alpha}_0) \equiv 1$. We multiply (3.8) by $\frac{\partial \tilde{\theta}_1}{\partial \eta}$ and integrate:

$$\left(\frac{\partial \tilde{\theta}_1}{\partial \eta}\right)^2\Bigg|_{+\infty} - \left(\frac{\partial \tilde{\theta}_1}{\partial \eta}\right)^2\Bigg|_{-\infty} = 2A^{-1}\int_{-\infty}^{\theta_1|_{z_1=-0}} \exp\left(\frac{\tau}{1+\tau\delta}\right) d\tau. \tag{3.40}$$

Subtracting (3.8) from (3.9) and integrating, we have:

$$\frac{\partial \tilde{\theta}_1}{\partial \eta}\Bigg|_{+\infty} - \frac{\partial \tilde{\theta}_1}{\partial \eta}\Bigg|_{-\infty} = -A^{-1}\left(\frac{\partial \zeta_0}{\partial t} + s\right), \tag{3.41}$$

where

$$s = \tilde{v}_{0x}\frac{\partial \zeta_0}{\partial x} + \tilde{v}_{0y}\frac{\partial \zeta_0}{\partial y} - \tilde{v}_{0z}.$$

From the last equations (3.40)-(3.41), we get the temperature jump conditions across the reaction front. Using the matching conditions, we have the following:

$$\theta \approx \theta_0, \quad \theta_1\,|_{z_1=-0} \approx Z\theta\,|_{z_1=+0}\,, \quad \zeta \approx \zeta_0, \quad v \approx v_0,$$

We can rewrite the jump conditions in the form (see [10], [13] and the references therein):

$$\left(\frac{\partial \theta}{\partial z_1}\right)^2\Bigg|_{+0} - \left(\frac{\partial \theta}{\partial z_1}\right)^2\Bigg|_{-0} = 2Z\left(1+\left(\frac{\partial \zeta}{\partial x}\right)^2+\left(\frac{\partial \zeta}{\partial y}\right)^2\right)^{-1}\int_{-\infty}^{\theta_1|_{z_1=-0}} \exp\left(\frac{\tau}{Z^{-1}+\tau\delta}\right) d\tau, \tag{3.42}$$

$$\frac{\partial \theta}{\partial z_1}\Bigg|_{z_1=+0} - \frac{\partial \theta}{\partial z_1}\Bigg|_{z_1=-0} = -\left(1+\left(\frac{\partial \zeta}{\partial x}\right)^2+\left(\frac{\partial \zeta}{\partial y}\right)^2\right)^{-1}\left(\frac{\partial \zeta}{\partial t} + \left(v_x\frac{\partial \zeta}{\partial x} + v_y\frac{\partial \zeta}{\partial y} - v_z\right)\Bigg|_{z_1=+0}\right), \tag{3.43}$$

$$v_z\,\Big|_{z_1=+0} = v_z\,\Big|_{z_1=-0}\,, \tag{3.44}$$

$$\frac{\partial v_z}{\partial z_1}\Bigg|_{z_1=+0} = \frac{\partial v_z}{\partial z_1}\Bigg|_{z_1=-0}, \tag{3.45}$$

$$\frac{\partial^2 v_z}{\partial z_1^2}\Bigg|_{z_1=+0} = \frac{\partial v_z}{\partial z_1^2}\Bigg|_{z_1=-0}, \tag{3.46}$$

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and

$$\left.\frac{\partial^3 v_z}{\partial z_1^3}\right|_{z_1=-0} - \left.\frac{\partial^3 v_z}{\partial z_1^3}\right|_{z_1=+0} = -R\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}$$
$$\times\left(\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}-1\right)(1+\lambda\sin(\mu t))\left.\frac{\partial\theta}{\partial z_1}\right|_{z_1=+0}. \quad (3.47)$$

Note that we consider only the matching condition for the velocity component $v_z$ since the other components are not used in the sequel.

