International Electronic Journal of Pure and Applied Mathematics
Volume 1 No. 2 2010, 177-194

A BÄCKLUND TRANSFORMATIONS FOR $u_y + \alpha u_{xx} + \beta u_{xxx} = f(x, y, u, u_x)$

R. Asokan
Department of Mathematics
Madurai Kamaraj University
Madurai, 625021, Tamil Nadu, INDIA

Abstract: Some results relating to Bäcklund transformations of the form $u = F(x, y, u', u'_x, u'_y)$ for $u_y + \alpha u_{xx} + \beta u_{xxx} = f(x, y, u, u_x)$ are obtained.

AMS Subject Classification: 35Q55, 35Q41, 35J10, 35K05
Key Words: Bäcklund transformation, Hopf-Cole transformation, Korteweg-de Vries equation, Schrödinger equation

1. Introduction

Nimmo and Crighton [7] proved that only the Burgers equations and their inhomogeneous version admit Bäcklund transformations (BTs), Rogers and Shadwick [9] also Anderson and Ibragimov [1] by studying $u_y + u_{xx} + H(x, y, u, u_x) = 0$. Later Kingston and Sophocleous [5] treated the generalized nonlinear Schrödinger equation, $iz_y + z_{xx} + f(z, z) = 0$ and Sophocleous and Kingston [11] investigated $u_{xy} = f(u, u_x)$ for their BTs. Sophocleous and Kingston [11] motivated by the work of Miura [6], Hopf [4], Cole [2] assumed the BT in the form $u = F(u', u'_x, u'_y)$. It is also possible to derive BTs through the study of Painlevé property [12]. For instance Nirmala, Vedan and Baby [8] obtained BTs for a variable coefficient KdV equation

$$u_t + at^n uu_x + \beta t^m u_{xxx} = 0, \tag{1}$$

where m, n are real numbers and α, β are constant parameters, as

$$u(x,t) = -\frac{12\beta}{\alpha} t^{m-n} \frac{\phi_x^2}{\phi^2} + \frac{12\beta}{\alpha} t^{m-n} \frac{\phi_{xx}}{\phi} + u_2, \tag{2}$$

or

$$u(x,t) = \frac{12\beta}{\alpha} t^{m-n} \frac{\partial^2}{\partial x^2}(\log\phi) + \mathrm{u}_2, \tag{3}$$

where u_2 and ϕ satisfy

Received: May 12, 2010

$$u_0 = -\frac{12\beta}{\alpha} t^{m-n} \phi_x^2, \tag{4}$$

$$u_t = -\frac{12\beta}{\alpha} t^{m-n} \phi_{xx}, \tag{5}$$

$$\frac{t^{-n}}{\alpha}\phi_x\phi_t + u_2\phi_x^2 - \frac{3\beta}{\alpha}t^{m-n}\phi_{xx}^2 + \frac{4\beta}{\alpha}t^{m-n}\phi_x\phi_{xxx} = 0, \tag{6}$$

$$\frac{t^{-n}}{\alpha}\phi_{xt} + \frac{m-n}{\alpha}t^{-n-1}\phi_x + u_2\phi_{xx} + \frac{\beta}{\alpha}t^{m-n}\phi_{xxxx} = 0, \tag{7}$$

$$u_{2,t} + \alpha t^n u_2 u_{2,x} + \beta t^m u_{2,xxx} = 0. \tag{8}$$

and $u_j = 0$ for $j \geq 3$.

A generalisation of the Hopf-Cole transformation is given by Sachdev [10]. Hongyou Wu [13] classified all the nonlinear partial differential equations of the form $u_{xxx} = F(u, u_x, u_t)$ that have BTs whose definition involves only u, u_x, u_{xx} and a function $\phi(u, u_x, u_{xx})$. Two BTs one each for the Hirota-Satsuma and the Tzitzéica equations are obtained by Conte, Musette and Michel Grundland [3].

The rest of the paper is organised as follows: In Section 2 we derive some results relating to Bäcklund transformations of the form $u = F(x, y, u', u'_x, u'_y)$ for $u_y + \alpha u_{xx} + \beta u_{xxx} = f(x, y, u, u_x)$. No Bäcklund transformations are obtained in section 3. The result of the present study is set forth in Section 4.

2. A Bäcklund Transformations

We consider the Bäcklund transformations of the form

$$u = F(x, y, u', u'_x, u'_y), \tag{9}$$

which relate the functions $f(x, y, u, p)$ and $f'(x, y, u', p')$, where we use the usual notation, $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$ and similarly for p', q', r', s', t' in terms of u'.

KdV-Burgers equations are

$$u_y + \alpha u_{xx} + \beta u_{xxx} = f(x, y, u, u_x), \tag{10}$$

$$u'_y + a u'_{xx} + b u'_{xxx} = f'(x, y, u', u'_x). \tag{11}$$

Rewriting equations (10)-(11), we obtain

$$q + \alpha r + \beta r_x = f(x, y, u, u_x), \tag{12}$$

$$q' + a r' + b r'_x = f'(x, y, u', u_x). \tag{13}$$

Differentiating (9) with respect to y, we get

$$q = t' F_{q'} + s' F_{p'} + q' F_{u'} + F_y. \tag{14}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 2 (2010)

Using (14) in equation (12) we find that

$$t' = \frac{1}{F_{q'}}(f - \alpha r - \beta r_x - F_y - q'F_{u'} - s'F_{p'}). \tag{15}$$

Differentiating (9) with respect to x thrice, we have

$$\begin{aligned}
p &= F_x + p'F_{u'} + r'F_{p'} + s'F_{q'}, \qquad (16)\\
r &= F_{xx} + 2p'F_{xu'} + p'^2F_{u'u'} + 2p'r'F_{u'p'} + 2p's'F_{u'q'} + r'F_{u'}\\
&\quad +2r'F_{xp'} + r'^2F_{p'p'} + 2r's'F_{p'q'} + r'_xF_{p'}\\
&\quad +2s'F_{xq'} + s'^2F_{q'q'} + s'_xF_{q'}, \qquad (17)\\
r_x &= F_{xxx} + p'^3F_{u'u'u'} + r'^3F_{p'p'p'} + s'^3F_{q'q'q'}\\
&\quad +3\Big(p'^2F_{x'u'u'} + r'^2F_{x'p'p'} + s'^2F_{x'q'q'} + p'^2r'F_{u'u'p'} + p'r'^2F_{u'p'p'}\\
&\quad +r'^2s'F_{p'p'q'} + p's'^2F_{u'q'q'} + r's'^2F_{p'q'q'} + p'^2s'F_{u'u'q'}\Big)\\
&\quad +6\left(p'r'F_{xu'p'} + p's'F_{xu'q'} + r's'F_{xp'q'} + p'r's'F_{u'p'q'}\right)\\
&\quad +2\Big(p's'F_{u'u'q'} + p'r'F_{u'u'} + r's'F_{u'q'} + r'^2F_{u'p'}\Big)\\
&\quad +r'F_{xu'} + r'_x\left[2\left(p'F_{u'p'} + r'F_{p'p'} + s'F_{q'p'}\right) + F'_u + F_{xp'}\right]\\
&\quad +r'_{xx}F'_p + s'_x\left[2\left(p'F_{u'q'} + r'F_{p'q'} + s'F_{q'q'}\right) + F_{xq'}\right] + s'_{xx}F'_q. \qquad (18)
\end{aligned}$$

