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UNIT GROUPS OF COMMUTATIVE GROUP ALGEBRAS

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Abstract: Let R be a commutative ring with identity and let G be an Abelian group. Denote by RG the group algebra of G over R. In the present paper we give a full description, up to isomorphism, of the group $V(RG)$ of normalized units in RG when:

(i) R is a direct product of m perfect fields of characteristic p, $m \in \mathbb{N}$, G is p-mixed Abelian group, the p-component G_p of G is simply presented and either G is splitting or G is of countable torsion free rank and

(ii) R is an infinite direct product of commutative indecomposable rings with identity, G is a finite Abelian group and the exponent of G is an invertible element in R.

These investigations extend some classical results of Berman (1967), Berman and Rossa (1968, 1975), Karpilovsky (1983), Chatzidakis and Pappas (1991) as well as some investigations of the authors (2006) and they correct some essential inaccuracies and incompleteness of results in this direction of Danchev (2004).

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1. Introduction

Let RG be the group algebra of an Abelian group G over a commutative ring R with identity. Denote by tG the torsion subgroup of G, by G_p be the p-component of G,

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by $U(RG)$ the multiplicative group of RG and by $S(RG)$ the Sylow p-subgroup of the group $V(RG)$ of normalized units in RG, that is the p-component of $V(RG)$. The investigations of the group $S(RG)$ begin with the fundamental papers of Berman [1] and [2] in which a complete description of $S(RG)$, up to isomorphism, is given when G is a countably infinite Abelian p-group and R is a countable field of characteristic p such that if G is not a restricted direct product of cyclic groups, then the field R is perfect. Further Mollov [16] and [17] calculates the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(RG)$ when G is an arbitrary Abelian group and R is a field of positive characteristic p. Let R be a commutative ring with identity of prime characteristic p. When R is without nilpotent elements Bovdi and Pataj [5] calculate the Ulm-Kaplansky invariants of $S(RG)$ under the following restriction: if the maximal divisible subgroup of G_p is not identity, then R is a p-divisible ring, that is, $R^p = R$. Nachev and Mollov [27] calculate the invariants $f_\alpha(S)$ with the only restriction G to be an Abelian p-group. Nachev [25] calculates the invariants $f_\alpha(S)$ without restrictions on G. Moreover, in all indicated cases the authors give a full description, up to isomorphism, of the maximal divisible subgroup of $S(RG)$.

Let G be an Abelian p-group and let K be a perfect field of characteristic p. May [13] proves that $S(KG)$ is simply presented if and only if G is a simply presented Abelian p-group. Therefore, if G is a simply presented Abelian p-group, then the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(KG)$ together with the description of the maximal divisible subgroup of $S(KG)$ give a full description, up to isomorphism, of the group $S(KG)$.

Let G be an Abelian p-group and let R be a commutative ring with identity which characteristic is different from p. Berman and Rossa [3] and [4] have given a description of the torsion subgroup $tV(RG)$ of $V(RG)$ when G is a countable Abelian p-group and R is a countable field. Nachev [23] and [24] has given a description of the torsion subgroup $tV(RG)$ of $V(RG)$ when G is an Abelian p-group and R contains the p^nth roots of the unity, $n \in \mathbb{N}$. Karpilovsky [10, 5.2.5 Theorem, p. 126] has determined the isomorphism class of $U(\mathbb{Q}G)$ when G is a finitely generated Abelian group. We define

$$G^{p^n} = \{g^{p^n} | g \in G\},\ n \in \mathbb{N},\ G^1 = \bigcap_{n=1}^{\infty} G^{p^n}.$$

Mollov [19] gives a full description, up to isomorphism, of the group $S(RG)$ when G is an arbitrary Abelian p-group and R is a field of the second kind with respect to p. Besides Mollov [19] and [20] has described, up to isomorphism, the torsion subgroup $tV(RG)$ of $V(RG)$ when G is restricted direct product of cyclic p-groups. Mollov [18] has also described $V(RG)$, up to isomorphism, when either (a) G is an infinite restricted direct product of cyclic p-groups and $R = \mathbb{Q}$ or (b) G is an Abelian p-group and $R = \mathbb{R}$. Chatzidakis and Pappas [6] have determined the isomorphic class of $U(RG)$ when the torsion Abelian group G is a restricted direct product of countable groups and R is a field. Nachev and Mollov [28] and [29] describe $U(RG)$, up to isomorphism, when G is an Abelian p-group and at least one of the following conditions (a) or (b) is fulfilled:

(a) the first Ulm factor G/G^1 of G is a restricted direct product of cyclic groups and R is a field of the first kind with respect to p;

(b) R is a field of the second kind with respect to p.