## 4. Interface Problem

We will study the following interface problem which approximates the original system (2.5)-(2.8):

In the reactant region $z > \zeta$:

$$\frac{\partial\theta}{\partial t} + (v.\nabla)\theta = \Delta\theta, \quad (4.1)$$

$$\alpha = 0, \quad (4.2)$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\nabla p + P\Delta v + Q(1+\lambda\sin\mu t)(\theta+\theta_0)\gamma, \quad (4.3)$$

$$div(v) = 0. \quad (4.4)$$

In the product region $z < \zeta$:

$$\frac{\partial\theta}{\partial t} + (v.\nabla)\theta = \Delta\theta, \quad (4.5)$$

$$\alpha = 1, \quad (4.6)$$

$$\frac{\partial v}{\partial t} + (v.\nabla)v = -\nabla p + P\Delta v + Q(1+\lambda\sin\mu t)(\theta+\theta_0)\gamma, \quad (4.7)$$

$$div(v) = 0. \quad (4.8)$$

At the interface $z = \zeta$:

$$\theta\,|_{\zeta-0} = \theta\,|_{\zeta+0}\,, \quad (4.9)$$

$$\left.\frac{\partial\theta}{\partial z}\right|_{\zeta=-0} - \left.\frac{\partial\theta}{\partial z}\right|_{\zeta=+0} = \left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}$$
$$\times\left(\frac{\partial\zeta}{\partial t}+\left.\left(v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}-v_z\right)\right|_{\zeta}\right), \quad (4.10)$$

$$\left(\frac{\partial\theta}{\partial z}\right)^2\Big|_{\zeta-0} - \left(\frac{\partial\theta}{\partial z}\right)^2\Big|_{\zeta_1+0}$$

$$= -2Z\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}\int_{-\infty}^{\theta|_\zeta}\exp\left(\frac{\tau}{Z^{-1}+\delta\tau}\right)d\tau, \quad (4.11)$$

$$v_z\,|_{\zeta=-0} = v_z\,|_{\zeta=+0}\,, \quad (4.12)$$

$$\frac{\partial v_z}{\partial z}\Big|_{\zeta=-0} = \frac{\partial v_z}{\partial z}\Big|_{\zeta=+0}, \quad (4.13)$$

$$\frac{\partial^2 v_z}{\partial z^2}\Big|_{\zeta=-0} = \frac{\partial^2 v_z}{\partial z^2}\Big|_{\zeta=+0}, \quad (4.14)$$

$$\frac{\partial^3 v_z}{\partial z^3}\Big|_{\zeta=-0} - \frac{\partial^3 v_z}{\partial z^3}\Big|_{\zeta=+0} = -R\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}\times$$

$$\left(\left(1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2\right)^{-1}-1\right)(1+\lambda\sin(\mu t))\left(\frac{\partial\theta}{\partial z}\right)\Big|_{\zeta=+0}. \quad (4.15)$$

Conditions at infinity:

$$z=-\infty: \quad \theta=0, \quad v=0,$$

$$z=+\infty: \quad \theta=-1, \quad v=0. \quad (4.16)$$

## 5. Linear Stability Analysis

In this section we perform the linear stability analysis of the steady-state solution for the interface problem (4.1)-(4.16). This problem has a travelling wave solution given by:

$$(\theta(x,y,z,t),\alpha(x,y,z,t),v) = (\theta_s(z-ut),\alpha_s(z-ut),0)\,,$$

$$(\theta_s(z-ut),\alpha_s(z-ut)) = \begin{cases}(0,1), & z_2<0,\\ (\exp(-uz_2)-1,0), & z_2>0\end{cases} \quad (5.1)$$

and

$$z_2 = z-ut, \qquad (u=c),$$

where $u$ is the speed of the reaction front. This solution will be referred to as a basic solution. It is a stationary solution of (4.1)-(4.16) written in the moving coordinates, where equations (4.1), (4.3), (4.5) and (4.7) are replaced respectively by :

$$\frac{\partial\theta}{\partial t}+(v.\nabla)\theta = \Delta\theta + u\frac{\partial\theta}{\partial z_2}, \quad (5.2)$$