Solving (11) for r'_x, we get

$$r'_x = \frac{1}{b}\left(f' - q' - ar'\right). \tag{19}$$

Differentiating (19) with respect to x, we get

$$r'_{xx} = \frac{1}{b}\left[f'_x + p'f'_{u'} + r'\left(f'_{p'} + \frac{a^2}{b}\right) - s' - \frac{a}{b}f' + \frac{a}{b}q'\right]. \tag{20}$$

Differentiating equation (11) with respect to y and solving for s'_{xx}, we have

$$s'_{xx} = \frac{1}{b}\left[f'_y + q'f'_{u'} + s'f'_{p'} - t' - as'_x\right]. \tag{21}$$

Substituting (14), (17) and (18) in (13), we get

$$\begin{aligned}
&F_y + p'F_{u'} + s'F_{p'} + t'F_{q'} + \alpha\Big[F_{xx} + 2p'F_{xu'} + p'^2F_{u'u'} + 2p'r'F_{u'p'}\\
&+2p's'F_{u'q'} + r'F_{u'} + 2r'F_{xp'} + r'^2F_{p'p'} + 2r's'F_{p'q'} + r'_xF_{p'}\\
&+2s'F_{xq'} + s'^2F_{q'q'} + s'_xF_{q'}\Big] + \beta\Big[F_{xxx} + p'^3F_{u'u'u'} + r'^3F_{p'p'p'}
\end{aligned}$$

$$
\begin{aligned}
&+s'^3F_{q'q'q'} + 3\left(p'^2F_{x'u'u'} + r'^2F_{x'p'p'} + s'^2F_{x'q'q'} + p'^2r'F_{u'u'p'}\right.\\
&\left.+p'r'^2F_{u'p'p'} + r'^2s'F_{p'p'q'} + p's'^2F_{u'q'q'} + r's'^2F_{p'q'q'} + p'^2s'F_{u'u'q'}\right)\\
&+6\left(p'r'F_{xu'p'} + p's'F_{xu'q'} + r's'F_{xp'q'} + p'r's'F_{u'p'q'}\right)\\
&+2\left(p's'F_{u'u'q'} + p'r'F_{u'u'} + r's'F_{u'q'} + r'^2F_{u'p'}\right)\\
&+r'F_{xu'} + r'_x\left[2\left(p'F_{u'p'} + r'F_{p'p'} + s'F_{q'p'}\right) + F'_u + F_{xp'}\right]\\
&+r'_{xx}F'_p + s'_x\left[2\left(p'F_{u'q'} + r'F_{p'q'} + s'F_{q'q'}\right) + F_{xq'}\right]\\
&+s'_{xx}F'_q\big] = f(x,y,u,p). \qquad (22)
\end{aligned}
$$

Replacing r'_x, r'_{xx} and s'_{xx} from (19)-(21) in (22), we have

$$
\begin{aligned}
&F_y + p'F_{u'} + s'F_{p'} + t'F_{q'} + \alpha\Big[F_{xx} + 2p'F_{xu'} + p'^2F_{u'u'}\\
&+2p'r'F_{u'p'} + 2p's'F_{u'q'} + r'F_{u'} + 2r'F_{xp'} + r'^2F_{p'p'}\\
&+2r's'F_{p'q'} + r'_xF_{p'} + 2s'F_{xq'} + s'^2F_{q'q'} + s'_xF_{q'}\Big]\\
&+\beta\Big[F_{xxx} + p'^3F_{u'u'u'} + r'^3F_{p'p'p'} + s'^3F_{q'q'q'}\\
&+3\left(p'^2F_{x'u'u'} + r'^2F_{x'p'p'} + s'^2F_{x'q'q'} + p'^2r'F_{u'u'p'} + p'r'^2F_{u'p'p'}\right.\\
&\left.+r'^2s'F_{p'p'q'} + p's'^2F_{u'q'q'} + r's'^2F_{p'q'q'} + p'^2s'F_{u'u'q'}\right)\\
&+6\left(p'r'F_{xu'p'} + p's'F_{xu'q'} + r's'F_{xp'q'} + p'r's'F_{u'p'q'}\right)\\
&+2\left(p's'F_{u'u'q'} + p'r'F_{u'u'} + r's'F_{u'q'} + r'^2F_{u'p'}\right) + r'F_{xu'}\\
&+\frac{1}{b}\left(f' - q' - ar'\right)\left[2\left(p'F_{u'p'} + r'F_{p'p'} + s'F_{q'p'}\right) + F'_u + F_{xp'}\right]\\
&+\frac{1}{b}\left[f'_x + p'f'_{u'} + r'\left(f'_{p'} + \frac{a^2}{b}\right) - s' - \frac{a}{b}f' + \frac{a}{b}q'\right]F'_p\\
&+s'_x\left[2\left(p'F_{u'q'} + r'F_{p'q'} + s'F_{q'q'}\right) + F_{xq'}\right]\\
&+\frac{1}{b}\left[f'_y + q'f'_{u'} + s'f'_{p'} - t' - as'_x\right]F'_q\Big] = f(x,y,u,p). \qquad (23)
\end{aligned}
$$

Equating the coefficient of t' to zero, we get

$$b = \beta. \qquad (24)$$

Inserting (24) in (23), we obtain

$$
\begin{aligned}
&F_y + [f' + (\alpha - a)r']F_{u'} + [(\alpha - a)s'_x + f'_y + q'f'_{u'} + s'f'_{p'}]F_{q'}\\
&+\left[f'_x + p'f'_{u'} + r'(f'_{p'} + \frac{a(a-\alpha)}{\beta} + \frac{\alpha - a}{\beta}f' + \frac{a-\alpha}{\beta}q'\right]F_{p'}
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha F_{xx} + (2\alpha p' + \beta r')F_{xu'} + \left(\alpha p'^2 + 2\beta p' r'\right) F_{u'u'} \\
&+2\left[\alpha p' r' + \beta r'^2 + p'(f' - q' - ar')\right] F_{u'p'} \\
&+\left[(\alpha + \alpha p' + 2\beta r')s' + 2\beta p' s'_x\right] F_{u'q'} \\
&+[2\alpha r' + (f' - q' - ar')]F_{xp'} + [\alpha r'^2 + 2(f' - q' - ar')r']F_{p'p'} \\
&+2\left[(\alpha r' + f' - q' - ar')s' + \beta r' s'_x\right] F_{p'q'} \\
&+(2\alpha s' + \beta s'_x)F_{xq'} + \left(\alpha s'^2 + 2\beta s' s'_x\right) F_{q'q'} \\
&+\beta\Big[F_{xxx} + p'^3 F_{u'u'u'} + r'^3 F_{p'p'p'} + s'^3 F_{q'q'q'} \\
&+3\Big(F_{xu'u'} p'^2 + F_{xp'p'} r'^2 + F_{xq'q'} s'^2 \\
&+p'^2 r' F_{u'u'p'} + p' r'^2 F_{u'p'p'} + p' s'^2 F_{u'q'q'} \\
&+\left(p'^2 s' + 2p' s'\right) F_{u'u'q'}\Big) + 6(p' r' F_{xu'p'} + p' s' F_{xu'q'} \\
&+r' s' F_{xp'q'} + p' r' s' F_{u'p'q'})\Big] = 0.
\end{aligned} \tag{25}
$$