If R is a direct product of m indecomposable rings, $m \in \mathbb{N}$, Mollov and Nachev [21] give a description, up to isomorphism, of the unit group $U(RG)$ of RG in the following cases:

(a) when G is a finite Abelian group of exponent n and n is an invertible element in R and

(b) when R is of characteristic zero, R has no nilpotent elements, tG is finite of exponent n and n is an invertible element in R.

Kuneva [11] gives a description of the maximal divisible subgroup of $V(RG)$ when G is a p-mixed Abelian group and the ring R is a direct product of n commutative indecomposable rings with identity of characteristic $p, n \in \mathbb{N}$.

The present paper continues the mentioned investigations of $V(RG)$. Namely, we give a full description, up to isomorphism, of the group $V(RG)$ when R is a direct product of m perfect fields of characteristic p, $m \in \mathbb{N}$, G is a p-mixed Abelian group, G_p is simply presented and either (i) G is splitting or (ii) G is of countable torsion free rank. Besides, we give a description, up to isomorphism, of the group $U(RG)$ when R is an infinite direct product of commutative indecomposable rings with identity, G is a finite Abelian group and the exponent of G is an invertible element in R.

2. Some Concepts and Preliminary Results

We recall some well known definitions. Let G be an Abelian group. We say that G has *exponent* n if $G^n = 1$ and n is the least natural with this property. The group G is called p-mixed if the torsion subgroup of G is p-primary. We use the signs $\dot{\sum}, \coprod$ and $\prod$ for a mark of direct sums of algebras, coproduct of groups and direct product of groups (algebras), respectively. Denote by $\coprod_{\alpha} G$ a coproduct of α copies of G, where α is a cardinal number. If $G \cong \coprod_\lambda Z(p^\infty)$ holds where λ is a cardinal, then λ is called an Ulm invariant of the divisible p-group G.

The Abelian group terminology is in agreement with the books of Fuchs [9].

Let R be a commutative ring with identity and let R^* be the multiplicative group of R. The ring R is called indecomposable, if it cannot be decomposed into a direct sum of two or more nontrivial ideals of R. The group algrbra RG is called modular if the characteristic of R is a prime number p. If α is an algebraic element over R, α is a root of the polynomial $f(x) \in R[x]$ and $f(x)$ is a polynomial of the least degree with this property then $f(x)$ is called *a minimal polynomial of* α *over* R. An algebraic element α over R is called *an integral algebraic element over* R if there exists a minimal polynomial of α over R which is monic. This definition differs from the definition in the theory of the algebraic numbers and in the field theory since an element α can be a root of a monic polynomial over R and α can have not a minimal polynomial over R which is monic.

For the formulation of the other results we shall introduce the following concepts. If $n, m \in \mathbb{Z}$, then n/m will denote n divides m. Let G be a finite Abelian group of exponent n and n be an invertible element in R. For every divisor d of n we denote $\lambda(d) = \mu(d)\nu(d)$, where $\mu(d)$ is the number of the cyclic subgroups of G of order d and $\nu(d)$ is the number of the monic irreducible divisors of the cyclotomic polynomial $\Phi_d(x)$ over R. Then the number $\lambda(d)$, where d/n, is called *a d-number of the group ring RG*. In the next result R_i is a commutative indecomposable ring with identity. We denote by ε_d *an integral algebraic element over* R_i *which is a root of a monic irreducible divisor of* $\Phi_d(x)$ *over* R_i, that is, ε_d is a root of a monic irreducible divisor $\varphi(x)$ of $\Phi_d(x)$ such that $\varphi(x)$ is the monic minimal polynomial of ε_d over R_i (see Mollov, Nachev [21]). The following results are well known.