and
$$\frac{\partial v}{\partial t} + (v\nabla)v = -\nabla p + P\Delta v + u\frac{\partial v}{\partial z_2} + Q(1+\lambda\sin(\mu t))(\theta+\theta_0)\gamma. \tag{5.3}$$
All other equations remain unchanged. We seek the solution of this problem in the form:
$$\theta = \theta_s + \tilde{\theta}, \quad p = p_s + \tilde{p}, \quad v = v_s + \tilde{v}, \tag{5.4}$$
$\tilde{\theta}$, $\tilde{p}$ and $\tilde{v}$ are small perturbation of the temperature, pressure and velocity, respectively, $\theta_s$ , $p_s$ and $v_s$ are given by the basic solution (5.1). We substitute (5.4) into (5.2)-(5.3) and obtain for the first-order terms:

$\boldsymbol{z_2 > \xi}$**:**
$$\frac{\partial\tilde{\theta}}{\partial t} = \Delta\tilde{\theta} + u\frac{\partial\tilde{\theta}}{\partial z_2} - \tilde{v}_z\theta_s', \tag{5.5}$$
$$\frac{\partial\tilde{v}}{\partial t} = -\nabla\tilde{p} + P\Delta\tilde{v} + u\frac{\partial\tilde{v}}{\partial z_2} + Q(1+\lambda\sin(\mu t))\tilde{\theta}\gamma, \tag{5.6}$$
$$div(\tilde{v}) = 0,$$

$\boldsymbol{z_2 < \xi}$**:**
$$\frac{\partial\tilde{\theta}}{\partial t} = \Delta\tilde{\theta} + u\frac{\partial\tilde{\theta}}{\partial z_2}, \tag{5.7}$$
$$\frac{\partial\tilde{v}}{\partial t} = -\nabla\tilde{p} + P\Delta\tilde{v} + u\frac{\partial\tilde{v}}{\partial z_2} + Q(1+\lambda\sin(\mu t))\tilde{\theta}\gamma, \tag{5.8}$$
$$div(\tilde{v}) = 0,$$
with $\xi = \zeta - ut$. We note that
$$(\tilde{\theta}, \tilde{v}_z) = \begin{cases} (\hat{\theta}_1, \hat{v}_{z1}) & \text{for } z_2 < \xi, \\ (\hat{\theta}_2, \hat{v}_{z2}) & \text{for } z_2 > \xi \end{cases}, \tag{5.9}$$
and consider the perturbation in the form:
$$\hat{\theta}_i = \theta_i(z_2, t)\exp(j(k_1x + k_2y)), \tag{5.10}$$
$$\hat{v}_{zi} = v_i(z_2, t)\exp(j(k_1x + k_2y)), \tag{5.11}$$
$$\xi = \epsilon_1(t)\exp(j(k_1x + k_2y)), \tag{5.12}$$
where $k_i$ $(i = 1, 2)$ are the wave numbers (in the $x$ and $y$ directions) and $j^2 = -1$. We then linearize the jump conditions (4.9)-(4.15). Taking into account that
$$\theta\,|_{\xi=\pm 0} = \theta_s(\pm 0) + \xi\theta_s'(\pm 0) + \tilde{\theta}(\pm 0),$$
$$\left.\frac{\partial\theta}{\partial z_2}\right|_{\xi=\pm 0} = \theta_s'(\pm 0) + \xi\theta_s''(\pm 0) + \left.\frac{\partial\tilde{\theta}}{\partial z_2}\right|_{\xi=\pm 0},$$

we obtain:

$$\hat{\theta}_2|_{z_2=0} - \hat{\theta}_1|_{z_2=0} = u\xi, \tag{5.13}$$

$$\left.\frac{\partial\hat{\theta}_2}{\partial z_2}\right|_{z_2=0} - \left.\frac{\partial\hat{\theta}_1}{\partial z_2}\right|_{z_2=0} = -u^2\xi - \frac{\partial\xi}{\partial t} + \tilde{v}_z|_{z_2=0}, \tag{5.14}$$

$$u^2\xi + \left.\frac{\partial\hat{\theta}_2}{\partial z_2}\right|_{z_2=0} = -\frac{Z}{u}\hat{\theta}_1|_{z_2=0}, \tag{5.15}$$