Setting the coefficients of $r'^3, s'^3, r'^2 s'$ and $r' s'^3$ in (25) to zero, we have

$$F_{p'p'p'} = 0, \tag{26}$$
$$F_{q'q'q'} = 0, \tag{27}$$
$$F_{p'p'q'} = 0, \tag{28}$$
$$F_{p'q'q'} = 0. \tag{29}$$

Equations (26)-(29) require that

$$
\begin{aligned}
u = F = {} & A(x,y,u')p'^2 + B(x,y,u')q'^2 + C(x,y,u')p'q' \\
& +D(x,y,u')p' + E(x,y,u')q' + G(x,y,u').
\end{aligned} \tag{30}
$$

On using (26)-(29) in (25), we obtain

$$
\begin{aligned}
&F_y + [f' + (\alpha - a)r']F_{u'} + [(\alpha - a)s'_x + f'_y + q' f'_{u'} + s' f'_{p'}]F_{q'} \\
&+\left[f'_x + p' f'_{u'} + r'(f'_{p'} + \frac{a(a-\alpha)}{\beta} + \frac{\alpha - a}{\beta} f' + \frac{a-\alpha}{\beta} q'\right] F_{p'} \\
&+\alpha F_{xx} + (2\alpha p' + \beta r')F_{xu'} + \left(\alpha p'^2 + 2\beta p' r'\right) F_{u'u'} \\
&+2\left[\alpha p' r' + \beta r'^2 + p'(f' - q' - ar')\right] F_{u'p'} \\
&+\left[(\alpha + \alpha p' + 2\beta r')s' + 2\beta p' s'_x\right] F_{u'q'} \\
&+[2\alpha r' + (f' - q' - ar')]F_{xp'} + [\alpha r'^2 + 2(f' - q' - ar')r']F_{p'p'} \\
&+2\left[(\alpha r' + f' - q' - ar')s' + \beta r' s'_x\right] F_{p'q'}
\end{aligned}
$$

$$
\begin{aligned}
&+(2\alpha s' + \beta s'_x)F_{xq'} + \left(\alpha {s'}^2 + 2\beta s' s'_x\right) F_{q'q'} \\
&+\beta \Big[F_{xxx} + {p'}^3 F_{u'u'u'} + 3\Big(F_{xu'u'}{p'}^2 + F_{xp'p'}{r'}^2 + F_{xq'q'}{s'}^2 \\
&+{p'}^2 r' F_{u'u'p'} + p'{r'}^2 F_{u'p'p'} + p'{s'}^2 F_{u'q'q'} \\
&+\left({p'}^2 s' + 2p's'\right) F_{u'u'q'}\Big) + 6(p'r'F_{xu'p'} + p's'F_{xu'q'} \\
&+r's'F_{xp'q'} + p'r's'F_{u'p'q'})\Big] = 0. \qquad (31)
\end{aligned}
$$

Substituting (30) in (31), we get

$$
\begin{aligned}
&A_y {p'}^2 + B_y {q'}^2 + C_y p'q' + D_y p' + E_y q' + G_y + [f' + (\alpha - a)r'] \Big(A_{u'}{p'}^2 \\
&+B_{u'}{q'}^2 + C_{u'}p'q' + D_{u'}p' + E_{u'}q' + G_{u'}\Big) + [(\alpha - a)s'_x + f'_y \\
&+q' f'_{u'} + s' f'_{p'}](2Bq' + Cp' + E) + \Big[f'_x + p' f'_{u'} \\
&r'(f'_{p'} + \frac{a(a-\alpha)}{\beta} + \frac{\alpha - a}{\beta} f' + \frac{a-\alpha}{\beta} q'\Big] (2Ap' + Cq' + D) \\
&+\alpha \left(A_{xx}{p'}^2 + B_{xx}{q'}^2 + C_{xx}p'q' + D_{xx}p' + E_{xx}q' + G_{xx}\right) \\
&+(2\alpha p' + \beta r') \left(A_{xu'}{p'}^2 + B_{xu'}{q'}^2 + C_{xu'}p'q' + D_{xu'}p' + E_{xu'}q' + G_{xu'}\right) \\
&+\left(\alpha {p'}^2 + 2\beta p'r'\right) \Big(A_{u'u'}{p'}^2 + B_{u'u'}{q'}^2 + C_{u'u'}p'q' + D_{u'u'}p' \\
&+E_{u'u'}q' + G_{u'u'}\Big) + 2\left[\alpha p'r' + \beta {r'}^2 + p'(f' - q' - ar')\right] (2A_{u'}p' \\
&+C_{u'}q' + D_{u'}) + [\alpha s'(1 + p') + 2\beta r's' + 2\beta p's'_x](2B_{u'}q' + C_{u'}p' + E_{u'}) \\
&+[2\alpha r' + (f' - q' - ar')](2A_x p' + C_x q' + D_x) + 2A[\alpha {r'}^2 + 2(f' - q' - ar')r'] \\
&+2C[(\alpha r' + f' - q' - ar')s' + \beta r's'_x] + (2\alpha s' + \beta s'_x)(2B_x q' + C_x p' + E_x) \\
&+2B(\alpha {s'}^2 + 2\beta s's'_x) + \beta \Big[A_{xxx}{p'}^2 + B_{xxx}{q'}^2 + C_{xxx}p'q' + D_{xxx}p' \\
&+E_{xxx}q' + G_{xxx} + A_{u'u'u'}{p'}^5 + B_{u'u'u'}{p'}^3{q'}^2 + C_{u'u'u'}{p'}^4 q' + D_{u'u'u'}{p'}^4 \\
&+E_{u'u'u'}{p'}^3 q' + G_{u'u'u'}{p'}^3 + 3\Big[A_{xu'u'}{p'}^4 + B_{xu'u'}{p'}^2{q'}^2 + C_{xu'u'}{p'}^3 q' + D_{xu'u'}{p'}^3 \\
&+E_{xu'u'}{p'}^2 q' + G_{xu'u'}{p'}^2 + 8A_x {r'}^2 + 2B_x {s'}^2 + {p'}^2 r'(2A_{u'u'}p' + C_{u'u'}q' + D_{u'u'}) \\
&+2A_{u'}p'{r'}^2 + 2B_{u'}p'{s'}^2 + \left({p'}^2 s' + 2p's'\right) (2B_{u'u'}q' + C_{u'u'}p' + E_{u'u'})\Big] \\
&+6[p'r'(2A_{xu'}p' + C_{xu'}q' + D_{xu'}) + p's'(2B_{xu'}q' + C_{xu'}p' + E_{xu'}) + C_x r's' \\
&+C_{u'}p'r's']] - f\Big(x, y, A{p'}^2 + B{q'}^2 + Cp'q' + Dp' + Eq' + G, A_x {p'}^2 + 2Ar'p' \\
&+B_x {q'}^2 + 2Bq's' + C_x p'q' + Cr's' + Cp's' + D_x p' + E_x q' + G_x\Big) = 0. \qquad (32)
\end{aligned}
$$

