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NEW INTEGRAL INEQUALITIES FOR s-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract: In this paper we establish some new inequalities for product of convex and s-convex functions in the second sense. We also prove several applications for special means.

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1. Introduction

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality (see [1], or [2]) which has generated a wide range of directions for extension and a rich mathematical literature. The following definitions are well known in the mathematical literature: a function $f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. Geometrically, this means that if P, Q and R are

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three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR.

In the paper [3] Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined in the following way: a function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f\left(tx + (1-t)y\right) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of s-convex functions in the second sense is usually denoted with K_s^2.

It can be easily seen that for $s = 1$ s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In the same paper [3] Hudzik and Maligranda proved that if $s \in (0, 1)$, $f \in K_s^2$ implies $f([0, \infty)) \subseteq [0, \infty)$, i.e., they proved that all functions from K_s^2, $s \in (0, 1)$, are nonnegative.

Example 1. (see [3]) Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0, \infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases} \tag{3}$$

It can be easily checked that:

(1) If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$.

(2) If $b > 0$ and $c < 0$, then $f \notin K_s^2$.

Many important inequalities are established for the class of convex functions, but one of the most famous is so called Hermit–Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as follows (see for example [5]): let f be a convex function on $[0, b] \subset \mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2}. \tag{4}$$

In [4] Pachpatte proved a variant of Hadamard's inequality which holds for convex functions.

Theorem 2. *Let f and g real-valued, nonnegative and convex functions on $[a, b]$. Then*

$$\begin{aligned} &\frac{3}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f\left(tx + (1-t)y\right) g\left(tx + (1-t)y\right) dtdydx \qquad (5) \\ \leq\ &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{8}\left[\frac{M(a,b) + N(a,b)}{(b-a)^2}\right] \end{aligned}$$

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and

$$\begin{aligned}&\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx\\ \leq\ &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{1}{4}\left(\frac{1+b-a}{b-a}\right)[M(a,b)+N(a,b)],\end{aligned}\tag{6}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ *and* $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$.

The main purpose of this paper is to establish new inequalities like those given in Theorems 2 but now for the class of s-convex functions.

2. Main Results

In the next our theorem we will also make use of Beta function of Euler type, which is for $x,y>0$ defined as

$$\beta(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt=\frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}$$

and

$$\beta(u,v)=\beta(v,u).$$

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}, a,b\in[0,\infty)$, $a<b$ be nonnegative and s-convex function in the second sense. If f is s_1-convex in the second sense and g is s_2-convex in the second sense on $[a,b]$ for some $t\in[0,1]$ and $s_1,s_2\in(0,1]$, then;*

$$\begin{aligned}&\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx\\ \leq\ &\frac{2(b-a)}{s_1+s_2+1}\int_a^b f(x)\,g(x)\,dx+\frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}[M(a,b)+N(a,b)],\end{aligned}\tag{7}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ *and* $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$.

Proof. Since f is s_1-convex and g is s_2-convex on $[a,b]$ we have

$$\begin{aligned}f(tx+(1-t)y)&\leq t^{s_1}f(x)+(1-t)^{s_1}f(y),\\ g(tx+(1-t)y)&\leq t^{s_2}g(x)+(1-t)^{s_2}g(y),\end{aligned}$$

for all $t\in[0,1]$. Because f and g are nonnegative, these above inequalities can be multiplied by side

$$f(tx+(1-t)y)\,g(tx+(1-t)y)$$

$$\leq t^{s_1+s_2} f(x) g(x) + (1-t)^{s_1+s_2} f(y) g(y) + t^{s_1}(1-t)^{s_2} f(x) g(y) + t^{s_2}(1-t)^{s_1} f(y) g(x).$$