$$\hat{v}_{2z}\,|_{z_2=0} = \hat{v}_{1z}\,|_{z_2=0}\ , \tag{5.16}$$

$$\left.\frac{\partial\hat{v}_{z2}}{\partial z_2}\right|_{z_2=0} = \left.\frac{\partial\hat{v}_{1z}}{\partial z_2}\right|_{z_2=0}, \tag{5.17}$$

$$\left.\frac{\partial^2\hat{v}_{2z}}{\partial z_2^2}\right|_{z_2=0} = \left.\frac{\partial^2\hat{v}_{1z}}{\partial z_2^2}\right|_{z_2=0}, \tag{5.18}$$

$$\left.\frac{\partial^3\hat{v}_{2z}}{\partial z_2^3}\right|_{z_2=0} = \left.\frac{\partial^3\hat{v}_{1z}}{\partial z_2^3}\right|_{z_2=0}. \tag{5.19}$$

We substitute (5.10)-(5.12) into (5.13)-(5.19) and obtain the equations:

$$\theta_2(0,t) - \theta_1(0,t) = u\epsilon_1(t), \tag{5.20}$$

$$\theta_2'(0,t) - \theta_1'(0,t) = -\epsilon_1(t)u^2 - \epsilon_1'(t) + v_1(0,t), \tag{5.21}$$

$$\epsilon_1(t)u^2 + \theta_2'(0,t) = -\frac{Z}{u}\theta_1(0,t), \tag{5.22}$$

$$v_2^{(m)}(0,t) = v_1^{(m)}(0,t), \quad m = 0,1,2,3. \tag{5.23}$$

Where

$$v_i^{(m)} = \frac{\partial^m v_i}{\partial z_2^m},\ \ \theta_i' = \frac{\partial\theta_i}{\partial z_2}, i = 1,2,\ \ \epsilon_1'(t) = \frac{d\epsilon_1(t)}{dt}.$$

We apply two times the operator *curl* to the Navier-Stokes equations (5.6) and (5.8) in order to eliminate the pressure taking into account that $rot(grad) = 0$. We consider only the $z$ component of velocity. Equations (5.6), (5.8) become:

$$\frac{\partial}{\partial t}\Delta\tilde{v}_z - u\frac{\partial}{\partial z_2}\Delta\tilde{v}_z = P\Delta\Delta\tilde{v}_z + Q(1+\lambda\sin(\mu t))\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{\theta}. \tag{5.24}$$

Substituting (5.1) in (5.10), (5.11) and (5.24) we obtain:

$\boldsymbol{z_2 < \xi}$**:**

$$\begin{cases} \frac{\partial}{\partial t}\left(v_1'' - k^2 v_1\right) - u(v_1''' - k^2 v_1') - P\left((v_1^{(4)} - k^2 v_1'') - k^2(v_1'' - k^2 v_1)\right) \\ \qquad\qquad\qquad\qquad\qquad = -Qk^2(1+\lambda\sin(\mu t))\theta_1, \\ \frac{\partial\theta_1}{\partial t} - \theta_1'' - u\theta_1' + k^2\theta_1 = 0, \end{cases}$$

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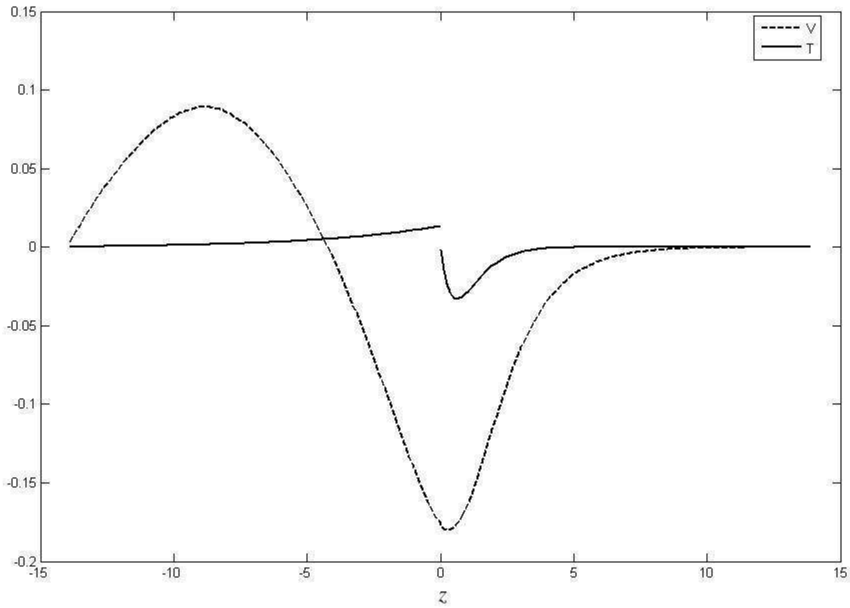