Equation (32) demands that

$$\begin{aligned} f'(x,y,u',p') &= L(x,y,u') + M(x,y,u')p' + N(x,y,u')p'^2 \\ &\quad + Q(x,y,u')p'^3 + R(x,y,u')p'^4. \end{aligned} \tag{33}$$

In view of (33) equation (32) becomes

$$\begin{aligned} & A_y p'^2 + B_y q'^2 + C_y p'q' + D_y p' + E_y q' + G_y + \Big[L + Mp' + Np'^2 + Qp'^3 \\ & + Rp'^4 + (\alpha - a)r'\Big] \Big(A_{u'}p'^2 + B_{u'}q'^2 + C_{u'}p'q' + D_{u'}p' + E_{u'}q' + G_{u'}\Big) \\ & + \Big[(\alpha - a)s'_x + L_y + M_y p' + N_y p'^2 + Q_y p'^3 + R_y p'^4 + q'\,\big(L_{u'} + M_{u'}p' \\ & + N_{u'}p'^2 + Q_{u'}p'^3 + R_{u'}p'^4\big) + s'\Big(M + 2Np' + 3Qp'^2 + 4Rp'^3\Big)\Big] \\ & (2Bq' + Cp' + E) + \Big[\Big(L_x + (M_x + L_{u'})p' + (N_x + M_{u'})p'^2 + (Q_x + N_{u'})p'^3 \\ & + (R_x + Q_{u'})p'^4 + R_{u'}p'^5\Big) + r'\Big(M + 2Np' + 3Qp'^2 + 4Rp'^3\Big) - \frac{a(\alpha - a)}{\beta} r' \\ & + \frac{\alpha - a}{\beta}\Big(L + Mp' + Np'^2 + Qp'^3 + Rp'^4 - q'\Big)\Big](2Ap' + Cq' + D) \\ & + \alpha\Big(A_{xx}p'^2 + B_{xx}q'^2 + C_{xx}p'q' + D_{xx}p' + E_{xx}q' + G_{xx}\Big) \\ & + (2\alpha p' + \beta r')\Big(A_{xu'}p'^2 + B_{xu'}q'^2 + C_{xu'}p'q' + D_{xu'}p' + E_{xu'}q' + G_{xu'}\Big) \\ & + \Big(\alpha p'^2 + 2\beta p'r'\Big)\Big(A_{u'u'}p'^2 + B_{u'u'}q'^2 + C_{u'u'}p'q' + D_{u'u'}p' + E_{u'u'}q' + G_{u'u'}\Big) \\ & + 2\Big[\alpha p'r' + \beta r'^2 + p'\Big(L + Mp' + Np'^2 + Qp'^3 + Rp'^4\Big) - q' - ar'\Big](2A_{u'}p' \\ & + C_{u'}q' + D_{u'}) + [\alpha s'(1 + p') + 2\beta r's' + 2\beta p's'_x](2B_{u'}q' + C_{u'}p' + E_{u'}) \\ & + \Big[(2\alpha - a)r' + L + Mp' + Np'^2 + Qp'^3 + Rp'^4 - q'\Big](2A_x p' + C_x q' + D_x) \\ & + 2A\Big[(\alpha - 2a)r'^2 + 2r'\Big(L + Mp' + Np'^2 + Qp'^3 + Rp'^4 - q'\Big)\Big] \\ & + 2C\Big[\Big(\alpha r' + L + Mp' + Np'^2 + Qp'^3 + Rp'^4 - q' - ar'\Big)s' + \beta r's'_x\Big] \\ & + (2\alpha s' + \beta s'_x)(2B_x q' + C_x p' + E_x) + 2B(\alpha s'^2 + 2\beta s's'_x) \\ & + \beta\Big[A_{xxx}p'^2 + B_{xxx}q'^2 + C_{xxx}p'q' + D_{xxx}p' + E_{xxx}q' + G_{xxx} + A_{u'u'u'}p'^5 \\ & + B_{u'u'u'}p'^3 q'^2 + C_{u'u'u'}p'^4 q' + D_{u'u'u'}p'^4 + E_{u'u'u'}p'^3 q' + G_{u'u'u'}p'^3\Big] \\ & + 3\Big[A_{xu'u'}p'^4 + B_{xu'u'}p'^2 q'^2 + C_{xu'u'}p'^3 q' + D_{xu'u'}p'^3 + E_{xu'u'}p'^2 q' \\ & + G_{xu'u'}p'^2 + 8A_x r'^2 + 2B_x s'^2 + p'^2 r'(2A_{u'u'}p' + C_{u'u'}q' + D_{u'u'}) \\ & + 2A_{u'}p'r'^2 + 2B_{u'}p's'^2 + \Big(p'^2 s' + 2p's'\Big)(2B_{u'u'}q' + C_{u'u'}p' + E_{u'u'})\Big] \end{aligned}$$

$$+6[p'r'(2A_{xu'}p' + C_{xu'}q' + D_{xu'}) + p's'(2B_{xu'}q' + C_{xu'}p' + E_{xu'})$$
$$+C_x r's' + C_{u'}p'r's'] - f\Big(x, y, Ap'^2 + Bq'^2 + Cp'q' + Dp'$$
$$+Eq' + G, A_x p'^2 + 2Ar'p' + B_x q'^2 + 2Bq's' + C_x p'q'$$
$$+Cr's' + Cp's' + D_x p' + E_x q' + G_x\Big) = 0. \quad (34)$$

Since (34) depends on p'^5 and f depends on p' through u and p (see (16) and (30)) we take

$$f(x, y, u, p) = H + Jp + Kp^2, \quad (35)$$

where

$$H = S(x, y) + T(x, y)u + V(x, y)u^2 + W(x, y)u^3,$$
$$J = Y(x, y) + Z(x, y)u \quad \text{and}\, K = I(x, y) + X(x, y)u. \quad (36)$$