Theorem 2.1. (see Mollov, Nachev [21]) *Let G be a finite Abelian group of exponent n and let R be a commutative ring with identity which is a direct product of m indecomposable rings R_i (e.g. R is a noetherian), m, $i \in \mathbb{N}$. Then*

$$RG \cong \sum_{i=1}^{.m} R_i G.$$

If n is an invertible element in R_i, then

$$R_i G \cong \sum_{d/n}^{.} \lambda_i(d) R_i[\varepsilon_d],$$

where $\lambda_i(d)$ is a d-number of the group ring R_iG. Therefore,

$$U(R_iG) \cong \coprod_{d/n} \coprod_{\lambda_i(d)} R_i[\varepsilon_d]^*.$$

Theorem 2.2. (see [14]) *Let F be a perfect field of prime characteristic p and let G be a p-mixed Abelian group of countable torsion free rank and simply presented torsion subgroup T. Then T is a direct factor of $S(FG)$ with simply presented complement and G is a direct factor of $V(FG)$ with the same complement.*

Theorem 2.3. (see [15]) *Let K be a perfect field of prime characteristic p, G an Abelian group and $KH \cong KG$ as K-algebras for some group H. Assume that G is a splitting group and that G_p is simply presented. Then $H_p \cong G_p$. If, in addition, G is p-mixed, then G_p is a direct factor of $S(KG)$ and G is a direct factor of $V(KG)$, each with the same simply presented complement.*

For any ordinal α we define G^{p^α} inductively by the following way:

$$G^{p^0} = G, \quad G^{p^{\alpha+1}} = (G^{p^\alpha})^p \quad \text{and} \quad G^{p^\alpha} = \bigcap_{\beta<\alpha} G^{p^\beta}$$

if α is a limit ordinal.

Proposition 2.4. (see [12]) *Let G be an Abelian group and let R be a finite commutative ring with identity of prime characteristic p without nilpotent elements. If α is any ordinal and G^{p^α} is finite, then*

$$f_\alpha(S/G_p)) = f_\alpha(S) - f_\alpha(G_p),$$

$f_\alpha(S) = (|G^{p^\alpha}| - 2|G^{p^{\alpha+1}}| + |G^{p^{\alpha+2}}|)\log_p |R|$.

Theorem 2.5. (see [7, Theorem 6, *(ii)*]) *Suppose $1 \neq G$ is an Abelian group and R is an unitary perfect commutative ring without nilpotent elements in prime characteristic p. Then:*

(ii) If $|R| \geq \aleph_0$ or $|G^{p^\sigma}| \geq \aleph_0$ for some ordinal σ,

$$f_\sigma(S(RG)/G_p) = \begin{cases} \max(|R|, |G^{p^\sigma}|) & \text{when } |G_p^{p^\sigma}| \neq 1 \quad \text{and} \quad G^{p^\sigma} \neq G^{p^{\sigma+1}}; \\ 0, & \text{when } G_p^{p^\sigma} = 1 \quad \text{or} \quad G^{p^\sigma} = G^{p^{\sigma+1}}. \end{cases}$$

3. Main Results

Let $R_i, i \in I$, be a system of rings and let G be an arbitrary group. If $a \in (\Pi_{i\in I} R_i)G$, then

$$a = \sum_{g\in G_a} a_g g, \quad a_g \in \Pi_{i\in I} R_i, \tag{3.1}$$

where G_a is a finite subset of G. We note that G_a and the systen $\{a_g | g \in G_a\}$ are defined identically from the element a. Besides, $a_g = (..., a_{gi}, ...)$ where $a_{gi} \in R_i$ for every $i \in I$ and every $g \in G_a$. We define a map

$$\varphi : (\Pi_{i\in I} R_i)G \to \Pi_{i\in I}(R_i G) \tag{3.2}$$

by

$$\varphi(a) = (..., \sum_{g\in G_a} a_{gi} g, ...). \tag{3.3}$$

It is not hard to see that φ is a natural injective homomorphism of R-algebras.

The following result is announced in Ph.D. Thesis of Nachev [26].

Proposition 3.1. *The homomorphism (3.2) of R-algebras, defined by (3.3), is an isomorphism of R-algebras if and only if either I is a finite set or G is a finite group.*

Proof. We shall prove that if either I is a finite set or G is a finite group, then φ is a surjective homomorphism, that is φ is an isomorphism of R-algebras.