Figure 1: Velocity (dashed line) and temperature (solid line) distributions without vibration for $R = 20$, $k = 0.7195$, $Z = 8$, $P = 0.5$.

$\boldsymbol{z_2 > \xi}$**:**

$$\begin{cases} \dfrac{\partial}{\partial t}\left(v_2'' - k^2 v_2\right) - u(v_2''' - k^2 v_2') - P\left((v_2^{(4)} - k^2 v_2'') - k^2(v_2'' - k^2 v_2)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = -Qk^2(1 + \lambda \sin(\mu t))\theta_2, \\ \dfrac{\partial \theta}{\partial t} - \theta_2'' - u\theta_2' + k^2\theta_2 = u\exp(-uz_2)v_2, \end{cases}$$

where $k = \sqrt{k_1^2 + k_2^2}$. We introduce the variables $w_i$ such that $w_i = v_i'' - k^2 v_i,\ i = 1, 2$. Then the previous systems can be written in the following form:

$\boldsymbol{z_2 < \xi}$**:**

$$\begin{cases} \dfrac{\partial w_1}{\partial t} - uw_1' - P(w_1'' - k^2 w_1) = -Qk^2(1 + \lambda \sin(\mu t))\theta_1, \\ w_1 = v_1'' - k^2 v_1, \\ \dfrac{\partial \theta_1}{\partial t} - \theta_1'' - u\theta_1' + k^2\theta_1 = 0, \end{cases} \tag{5.25}$$

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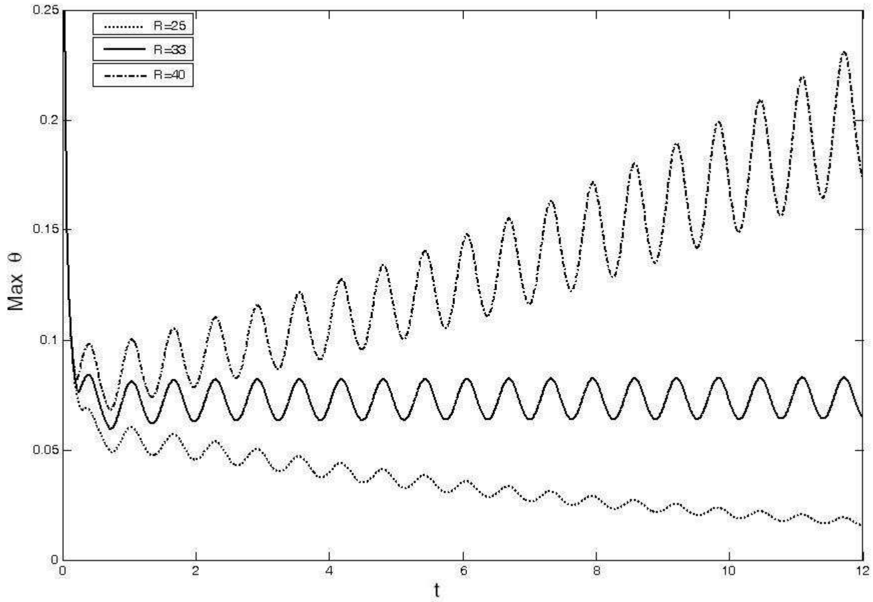


Figure 2: Maximal temperature in the case with vibration for different values of $R$ with $k = 0.7$, $Z = 8$, $P = 0.5$, $\lambda = 5$ and $\mu = 10$.