Inserting (35) in equation (34), we get

$$A_y p'^2 + B_y q'^2 + C_y p'q' + D_y p' + E_y q' + G_y$$
$$+\Big[L + Mp' + Np'^2 + Qp'^3 + Rp'^4$$
$$+(\alpha - a)r')\Big]\Big(A_{u'}p'^2 + B_{u'}q'^2 + C_{u'}p'q' + D_{u'}p' + E_{u'}q' + G_{u'}\Big)$$
$$+\Big[(\alpha - a)s'_x + L_y + M_y p' + N_y p'^2 + Q_y p'^3 + R_y p'^4 + q'\big(L_{u'} + M_{u'}p'$$
$$+N_{u'}p'^2 + Q_{u'}p'^3 + R_{u'}p'^4\Big) + s'\Big(M + 2Np' + 3Qp'^2 + 4Rp'^3\Big)\Big]$$
$$(2Bq' + Cp' + E) + \Big[\Big(L_x + (M_x + L_{u'})p' + (N_x + M_{u'})p'^2$$
$$+(Q_x + N_{u'})p'^3 + (R_x + Q_{u'})p'^4 + R_{u'}p'^5\Big)$$
$$+\left(M + 2Np' + 3Qp'^2 + 4Rp'^3 - \frac{a(\alpha - a)}{\beta}\right)r' + \frac{\alpha - a}{\beta}\Big(L + Mp' + Np'^2$$
$$+Qp'^3 + Rp'^4 - q'\Big)\Big](2Ap' + Cq' + D) + \alpha\Big(A_{xx}p'^2 + B_{xx}q'^2$$
$$+C_{xx}p'q' + D_{xx}p' + E_{xx}q' + G_{xx}\big) + (2\alpha p' + \beta r')\Big(A_{xu'}p'^2 + B_{xu'}q'^2$$
$$+C_{xu'}p'q' + D_{xu'}p' + E_{xu'}q' + G_{xu'}\big)$$
$$+\Big(\alpha p'^2 + 2\beta p'r'\Big)\Big(A_{u'u'}p'^2$$
$$+B_{u'u'}q'^2 + C_{u'u'}p'q' + D_{u'u'}p'$$
$$+E_{u'u'}q' + G_{u'u'}\big) + 2\Big[\alpha p'r' + \beta r'^2 + p'\Big(L + Mp' + Np'^2$$
$$+Qp'^3 + Rp'^4\Big) - q' - ar'\big)\Big](2A_{u'}p'$$
$$+C_{u'}q' + D_{u'}) + [\alpha s'(1 + p') + 2\beta r's' + 2\beta p's'_x](2B_{u'}q' + C_{u'}p' + E_{u'})$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 2 (2010)

$$
\begin{aligned}
&+\left[(2\alpha-a)r'+L+Mp'+Np'^2+Qp'^3+Rp'^4-q'\right](2A_xp'+C_xq'+D_x)\\
&+2A\left[(\alpha-2a)r'^2+2r'\left(L+Mp'+Np'^2+Qp'^3+Rp'^4-q'\right)\right]\\
&+2C\left[\left(\alpha r'+L+Mp'+Np'^2+Qp'^3+Rp'^4-q'-ar'\right)s'+\beta r's'_x\right]\\
&+(2\alpha s'+\beta s'_x)(2B_xq'+C_xp'+E_x)\\
&+2B(\alpha s'^2+2\beta s's'_x)+\beta\left[A_{xxx}p'^2+B_{xxx}q'^2+C_{xxx}p'q'+D_{xxx}p'\right.\\
&+E_{xxx}q'+G_{xxx}+A_{u'u'u'}p'^5+B_{u'u'u'}p'^3q'^2+C_{u'u'u'}p'^4q'+D_{u'u'u'}p'^4\\
&+E_{u'u'u'}p'^3q'+G_{u'u'u'}p'^3+3\left[A_{xu'u'}p'^4+B_{xu'u'}p'^2q'^2+C_{xu'u'}p'^3q'+D_{xu'u'}p'^3\right.\\
&+E_{xu'u'}p'^2q'+G_{xu'u'}p'^2+8A_xr'^2+2B_xs'^2+p'^2r'(2A_{u'u'}p'+C_{u'u'}q'+D_{u'u'})\\
&\left.+2A_{u'}p'r'^2+2B_{u'}p's'^2+\left(p'^2s'+2p's'\right)(2B_{u'u'}q'+C_{u'u'}p'+E_{u'u'})\right]\\
&+6[p'r'(2A_{xu'}p'+C_{xu'}q'+D_{xu'})+p's'(2B_{xu'}q'+C_{xu'}p'+E_{xu'})+C_xr's'\\
&+C_{u'}p'r's']-H-J\left(A_xp'^2+2Ar'p'+B_xq'^2+2Bq's'\right.\\
&\left.+C_xp'q'+Cr's'+Cp's'+D_xp'+E_xq'+G_x\right)-K\left(A_xp'^2+2Ar'p'+B_xq'^2\right.\\
&\left.+2Bq's'+C_xp'q'+Cr's'+Cp's'+D_xp'+E_xq'+G_x\right)^2=0. \qquad (37)
\end{aligned}
$$

Equating the coefficients of $s'_x, r'^2, s'^2, s's'_x, r's'_x, r', s'$ and rest to zero, we have

$$
\begin{aligned}
&(\alpha-a)(2Bq'+Cp'+E)+2\beta p'(2B'_uq'+C'_uq'+E'_u)\\
&+\beta(2B_xq'+C_xq'+E_x)=0, \qquad (38)\\
&(\alpha p'+2\beta)(2A'_up'+C'_uq'+D'_u)+2A(\alpha-2a)+6\beta A_x\\
&+6\beta p'A'_u-K(4A^2p'^2+D^2+4ADp')=0, \qquad (39)\\
&\alpha B+\beta B_x+3p'B'_u-KE^2=0, \qquad (40)\\
&B=0, \qquad (41)\\
&C=0, \qquad (42)\\
&(\alpha-a)G'_u+\left[M+2Np'+3Qp'^2+4Rp'^3+\frac{2(a-\alpha)}{\beta}\right]\\
&(2Ap'+D)+\beta G_{xu'}+2\beta p'G_{u'u'}+(2\alpha-a)(2A_xp'+D_x)\\
&+4A\left(L+Mp'+Np'^2+Qp'^3+Rp'^4-q'\right)\\
&-(2Ap'+D)\left[Y+Z(Ap'^2+Dp'+Eq'+G\right]=0, \qquad (43)\\
&(M+2Np'+3Qp'^2+4Rp'^3)E+2\alpha E_x\\
&-[Y+Z(Ap'^2+Dp'+Eq'+G]E=0, \qquad (44)
\end{aligned}
$$

$$A_yp'^2+D_yp'+E_yq'+G_y+(L+Mp'+Np'^2+Qp'^3+Rp'^4)G'_u$$

$$
\begin{aligned}
&+\Big[L_y + L_{u'}q' + (M_y + M_{u'}q')p' + (N_y + N_{u'}q')p'^2 \\
&+(Q_y + Q_{u'}q')p'^3 + (R_y + R_{u'}q')p'^4)\Big] E + [L_x + (M_x + L_{u'})p' \\
&+(N_x + M_{u'})p'^2 + (Q_x + N_{u'})p'^3 \\
&+(R_x + Q_{u'})p'^4 + R_{u'}p'^5](2Ap' + D) + (L + Mp' + Np'^2 \\
&+Qp'^3 + Rp'^4 - q')\left[\frac{\alpha - a}{\beta}(2Ap' + D) + 2A_xp' + D_x\right] \\
&+\alpha(A_{xx}p'^2 + D_{xx}p' + E_{xx}q' + G_{xx} + 2p'G_{xu'} + G_{u'u'}p'^2) \\
&+\beta(A_{xxx}p'^2 + D_{xxx}p' + E_{xxx}q' + G_{xxx} + 3G_{xu'u'}p'^2) \\
&-S - T(Ap'^2 + Dp' + Eq' + G) - V[A^2p'^4 + D^2p'^2 + E^2q'^2 + G^2 \\
&+2Ap'^2(Dp' + Eq' + G) + 2Dp'(Eq' + G) + 2(EGq'] - W[A^3p'^6 \\
&+3A^2Dp'^5 + 3A(D^2 + AG + AEq')p'^4 + (D^3 + 6ADEq')p'^3 \\
&+3(D^2G + AG^2 + 2ADG + D^2E + 2AEGq' + AE^2q'^2)p'^2 \\
&+3(DG^2 + 2DEGq' + DE^2q'^2)p' + G^3 + 3EG^2q' \\
&+3GE^2q'^2 + E^3q'^3] \\
&-\Big[Y + Z(Ap'^2 + Dp' + Eq' + G)\Big](A_xp'^2 + D_xp' \\
&+E_xq' + G_x) = 0.
\end{aligned}
\tag{45}
$$