Let I be a finite set, $I = \{1, 2, ..., n\}$ and let $z \in \Pi_{i\in I}(R_iG)$. Then

$$z = (z_1, z_2, ..., z_n), \quad z_i \in R_iG, \quad i = 1, 2, ..., n.$$

We put $G_z = \bigcup_{i=1}^{n} G_i$, where G_i is the support of z_i. Then

$$z_i = \sum_{g\in G_z} z_{gi}g, \quad z_{gi} \in R_i, \quad i = 1, 2, ..., n, \tag{3.4}$$

where $z_{gi} = 0$ if $g \notin G_i$. Formula (3.4) has a sense, since G_z is a finite set of G. Now we put

$$x = \sum_{g\in G_z} z_g g, \quad z_g = (z_{g1}, z_{g2}, ..., z_{gn}).$$

Then, by (3.3), we have $\varphi(x) = z$, that is φ is a surjective homomorphism.

Now let G be a finite group and let again $z \in \Pi_{i\in I}(R_iG)$. Then we have

$$z = (..., z_i, ...), \quad z_i \in R_iG.$$

Since G is finite, then

$$z_i = \sum_{g\in G} z_{gi}g, \quad z_{gi} \in R_i, \quad i \in I.$$

If we put

$$y = \sum_{g\in G} z_g g, \quad z_g = (..., z_{gi}, ...),$$

then (3.3) implies $\varphi(y) = z$, that is φ is a surjective homomorphism.

In the end we shall prove that if I and G are infinite, then the homomorphism φ is not surjective. Indeed, then I can be represented in the kind $I = I_1 \bigcup I_2$ where I_1 is countably infinite and I_2 is its complement in I. We choose also an infinite countable subset M of G which elements g_i are indexed by naturals. Now we consider the element $r = (..., r_i, ...) \in \Pi_{i\in I}(R_iG)$ which is constructed by the following way:

$$r_i = \begin{cases} g_i, & \text{if} \quad i \in I_1; \\ 0, & \text{if} \quad i \in I_2. \end{cases}$$

We shall prove that r does not have preimage. Suppose the contrary. Let $\varphi(a) = r$. Then a has a form (3.1). Since G_a is finite, then there exists an element $g_i \in M$ such that $g_i \notin G_a$. Then, by (3.3), for the indicated i we have

$$\sum_{g\in G_a} a_{gi}g = g_i.$$

In the right part of this equality the element g_i participates with a coefficient 1 and in the left part this coefficient is 0, since $g_a \notin G_a$, which is a contradiction. Therefore, the homomorphism φ is not surjective. □

Proposition 3.2. *Let G be an Abelian group and let R be a commutative perfect ring with identity of prime characteristic p without nilpotent elements. Then for the maximal divisible subgroup dT of $T = S(RG)/G_p$ the following holds: if α is the first ordinal such that $G^{p^\alpha} = G^{p^{\alpha+1}}$, then $dT = S(RG^{p^\alpha})G_p/G_p$ and $dT \cong S(RG^{p^\alpha})/G_p^{p^\alpha}$. If $G_p^{p^\alpha} = 1$, then $dT = 1$. If $G_p^{p^\alpha} \neq 1$, then*

$$dT \cong \coprod_\lambda Z(p^\infty), \quad \lambda = \max(|R|, |G^{p^\alpha}|). \tag{3.5}$$

Proof. Since, by the paper of Kuneva, Mollov and Nachev [12], G_p is nice in $S(RG)$, then

$$(S(RG)/G_p)^{p^\alpha} = S(RG^{p^\alpha})G_p/G_p = S(RG^{p^{\alpha+1}})G_p/G_p = (S(RG)/G_p)^{p^{\alpha+1}}.$$

Consequently,

$$dT = S(RG^{p^\alpha})G_p/G_p \cong S(RG^{p^\alpha})/S(RG^{p^\alpha}) \bigcap G_p = S(RG^{p^\alpha})/G_p^{p^\alpha},$$

that is, $dT = S(RG^{p^\alpha})G_p/G_p$ and $dT \cong S(RG^{p^\alpha})/G_p^{p^\alpha}$. Since G^{p^α} is the maximal divisible subgroup of G, then $G_p^{p^\alpha}$ is the maximal divisible subgroup of G_p and $S(RG^{p^\alpha})$ is the maximal divisible subgroup of $S(RG)$. If $G_p^{p^\alpha} = 1$, then, by the result of Nachev [25], every Ulm-Kaplansky invariants of $S(RG^{p^\alpha})$ are zero and the maximal divisible subgroup of $S(RG^{p^\alpha})$ is equal to 1. Hence $S(RG^{p^\alpha}) = 1$ and $dT = 1$.