Integrating both sides of the above inequalities over $[0,1]$ we obtain

$$\begin{aligned}
&\int_0^1 f(tx+(1-t)y)\, g(tx+(1-t)y)\, dt \\
\leq\ & f(x) g(x) \int_0^1 t^{s_1+s_2} dt + f(y) g(y) \int_0^1 (1-t)^{s_1+s_2} dt \\
&+ f(x) g(y) \int_0^1 t^{s_1}(1-t)^{s_2} dt + f(y) g(x) \int_0^1 t^{s_2}(1-t)^{s_1} dt \\
&= \frac{1}{s_1+s_2+1} [f(x) g(x) + f(y) g(y)] \\
&+ \beta(s_1+1, s_2+1) f(x) g(y) + \beta(s_2+1, s_1+1) f(y) g(x) \\
=\ & \frac{1}{s_1+s_2+1} [f(x) g(x) + f(y) g(y)] \\
&+ \beta(s_1+1, s_2+1) [f(x) g(y) + f(y) g(x)] \\
=\ & \frac{1}{s_1+s_2+1} [f(x) g(x) + f(y) g(y)] \\
&+ \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} [f(x) g(y) + f(y) g(x)].
\end{aligned}$$

Integrating both sides of the above inequalities on $([a,b]\times[a,b])$ we obtain

$$\begin{aligned}
&\int_a^b \int_a^b \int_0^1 f(tx+(1-t)y)\, g(tx+(1-t)y)\, dt dy dx \\
\leq\ & \frac{1}{s_1+s_2+1} \left[\int_a^b \int_a^b f(x) g(x)\, dy dx + \int_a^b \int_a^b f(y) g(y)\, dy dx\right] \\
&+ \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} \left[\int_a^b \int_a^b f(x) g(y)\, dy dx + \int_a^b \int_a^b f(y) g(x)\, dy dx\right] \\
\leq\ & \frac{1}{s_1+s_2+1} (b-a) \left[\int_a^b f(x) g(x)\, dx + \int_a^b f(y) g(y)\, dy\right] \\
&+ \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} \left[\int_a^b f(x)\, dx \int_a^b g(y)\, dy + \int_a^b f(y)\, dy \int_a^b g(x)\, dx\right]
\end{aligned}$$

by using the right half of the Hadamard's inequality on the right side of the above inequality, we have

$$\begin{aligned}
&\int_a^b \int_a^b \int_0^1 f(tx+(1-t)y)\, g(tx+(1-t)y)\, dt dy dx \\
\leq\ & \frac{2(b-a)}{s_1+s_2+1} \int_a^b f(x) g(x)\, dx
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}(b-a)^2\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}\\
&+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}(b-a)^2\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}\\
=\;&\frac{2(b-a)}{s_1+s_2+1}\int_a^b f(x)\,g(x)\,dx\\
&+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}(b-a)^2\left[M(a,b)+N(a,b)\right].
\end{aligned}
$$

The proof is complete. □

Remark 4. If in Theorem 3 we choose $s_1 = s_2 = 1$, then (7) reduces to (4):

$$
\begin{aligned}
\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} &= \frac{\Gamma(1+1)\Gamma(1+1)}{\Gamma(1+1+2)}\\
&= \frac{1}{\Gamma(s_1+s_2+2)}\Gamma(s_1+1)\Gamma(s_2+1) = \frac{1}{6}.
\end{aligned}
$$

Theorem 5. *Let $f, g : [a, b] \to \mathbb{R}, a, b \in [0,\infty)$, $a < b$ be nonnegative and s-convex function in the second sense. If f is s_1-convex in the second sense and g is s_2-convex in the second sense on $[a, b]$ for some $t \in [0, 1]$ and $s_1, s_2 \in (0, 1]$, then;*

$$
\begin{aligned}
&\int_a^b\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)dtdx \qquad (8)\\
\leq\;&\frac{1}{s_1+s_2+1}\int_a^b f(x)\,g(x)\,dx\\
&+\left[\frac{1}{4(s_1+s_2+1)}+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}\right](b-a)\left[M(a,b)+N(a,b)\right],
\end{aligned}
$$

where $M(a, b)$ and $N(a, b)$ are as defined in Theorem 3.