Using (41) in (40), we get $KE = 0$ and to satisfy this equation we choose

$$K = 0. \tag{46}$$

Substituting (41) and (42) in (38) and equating the coefficients of p' and p'^0 to zero, we obtain $E_{u'} = 0$ and

$$(\alpha - a)E + \beta E_x = 0. \tag{47}$$

Inserting $B = C = K = 0$ in (39) and equating the coefficients of p'^2, p' and p'^0 to zero, we get $A_{u'} = 0 = D_{u'}$ and

$$(\alpha - 2a)A + 3\beta A_x = 0. \tag{48}$$

Equating the coefficients of p'^4, p'^3, p'^2, p' and p'^0 on both sides in (43) to zero, we have

$$AR = 0, \tag{49}$$

$$Q - AZ = 0, \tag{50}$$

$$8AN + 3DQ - 3ADZ = 0, \tag{51}$$

$$2DN + 6AM + 2A\frac{a(a-\alpha)}{\beta} + 2\beta G_{u'u'} + 2(2\alpha - a)A_x$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 2 (2010)

$$-2A(Y+ZG+EZq')-D^2Z=0, \quad (52)$$
$$(\alpha-a)G'_u+\left[M+\frac{a(a-\alpha)}{\beta}\right]D+\beta G_{xu'}+(2\alpha-a)D_x$$
$$+4AL-4Aq'-DY-DGZ=0. \quad (53)$$

Setting the coefficients q' and q'^0 equal to zero in 52, we get

$$AEZ=0, \quad (54)$$
$$2DN+6AM+2\frac{a(a-\alpha)}{\beta}A+2\beta G_{u'u'}+2(2\alpha-a)A_x$$
$$-2A(Y+ZG)-D^2Z=0. \quad (55)$$

The coefficients of q' and q'^0 in (53) give

$$A=0, \quad (56)$$
$$(\alpha-a)G'_u+\left[M+\frac{a(a-\alpha)}{\beta}\right]D+\beta G_{xu'}+(2\alpha-a)D_x$$
$$-D(Y+GZ)=0. \quad (57)$$

In (44) equating the coefficients of p'^3, p'^2, p' and p'^0 on both sides to zero, we have

$$RE=0, \quad (58)$$
$$3QE-AEZ=0, \quad (59)$$
$$2NE-DEZ=0, \quad (60)$$
$$EM+2\alpha E_x-E[Y+Z(Eq'+G)]=0. \quad (61)$$

In (61), the coefficient of q' leads to

$$E=0, \quad \text{since} \quad Z\neq 0. \quad (62)$$

From (46) equating the coefficients of $p'^5, p'^4, p'^3, p'^2, p'$ and p'^0 on both sides to zero, we get

$$DR_{u'}=0, \quad (63)$$
$$(R_x+Q_{u'})D+\left[G_{u'}+D_x+\frac{\alpha-a}{\beta}D\right]R=0. \quad (64)$$
$$DN_{u'}-WD^3=0, \quad (65)$$
$$NG_{u'}+(N_x+M_{u'})D+\frac{\alpha-a}{\beta}DN+D_xN$$
$$-VD^2-3WD^2G-ZDD_x=0, \quad (66)$$
$$D_y+MG_{u'}+\left(M_x+L_{u'}+\frac{\alpha-a}{\beta}M\right)D+D_xM$$

$$+\alpha(D_{xx} + 2G_{xu'}) - TD - 2DGV - 3WDG^2$$
$$-YD_x - ZDG_x - ZGD_x = 0, \tag{67}$$
$$G_y + LG_{u'} + DL_x + \left[\frac{\alpha - a}{\beta} D + D_x\right](L - q')$$
$$+\alpha G_{xx} + \beta G_{xxx} - S - TG - VG^2$$
$$-WG^3 - YG_x - ZGG_x = 0. \tag{68}$$

We obtain from (51) and (63) that

$$Q = R_{u'} = 0. \tag{69}$$

From equation (68) equating the coefficients q' and q'^0 on both sides to zero, we obtain

$$\frac{\alpha - a}{\beta} D + D_x = 0, \tag{70}$$
$$G_y + LG_{u'} + \left[\frac{\alpha - a}{\beta} L + L_x\right] D + LD_x + \alpha G_{xx} + \beta G_{xxx}$$
$$-S - TG - VG^2 - WG^3 - YG_x - ZGG_x = 0. \tag{71}$$

Inserting (56), (41), (42) and (62) in equation (30), we get

$$u = Dp' + G. \tag{72}$$

Differentiating (72) with respect to y, we get

$$q = Ds' + D_y p' + G_y + G_{u'}. \tag{73}$$

Again differentiating (72) with respect to x thrice, we have

$$p = (D_x + G'_u)p' + Dr' + G_x, \tag{74}$$
$$r = (D_{xx} + 2G_{xu'} + G_{u'u'}p')p' + (2D_x + G_{u'})r'$$
$$+Dr'_x + G_{xx}, \tag{75}$$
$$r_x = [D_{xxx} + 3(G_{xxu'} + G_{xu'u'}p') + G_{u'u'u'}p'^2]p'$$
$$+3(D_{xx} + G_{xu'} + G_{u'u'}p')r' + (3D_x + G_{u'})r'_x$$
$$+Dr'_{xx} + G_{xxx}. \tag{76}$$

Using (51) in (33) equation (13) can be written as

$$q' + ar' + \beta r'_x = L + Mp' + Np'^2 + Rp'^4. \tag{77}$$

Differentiating (77) with respect to x and rewriting we get

$$r'_{xx} = \frac{1}{\beta}[L_x + M_x p' + N_x p'^2 + R_x p'^4$$

$$+(M+2Np'+4Rp'^3)r'-s'-ar'_x]. \tag{78}$$

Substituting (78) in (76), the latter becomes

$$\begin{aligned} &r_x=[D_{xxx}+3(G_{xxu'}+G_{xu'u'}p')+G_{u'u'u'}p'^2]p' \\ &+3(D_{xx}+G_{xu'}+G_{u'u'}p')r'+(3D_x+G_{u'})r'_x \\ &+\frac{1}{\beta}[L_x+M_xp'+N_xp'^2+R_xp'^4 \\ &+(M+2Np'+4Rp'^3)r'-s'-ar'_x]D+G_{xxx}. \end{aligned} \tag{79}$$