Let $G_p^{p^\alpha} \neq 1$. Further, for a simplycity, we shall designate G^{p^α} with G and $G_p^{p^\alpha}$ with $G_p \neq 1$. Therefore, $|G| \geq \aleph_0$ and $dT \cong S(RG)/G_p$. For the description of dT we shall consider two cases: 1) $|R| \geq |G|$ and 2) $|G| > |R|$.

1) Let $|R| \geq |G|$. Denote $R' = R \setminus \{0, 1\}$. Obviously $|R'| \geq \aleph_0$. We form the set

$$X = \{x_r = [1 + rg_1(g-1)]G_p | g \in G[p], g \neq 1, g_1 \in G, g_1^p = g, r \in R'\}.$$

We note that since $G_p \neq 1$ is a divisible group, then there exists $g \in G[p], g \neq 1$ and an element $g_1 \in G$ such that $g_1^p = g$. Obviously, $X \subseteq (dT)[p]$. If r and s are different elements of R', then $x_r \neq x_s$. Otherwise we obtain

$$1 - rg_1 + rg_1g = h - shg_1 + shg_1g. \tag{3.6}$$

Obviously the left and the right part of this equality are in a canonical form. Therefore, we have the following three cases: *a)* $h = 1$, *b)* $hg_1 = 1$, that is, $h = g_1^{-1}$ and *c)* $hg_1g = 1$, that is, $h = g_1^{-1}g^{-1}$.

a) Let $h = 1$. Then (3.6) obtains the form $-rg_1 + rg_1g = -sg_1 + sg_1g$. Hence $r = s$, which is a contradiction.

b) Let $h = g_1^{-1}$. Then (3.6) obtains the form $1 - rg_1 + rg_1g = g_1^{-1} - s + sg$. Consequently, $p = 2$ and $s = 1$, which is a contradiction.

c) Let $h = g_1^{-1}g^{-1}$. Then (3.6) obtains the form $1-rg_1+rg_1g = g_1^{-1}g^{-1}-sg^{-1}+s$. Hence $s = 1$, which is a contradiction.

In this way we obtain that $|X| \geq |R|$ and $|(dT)[p]| \geq |R|$. Since $|RG| = |R|$, then $|(dT)[p]| = |R|$. Therefore, formula (3.5) holds.

2) Let $|G| > |R|$. Denote by g a fixed element of $G[p] \setminus \{1\}$ and

$$\Pi = \Pi(G/\langle g\rangle), \quad 1 \in \Pi, \quad \Pi' = \Pi \setminus \{1\}.$$

We form the set

$$X = \{x_a = [1 + a(g-1)]G_p, \quad a \in \Pi'\}.$$

Obviously, $X \subseteq (dT)[p]$. If a and b are different elements from Π', then $x_a \neq x_b$. Otherwise we obtain

$$1 - a + ag = h - bh + bgh, \quad h \in G_p. \tag{3.7}$$

Obviously the left and the right part of this equality are in a canonical form. Therefore, we have the following three cases: *a)* $h = 1$, *b)* $bh = 1$, that is, $h = b^{-1}$, *c)* $bgh = 1$, that is, $h = b^{-1}g^{-1}$.

a) Let $h = 1$. Then (3.7) obtains the form $-a + ag = -b + bg$. The elements of the left and the right part of this equality belong to the classes $a\langle g\rangle$ and $b\langle g\rangle \neq a\langle g\rangle$ respectively, which is a contradiction.

b) Let $h = b^{-1}$. Consequently, (3.7) obtains the form $1 - a + ag = b^{-1} - 1 + g$. Hence p=2 and we obtain $a + ag = b^{-1} + g$. In this equality only the element g belongs to the class $\langle g\rangle$ which is a contradiction.

c) Let $h = b^{-1}g^{-1}$. Consequently, (3.7) obtains the form $-a+ag = b^{-1}g^{-1}-g^{-1}$. In this equality only the element g^{-1} belongs to the class $\langle g\rangle$ which is a contradiction.