$\boldsymbol{z_2 > \xi}$**:**

$$\begin{cases} \dfrac{\partial w_2}{\partial t} - uw_2' - P(w_2'' - k^2 w_2) = -Qk^2(1 + \lambda \sin(\mu t)), \\ w_2 = v_2'' - k^2 v_2, \\ \dfrac{\partial \theta_2}{\partial t} - \theta_2'' - u\theta_2' + k^2\theta_2 = u\exp(-uz_2)v_2, \end{cases} \tag{5.26}$$

with the following conditions:

$$\theta_2(0,t) - \theta_1(0,t) = u\epsilon_1(t), \tag{5.27}$$

$$\theta_2'(0,t) - \theta_1'(0,t) = -u^2\varepsilon_1(t) - \varepsilon_1'(t) + v_1(0,t), \tag{5.28}$$

$$\theta_2'(0,t) + \frac{Z}{u}\theta_1'(0,t) = -u^2\varepsilon_1(t), \tag{5.29}$$

$$v_1^{(m)}(0,t) = v_2^{(m)}(0,t) \quad m = 0,1,2,3. \tag{5.30}$$

From the equations (5.27), (5.28) and (5.29), we have:

$$\theta_1'(0,t) + \frac{Z}{u}\theta_1(0,t) = \frac{1}{u}\left(\theta_2(0,t) - \theta_1(0,t)\right)_t' - v_1(0,t),. \tag{5.31}$$

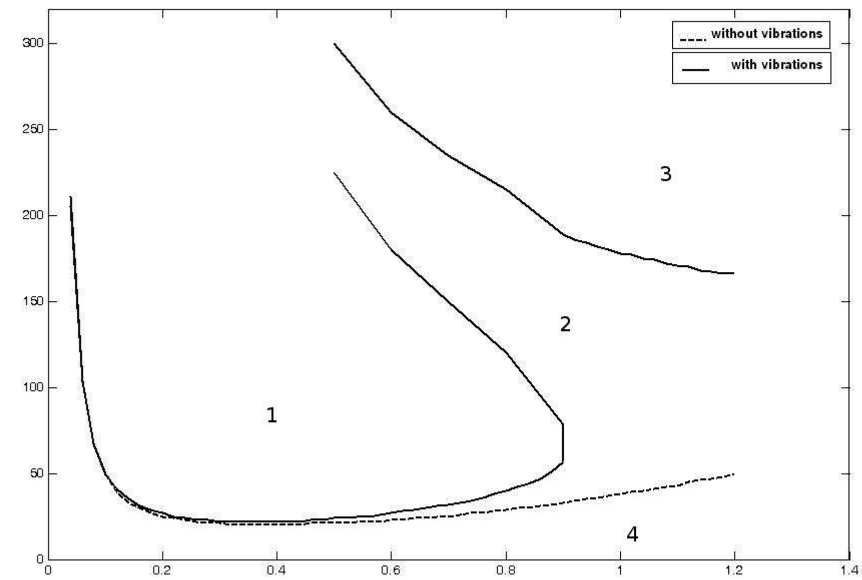


Figure 3: Convective instability boundary: critical Rayleigh number as a function of the wave number for $Z = 40$, $P = 0.5$, $\lambda = 0.5$ and $\mu = 10$.

Taking into account the previous development, we introduce the linear operators $L_1$ and $L_2$ defined by:

$$L_1 w = \frac{\partial w}{\partial t} - uw' - P(w'' - k^2 w),$$

and

$$L_2 \theta = \frac{\partial \theta}{\partial t} - \theta'' - u\theta' + k^2 \theta.$$

We rewrite (5.25) and (5.26) in the form:

$$\begin{cases} L_1 w_i = -Q(1 + \lambda \sin(\mu t)) k^2 \theta_i \quad , \quad i = 1, 2 \\ L_2 \theta_2 = u \exp(-u z_2) v_2, \\ L_2 \theta_1 = 0. \end{cases} \tag{5.32}$$

In the next section we solve numerically the problem (5.32) subject tot he conditions (5.27)-(5.31) using the finite-difference approximation with implicit scheme except for the boundary condition where the velocity $v$ is taken from the previous time step (see [2] and references therein).