On using (12) in (35) also inserting (72)-(75) and (79) in equation (12), we have

$$\begin{aligned} &G_{u'}q'+[\alpha(2D_x+G_{u'})+3\beta(D_{xx}+G_{xu'}+G_{u'u'}p') \\ &+D(M+2Np'+4Rp'^3)-D(Y+ZG+ZDp')]r' \\ &+[(\alpha-a)D+\beta(3D_x+G_{u'})]r'_x+[G_y+G_{xx}+G_{xxx} \\ &+DL_x-S-TG-VG^2-WG^3-(Y+ZG)G_x]p'^0 \\ &+[D_y+\alpha(D_{xx}+2G_{xu'})\beta(D_{xxx}+3G_{xxu'})+DM_x \\ &-(Y+ZG)(D_x+G_{u'})-ZDG_x-TD-2DG-3DWG^2]p' \\ &+[3\beta G_{xu'u'}+\alpha G_{u'u'}+DN_x-VD^2-3WGD^2 \\ &-ZD(D_x+G_{u'})]p'^2+(\beta G_{u'u'u'}-WD^3)p'^3+DR_xp'^4=0. \end{aligned} \tag{80}$$

Dividing (80) by the coefficient of q' and comparing the resulting equation with (9), we get

$$\begin{aligned} &\frac{1}{G_{u'}}\big[\alpha(2D_x+G_{u'})+3\beta(D_{xx}+G_{xu'}+G_{u'u'}p') \\ &+D(M+2Np'+4Rp'^3)\big]=a, \end{aligned} \tag{81}$$

$$\frac{1}{G_{u'}}[(\alpha-a)D+\beta(3D_x+G_{u'})]=\beta, \tag{82}$$

$$\begin{aligned} &\frac{1}{G_{u'}}[G_y+G_{xx}+G_{xxx}+DL_x-S-TG \\ &-VG^2-WG^3-(Y+ZG)G_x\big]=-L, \end{aligned} \tag{83}$$

$$\begin{aligned} &\frac{1}{G_{u'}}[D_y+\alpha(D_{xx}+2G_{xu'})\beta(D_{xxx}+3G_{xxu'})+DM_x \\ &-(Y+ZG)(D_x+G_{u'})-ZDG_x-TD-2DG-3DWG^2\big]=-M, \end{aligned} \tag{84}$$

$$\begin{aligned} &\frac{1}{G_{u'}}\big[3\beta G_{xu'u'}+\alpha G_{u'u'}+DN_x-VD^2-3WGD^2 \\ &-ZD(D_x+G_{u'})]=-N, \end{aligned} \tag{85}$$

$$\beta G_{u'u'u'}-WD^3=0, \tag{86}$$

$$\frac{1}{G_{u'}} DR_x = -R. \tag{87}$$

Using (56),(62) and (69), we find that (47)-(51), (54), (58)-(61) and (63) are satisfied and (55), (57), (64)-(67), (70) and (71) take the form

$$2DN + 2\beta G_{u'u'} - D^2 Z = 0, \tag{88}$$

$$(\alpha - a)G_{u'} + \left[M + \frac{a(a-\alpha)}{\beta}\right] D + \beta G_{xu'} + (2\alpha - a)D_x$$
$$Y + GZ = 0, \tag{89}$$

$$DR_x + \left[G_{u'} + D_x + \frac{\alpha - a}{\beta} D\right] R = 0, \tag{90}$$

$$DN_{u'} - WD^3 + \beta G_{u'u'u'} = 0, \tag{91}$$

$$NG_{u'} + 3G_{xu'u'}(N_x + M_{u'})D + \left[\frac{\alpha - a}{\beta} D + D_x\right] N$$
$$-VD^2 - 3WD^2G - ZDD_x = 0, \tag{92}$$

$$D_y + MG_{u'} + \left(M_x + L_{u'} + \frac{\alpha - a}{\beta} M\right) D + D_x M$$
$$+\alpha(D_{xx} + 2G_{xu'}) - TD - 2DGV - 3WDG^2$$
$$-YD_x - ZDG_x - ZGD_x = 0, \tag{93}$$

$$\frac{\alpha - a}{\beta} D + D_x = 0, \tag{94}$$

$$G_y + LG_{u'} + \left[\frac{\alpha - a}{\beta} L + L_x\right] D + LD_x + \alpha G_{xx} + \beta G_{xxx}$$
$$-S - TG - VG^2 - WG^3 - YG_x - ZGG_x = 0. \tag{95}$$

Thus (72) is a Bäcklund transformation between (10)-(11) where as governed by (88)-(95).

3. No Bäcklund Transformations

We rewrite (21) as

$$s'_{xx} + \frac{1}{b} t' = \frac{1}{b} \left[f'_y + q' f'_{u'} + s' f'_{p'} - a s'_x\right]. \tag{96}$$

Using (17) and (18) in (15), we get

$$\beta s'_{xx} + t' = \frac{f}{F_{q'}} - \frac{\alpha}{F_{q'}} \Big[F_{xx} + 2p' F_{xu'} + p'^2 F_{u'u'}$$
$$+2p'r' F_{u'p'} + 2p's' F_{u'q'} + r' F_{u'}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 2 (2010)

$$
\begin{aligned}
&+2r'F_{xp'}+r'^2F_{p'p'}+2r's'F_{p'q'}+r'_xF_{p'}\\
&+2s'F_{xq'}+s'^2F_{q'q'}+s'_xF_{q'}\Big]\\
&-\frac{\beta}{F_{q'}}\Big[F_{xxx}+p'^3F_{u'u'u'}+r'^3F_{p'p'p'}+s'^3F_{q'q'q'}\\
&+3\Big(p'^2F_{x'u'u'}+r'^2F_{x'p'p'}+s'^2F_{x'q'q'}+p'^2r'F_{u'u'p'}\\
&+p'r'^2F_{u'p'p'}+r'^2s'F_{p'p'q'}+p's'^2F_{u'q'q'}+r's'^2F_{p'q'q'}\\
&+p'^2s'F_{u'u'q'}\Big)+6(p'r'F_{xu'p'}+p's'F_{xu'q'}+r's'F_{xp'q'}\\
&+p'r's'F_{u'p'q'})+2(p's'F_{u'u'q'}+p'r'F_{u'u'}+r's'F_{u'q'}\\
&+r'^2F_{u'p'})+r'F_{xu'}+r'_x[2(p'F_{u'p'}+r'F_{p'p'}+s'F_{q'p'})\\
&+F_{u'}+F_{xp'}]+r'_{xx}F'_p+s'_x[2(p'F_{u'q'}+r'F_{p'q'}\\
&+s'F_{q'q'})+F_{xq'}]\Big]-\frac{1}{F_{q'}}(F_y+q'F_{u'}+s'F_{p'}). \qquad (97)
\end{aligned}
$$