In this way we obtain that $|X| \geq |G|$ and $|(dT)[p]| \geq |G|$. Since $|RG| = |G|$, then $|(dT)[p]| = |G|$. Therefore, formula (3.5) holds. □

The following theorem extends results of the authors [21]. This theorem gives a full description, up to isomorphism, of the group $V(RG)$ when R is a direct product of m perfect fields, $m \in \mathbb{N}$, G is p-mixed Abelian group, G_p is simply presented and either (i) G is splitting or (ii) G is of countable torsion free rank.

Theorem 3.3. *Let G be a p-mixed Abelian group and let a ring R be a direct product of m perfect fields F_i of characteristic p, $m \in \mathbb{N}$. Suppose that G_p is simply presented and either (i) G is splitting or (ii) G is of countable torsion free rank. Then there exist p-subgroups T_i of $V(F_iG), i = 1, 2, ..., m$, such that*

$$V(RG) \cong \coprod_m G \times \coprod_{i=1}^m T_i, \quad T_i \cong S(F_iG)/G_p. \tag{3.8}$$

Every group T_i is simply presented and it is described, up to isomorphism, by its Ulm-Kaplansky invariants $f_\alpha(T_i)$ and by its maximal divisible subgroup dT_i. The invariants $f_\alpha(T_i)$ are the following:

(a) if F_i and G^{p^α} are finite, then

$$f_\alpha(T_i) = (|G^{p^\alpha}| - 2|G^{p^{\alpha+1}}| + |G^{p^{\alpha+2}}|)\log_p |F_i| - f_\alpha(G_p). \tag{3.9}$$

(b) If either $|F_i| \geq \aleph_0$ or $|G^{p^\alpha}| \geq \aleph_0$, then

$$f_\alpha(T_i) = \begin{cases} \max(|F_i|, |G^{p^\alpha}|), & \text{if } |G_p^{p^\alpha}| \neq 1 \quad \text{and} \quad G^{p^\alpha} \neq G^{p^{\alpha+1}}; \\ 0, & \text{if either } G_p^{p^\alpha} = 1 \quad \text{or} \quad G^{p^\alpha} = G^{p^{\alpha+1}}. \end{cases} \tag{3.10}$$

For the maximal divisible subgroup of $S(F_iG)/G_p \cong T_i$ the following assertions hold: if α is the first ordinal such that $G^{p^\alpha} = G^{p^{\alpha+1}}$, then $d(S(F_iG)/G_p) = S(F_iG^{p^\alpha})G_p/G_p$ and $dT_i \cong S(F_iG^{p^\alpha})/G_p^{p^\alpha}$; if $G_p^{p^\alpha} = 1$, then $dT_i = 1$; if $G_p^{p^\alpha} \neq 1$, then

$$dT_i \cong \coprod_\lambda Z(p^\infty), \quad \lambda = \max(|F_i|, |G^{p^\alpha}|).$$

Proof. We obtain, by Proposition 3.1, $RG \cong \Pi_{i=1}^m (F_iG)$. Therefore,

$$U(RG) \cong \coprod_{i=1}^m U(F_iG), \quad V(RG) \cong \coprod_{i=1}^m V(F_iG). \tag{3.11}$$

Since either the condition *(i)* or the condition *(ii)* of the theorem are valid, then, by Theorem 2.3 and Theorem 2.2, respectively, we obtain

$$V(F_iG) = G \times T_i, \quad T_i \cong V(F_iG)/G, \tag{3.12}$$

where T_i is a simply presented Abelian p-group. Then (3.11) and (3.12) imply the first formula of (3.8). Since, by a result of Mollov and Nachev [22], $V(F_iG) = GS(F_iG)$ is fulfilled, then

$$T_i \cong V(F_iG)/G = GS(F_iG)/G \cong S(F_iG)/S(F_iG) \bigcap G = S(F_iG)/G_p,$$

that is,

$$T_i \cong S(F_iG)/G_p. \tag{3.13}$$

Consequently, the second formula of (3.8) holds. Since T_i is simply presented, then $T_i \cong S(F_iG)/G_p$ is described, up to isomorphism, by its Ulm-Kaplansky invariants $f_\alpha(T_i)$ and by its maximal divisible subgroup dT_i. In the case *(a)* of the theorem the invariants $f_\alpha(T_i)$ are described by Proposition 2.4 and in the case *(b)* they are described by Theorem 2.5. Therefore, for $F_\alpha(T_i)$ (3.9) and (3.10) are fulfilled. Proposition 3.2 implies that the indicated description of dT_i holds. □