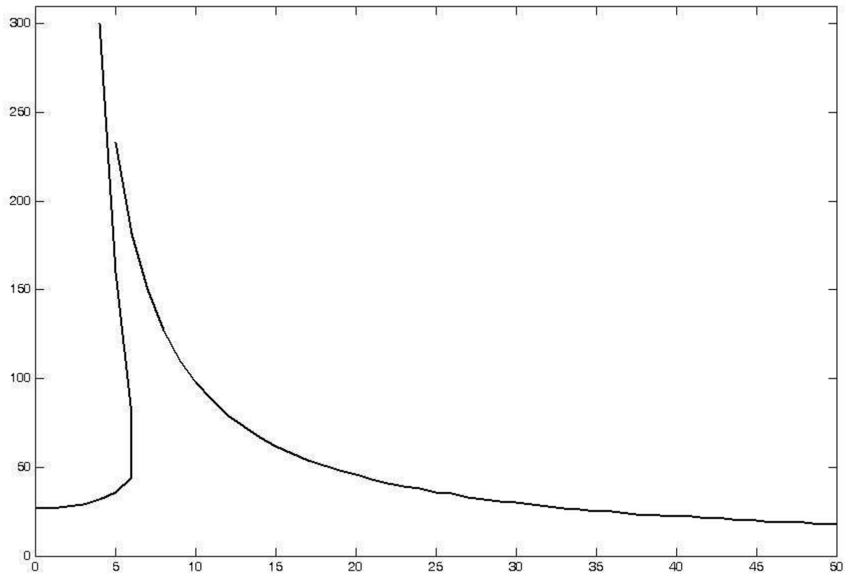


Figure 4: Convective instability boundary : critical Rayleigh number as a function of the amplitude of vibrations for $k = 0.7$, $Z = 8$, $P = 0.5$ and $\mu = 10$. Instability region is above the curves.

## 6. Stability Boundary

In the case without vibrations ($\lambda = 0$), we obtain the eigenvalue problem which determines stability of reaction fronts with respect to natural convection. Figure 1 shows the eigenfunctions (temperature and velocity) corresponding to the stability boundary. These results coincide with the results founded in [6] where the problem without vibrations is studied.

We now study the influence of vibrations on the stability boundary. For fixed $Z$ and $P$ we vary $R$. If the Rayleigh number $R$ is less than a critical value $R_{cr}$, then the solution is a decreasing function of time (see Figure 2), that is the perturbation decays. If the Rayleigh number $R$ is greater than a critical value $R_{cr}$, then the perturbation grows. The critical value of Rayleigh number is found as the limit between the two cases. The curves in Figure 2 are oscillating because of the time dependence of the perturbed solution of our problem.

Figure 3 shows the stability boundary, i.e. the critical value of the Rayleigh number is given as a function of the wavenumber $k$ in the cases with and without vibrations: $Z = 8$, $P = 0.5$, $\lambda = 0$ (dotted line, no vibrations), $\lambda = 5$, $\mu = 10$ (solid line, vibrations). In the case of the upward propagating front, vibration stabilizes it. The critical value of the Rayleigh number becomes higher. It is interesting to note

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that vibrations can have a strong stabilizing effect for certain wavenumbers. For the propagating front with vibrations, the regions of instability are 1 and 3, while the regions of stablity are 2 and 4. For the propagating front without vibrations, only the region 4 represents the stability of the reaction front.

Figure 3 shows the critical value of the Rayleigh number as a function of the amplitude of vibrations. For $\lambda = 0$, we obtain the same value for the critical Rayleigh number $R_{cr} \sim 27$ as for the case without vibrations (see [6]). For small positive values of the amplitude $\lambda$, vibrations stabilize the front. The critical values of Rayleigh number is greater than without vibrations. When the amplitude is sufficiently large, the front becomes less stable. We note that for certain values of the amplitudes, the front remains stable for very high values of the Rayleigh number.