Inserting (19) and (20) in (96)-(97) and solving for t' and s'_{xx}, we have

$$
\begin{aligned}
&t'=\frac{-\beta}{b-\beta}\left[f'_y+q'f'_{u'}+s'f'_{p'}-as'_x\right]+\frac{bf}{(b-\beta)F_{q'}}\\
&-\frac{\alpha b}{(b-\beta)F_{q'}}\Big[F_{xx}+2p'F_{xu'}+p'^2F_{u'u'}\\
&+2p'r'F_{u'p'}+2p's'F_{u'q'}+r'F_{u'}\\
&+2r'F_{xp'}+r'^2F_{p'p'}+2r's'F_{p'q'}+\frac{1}{b}(f'-q'-ar')F_{p'}\\
&+2s'F_{xq'}+s'^2F_{q'q'}+s'_xF_{q'}\Big]\\
&-\frac{\beta b}{(b-\beta)F_{q'}}\Big[F_{xxx}+p'^3F_{u'u'u'}+r'^3F_{p'p'p'}+s'^3F_{q'q'q'}\\
&+3\Big(p'^2F_{x'u'u'}+r'^2F_{x'p'p'}+s'^2F_{x'q'q'}+p'^2r'F_{u'u'p'}\\
&+p'r'^2F_{u'p'p'}+r'^2s'F_{p'p'q'}+p's'^2F_{u'q'q'}+r's'^2F_{p'q'q'}\\
&+p'^2s'F_{u'u'q'}\Big)+6(p'r'F_{xu'p'}+p's'F_{xu'q'}+r's'F_{xp'q'}\\
&+p'r's'F_{u'p'q'})+2(p's'F_{u'u'q'}+p'r'F_{u'u'}+r's'F_{u'q'}\\
&+r'^2F_{u'p'})+r'F_{xu'}+\frac{1}{b}(f'-q'-ar')[2(p'F_{u'p'}+r'F_{p'p'}+s'F_{q'p'})\\
&+F_{u'}+F_{xp'}]+\frac{1}{b}\left[f'_x+p'f'_{u'}+r'(f'_{p'}+\frac{a^2}{b})\right.\\
&\left.-s'-\frac{a}{b}f'+\frac{a}{b}q'\right]F'_p+s'_x[2(p'F_{u'q'}+r'F_{p'q'}
\end{aligned}
$$

$$+s'F_{q'q'}) + F_{xq'}]] - \frac{b}{(b-\beta)F_{q'}}(F_y + q'F_{u'} + s'F_{p'}), \quad (98)$$

$$s'_{xx} = \frac{1}{b-\beta}[f'_y + q'f'_{u'} + s'f'_{p'} - as'_x] - \frac{f}{(b-\beta)F_{q'}}$$
$$+\frac{\alpha}{(b-\beta)F_{q'}}\Big[F_{xx} + 2p'F_{xu'} + p'^2F_{u'u'}$$
$$+2p'r'F_{u'p'} + 2p's'F_{u'q'} + r'F_{u'}$$
$$+2r'F_{xp'} + r'^2F_{p'p'} + 2r's'F_{p'q'} + \frac{f'-q'-ar'}{b}F_{p'}$$
$$+2s'F_{xq'} + s'^2F_{q'q'} + s'_xF_{q'}\Big]$$
$$+\frac{\beta}{(b-\beta)F_{q'}}\Big[F_{xxx} + p'^3F_{u'u'u'} + r'^3F_{p'p'p'} + s'^3F_{q'q'q'}$$
$$+3\Big(p'^2F_{x'u'u'} + r'^2F_{x'p'p'} + s'^2F_{x'q'q'} + p'^2r'F_{u'u'p'}$$
$$+p'r'^2F_{u'p'p'} + r'^2s'F_{p'p'q'} + p's'^2F_{u'q'q'} + r's'^2F_{p'q'q'}$$
$$+p'^2s'F_{u'u'q'}\Big) + 6(p'r'F_{xu'p'} + p's'F_{xu'q'} + r's'F_{xp'q'}$$
$$+p'r's'F_{u'p'q'}) + 2(p's'F_{u'u'q'} + p'r'F_{u'u'} + r's'F_{u'q'}$$
$$+r'^2F_{u'p'}) + r'F_{xu'} + \frac{1}{b}(f'-q'-ar')[2(p'F_{u'p'} + r'F_{p'p'} + s'F_{q'p'})$$
$$+F_{u'} + F_{xp'}] + \frac{1}{b}\Big[f'_x + p'f'_{u'} + r'(f'_{p'} + \frac{a^2}{b})$$
$$-s' - \frac{a}{b}f' + \frac{a}{b}q'\Big]F'_p + s_{x'}[2(p'F_{u'q'} + r'F_{p'q'}$$
$$+s'F_{q'q'}) + F_{xq'}]] + \frac{1}{(b-\beta)F_{q'}}(F_y + q'F_{u'} + s'F_{p'}). \quad (99)$$

Substituting (98)-(99) in (22), the latter becomes

$$f(x,y,u,u_x) - \alpha\Big[F_{xx} + 2p'F_{xu'} + p'^2F_{u'u'} + 2p'r'F_{u'p'}$$
$$+2p's'F_{u'q'} + r'F_{u'} + 2r'F_{xp'} + r'^2F_{p'p'} + 2r's'F_{p'q'}$$
$$+\frac{1}{b}(f'-q'-ar')F_{p'} + 2s'F_{xq'} + s'^2F_{q'q'} + s'_xF_{q'}\Big] = 0, \quad (100)$$

which, inview of (17), is $f - \alpha r = 0$. We therefore conclude that this approach will not yield any fruitful result.

4. Results

In the present work we obtain a Bäcklund transformation (72) between the equations (10)-(11) where as governed by (88)-(95). In Section 3 we conclude that will not yield any fruitful result.

References

[1] R.L. Anderson, N.H. Ibragimov, *Lie-Bäcklund Transformations in Applications*, SIAM, Philadelphia (1979).

[2] J.D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.*, **9** (1951), 225-236.

[3] R. Conte, M. Musette, A. Michel Grundland, Bäcklund transformation of partial differential equations from the Painlevé-Gambier classification. II: Tzitzéica, *J. Math. Phys.*, **40** (1999), 2092-2106.

[4] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Commn. Pure Appl. Math.*, **3** (1950), 201-230.

[5] J.G. Kingston,C. Sophocleous, Bäcklund transformations for generalized Schrödinger equations, *J. Math. Phys.*, **31** (1990), 2597-2602.

[6] R.M. Miura, Korteweg-de Vries equation and generalizations. I: A remarkable explicit nonlinear transformation, *J. Math. Phys.*, **9** (1968), 1202-1210.

[7] J.J.C. Nimmo, D.G. Crighton, Bäcklund transformations for nonlinear parabolic equations: the generlal results, *Proc. R. Soc. Lond., A*, **384** (1982), 381-401.

[8] N. Nirmala, M.J. Vedan, B.V. Baby, Auto-Bäcklund transformation, Lax pairs, and Painlevé property of a variable coefficient Korteweg-de Vries equation. I, *J. Math. Phys.*, **27** (1986), 2640-2643.

[9] C. Rogers, W.F. Shadwick, *Bäcklund Transformations and their Applications*, Mathematics in Science and Engineering, **161**, Academic Press, New York (1982).

[10] P.L. Sachdev, A generalised Hopf-Cole transformations for nonlinear parabolic and hyberbolic equations, *ZAMP*, **29** (1978), 963-970.

[11] C. Sophocleous, J.G. Kingston, A class of Bäcklund transformations for equations of the type $u_{xy} = f(u, u_x)$, *J. Math. Phys.*, **31** (1991), 3176-3183.

[12] J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.*, **24** (1951), 522-526.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 2 (2010)

[13] Hongyou Wu, On Bäcklund transformations for nonlinear partial differential equations, *J. Math. Anal. Appl.*, **192** (1995), 151-179.