Remark 1. For the calculation of the Ulm-Kaplansky invariants of the group $S(FG)/G_p$ in the finite case, that is, in case (a) of Theorem 3.3, we use Proposition 2.4 and we do not use the result of Danchev [7, Theorem 6, case *(i)*], since the last result is inexact and it is not complete (see Kuneva, Mollov and Nachev [12]).

Remark 2. Let G and F_i be as in Theorem 3.3. We obtain, by (3.12) and (3.13), the following decomposition:

$$V(F_iG) \cong G \times (S(F_iG)/G_p).$$

Since the p-components of isomorphic groups are isomorphic, then $S(F_iG) \cong G_p \times (S(F_iG)/G_p)$ and

$$dS(F_iG) \cong dG_p \times d(S(F_iG)/G_p). \tag{3.14}$$

The description of $dS(F_iG)$, up to isomorphism, is given by Mollov [17] in terms of the Ulm invariant of $dS(F_iG)$. Let G^* be the maximal p-divisible subgroup of G. We note that when $dG_p \neq 1$ and $|dG_p| = |G^*| \geq |F_i|$, then the Ulm invariants of $dS(F_iG)$ and dG_p are infinite and equal, since they coincide with $\max(|F_i|, |G^*|) = |dG_p|$ (see either Mollov [17] or Nachev [25]). Now (3.14) shows that when $dS(F_iG)$ and dG_p have equal infinite Ulm invariants, then *we cannot calculate the Ulm invariant of $d(S(F_iG)/G_p)$ by an using of the Ulm invariants of $dS(F_iG)$ and dG_p.* Therefore, *the description of $dS(F_iG)$ (and the description of $dV(F_iG)$), up to isomorphism, does not imply the description of the maximal divisible subgroup of $S(F_iG)/G_p$, which is done in Theorem 3.3.*

In an analogous way we can see that the description of

$$d[S(RG)/G_p] \cong S(R_dG^*)/(G^*)_p$$

in Proposition 1 of the paper of Danchev [8] does not be obtained by the description of Nachev [25] of $S(R_dG^*)$ when the groups $S(R_dG^*)$ and $(G^*)_p$ have the same infinite Ulm invariants (for this notation see Danchev [8]). Because of this reason the conclusions of Theorem 7 and Corollary 8 of Danchev [7] that the structure of $S(RG)$ and $V(RG)$ are completely determined are ungrounded and incorrect, that is, *the structures of $S(RG)$ and $V(RG)$ in the mentioned two statements of Danchev [7] are not completely determined.*

The following theorem extends results of the authors [21] giving a full description, up to isomorphism, of the group $U(RG)$ when R is an infinite direct product of commutative indecomposable rings with identity, G is a finite Abelian group and the exponent of G is an invertible element in R.

Theorem 3.4. *Let $R = (\Pi_{i\in I} R_i)$, R_i be commutative indecomposable rings with identity and let G be a finite Abelian group of an exponent n such that n is an invertible element in R. Then*

$$U(RG) \cong \Pi_{i\in I} \coprod_{d/n} \coprod_{\lambda_i(d)} R_i[\varepsilon_d]^*. \tag{3.15}$$

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where $\lambda_i(d)$ is a d-number of the group ring R_iG.

Proof. We obtain, by Proposition 3.1, $RG \cong \Pi_{i\in I}(R_iG)$ and

$$U(RG) \cong \Pi_{i\in I}U(R_iG). \tag{3.16}$$

Since n is an invertible element in R_i, then, by Theorem 2.1,

$$U(R_iG) \cong \coprod_{d/n}\coprod_{\lambda_i(d)} R_i[\varepsilon_d]^*.$$

This formula and (3.16) imply (3.15). □

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