## 7. Conclusion

In this work we study the influence of vibrations on the convective instability of reaction fronts in a liquid medium. The model consists of a reaction-diffusion system coupled with the Navier-Stokes equations under the Boussinesq approximation. We use narrow reaction zone method which allows the reduction of this problem to a free boundary problem. Its steady propagating front can be found explicitly. This allows us to fulfill linear stability analysis and to find the stability boundary.

If the amplitude of vibrations is zero, which means that no vibrations are considered, we find the same results as in [6]. For small vibrations frequency or amplitude the reaction front becomes more stable and loses its stability when one of two our parameters of vibrations becomes sufficiently high. We note finally that in the case when both of the reactant and the product are liquids, the results are different in comparison with the case of the liquid-solid front.

## References

[1] A.P. Aldushin, S.G. Kasparyan, Thermodiffusional instability of a combustion front, *Soviet Physics Doklady*, **24** (1979), 29-31.

[2] K. Allali, J. Pojman, V. Volpert, Influence of vibrations on convective instability of polymerization fronts, *J. of Engineering Mathematics*, **41** (2001) 13-31.

[3] G.I. Barenblatt, Ya. B. Zeldovich, A.G. Istratov, Diffusive-thermal stability of a laminar flame, *Zh. Prikl. Mekh. Tekh. Fiz.*, **4**, No. 21 (1962), In Russian.

[4] P. Clavin, Dynamic behavior of premixed flame fronts in laminar and turbulent flows, *Progr. Energy Comb. Sci.*, **11** (1985), 1-59.

[5] M. Garbey, A. Taik, V. Volpert, Linear stability analysis of reaction fronts in liquids, *Quart. Appl. Math.*, **54** (1996), 225-247.

[6] M. Garbey, A. Taik, V. Volpert, Influence of natural convection on stability of reaction fronts in liquids, *Quart. Appl. Math.*, **53** (1998), 1-35.

[7] A.G. Istratov, V.B. Librovich, Effect of the transfer processes on stability of a planar flame front, *J. Appl. Math. Mech.*, **30** (1966), 451-466, In Russian.

[8] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, Pergamon, New York (1987), 683 pp.

[9] S.B. Margolis, H.G. Kaper, G.K. Leaf, B.J. Matkowsky, Bifurcation of pulsating and spinning reaction fronts incondensed two-phase combustion, *Combust. Sci. Technol.*, **43** (1985), 127-165.

[10] B.J. Matkowsky, G.I. Sivashinsky, Propagation of a pulsating reaction front in solid fuel combustion, *SIAM J. Appl. Math.*, **35** (1978), 465-478.

[11] B.J. Matkowsky, G.I. Sivashinsky, An asymptotic derivation of two models in flame theory associated with the constant density approximation, *SIAM J. Appl. Math.*, **37** (1979), 686-699.

[12] J. Pojman, V. Volpert, T. Dumont, *Bubble Behavior and Convection in Frontal Polymerization on the KC-135 Aircraft*, AIAA Conference, Reno (2000).

[13] D.A. Schult, Matched asymptotic expansions and the closure problem for combustion waves, *SIAM J. Appl. Math.*, **60** (2000), 136-155.

[14] K.G. Shkadinsky, B.I. Khaikin, A.G. Merzhanov, Propagation of a pulsating exothermic reaction front in the condensed phase, *Combustion, Explosion, and Shock Waves*, **7** (1971), 15-22.

[15] A. Volpert, V. Volpert, V. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Amer. Math. Soc. (1994) 448 pp.

[16] Ya.B. Zeldovich, D.A. Frank-Kamenetskii, A theory of thermal propagation of flame, *Acta Physicochim.*, USSR, **9** (1938), 341-350, In Russian.

[17] Ya.B. Zeldovich, G.I. Barenblatt, V.B. Librovich, G.M. Makhviladze, *The Mathematical Theory of Combustion and Explosions*, Consultants Bureau, New York (1985), 597 pp.