Proof. Since f is s_1-convex and g is s_2-convex on $[a, b]$ we have

$$
\begin{aligned}
f\left(tx+(1-t)\frac{a+b}{2}\right) &\leq t^{s_1}f(x)+(1-t)^{s_1}f\left(\frac{a+b}{2}\right),\\
g\left(tx+(1-t)\frac{a+b}{2}\right) &\leq t^{s_2}g(x)+(1-t)^{s_2}g\left(\frac{a+b}{2}\right),
\end{aligned}
$$

for x in $[a, b]$ and all $t \in [0, 1]$. Because f and g are nonnegative, these above inequalities can be multiplied by side

$$
f\left(tx+(1-t)\frac{a+b}{2}\right)g\left(tx+(1-t)\frac{a+b}{2}\right)
$$

$$
\begin{aligned}
&\le f(x)\,g(x)\,t^{s_1+s_2} + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)(1-t)^{s_1+s_2} \\
&\quad + f(x)\,g\left(\frac{a+b}{2}\right) t^{s_1}(1-t)^{s_2} + f\left(\frac{a+b}{2}\right) g(x)\,t^{s_2}(1-t)^{s_1}.
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$ we obtain

$$
\begin{aligned}
&\int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right) g\left(tx+(1-t)\frac{a+b}{2}\right) dt \\
\le\ & f(x)\,g(x)\int_0^1 t^{s_1+s_2}dt + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\int_0^1 (1-t)^{s_1+s_2}\,dt \\
& + f(x)\,g\left(\frac{a+b}{2}\right)\int_0^1 t^{s_1}(1-t)^{s_2}\,dt + f\left(\frac{a+b}{2}\right) g(x)\int_0^1 t^{s_2}(1-t)^{s_1}\,dt \\
& = \frac{1}{s_1+s_2+1}\left[f(x)\,g(x) + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\right] \\
& + \beta(s_1+1, s_2+1)\, f(x)\,g\left(\frac{a+b}{2}\right) + \beta(s_2+1, s_1+1)\, f\left(\frac{a+b}{2}\right) g(x) \\
=\ & \frac{1}{s_1+s_2+1}\left[f(x)\,g(x) + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\right] \\
& + \frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}\left[f(x)\,g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) g(x)\right].
\end{aligned}
$$

As explained in the proof (7) given above, fg is integrable on $[a,b]$. Now integrating both sides of the above inequality on $[a,b]$, using the right half of the Hadamard's inequality and the s-convexity of f and g we observe that

$$
\begin{aligned}
&\int_a^b \int_0^1 f\left(tx+(1-t)\frac{a+b}{2}\right) g\left(tx+(1-t)\frac{a+b}{2}\right) dt dx \\
\le\ & \frac{1}{s_1+s_2+1}\int_a^b f(x)\,g(x)\,dx + \frac{(b-a)}{s_1+s_2+1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
& + \frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}\left[g\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right] \\
=\ & \frac{1}{s_1+s_2+1}\int_a^b f(x)\,g(x)\,dx + \frac{(b-a)}{s_1+s_2+1}\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2} \\
& + \frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}(b-a)\frac{g(a)+g(b)}{2}\frac{f(a)+f(b)}{2} \\
& + \frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}(b-a)\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2} \\
=\ & \frac{1}{s_1+s_2+1}\int_a^b f(x)\,g(x)\,dx + \frac{(b-a)}{4(s_1+s_2+1)}\left[M(a,b), N(a,b)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}(b-a)[M(a,b),N(a,b)] \\
= & \frac{1}{s_1+s_2+1}\int_a^b f(x)g(x)\,dx \\
&+\left[\frac{1}{4(s_1+s_2+1)}+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}\right](b-a)[M(a,b),N(a,b)].
\end{aligned}
$$

The proof is complete. □

Theorem 6. *Let $f,g:[a,b]\to\mathbb{R}, a,b\in[0,\infty)$, $a<b$ be nonnegative and convex function. If f and g are convex on $[a,b]$ for some $t\in[0,1]$, then;*

$$
\begin{aligned}
&\frac{3}{b-a}\int_a^b\int_0^1 f\left(t\frac{a+b}{2}+(1-t)y\right)g\left(t\frac{a+b}{2}+(1-t)y\right)dtdy \qquad (9)\\
\leq & \frac{1}{b-a}\int_a^b f(y)g(y)\,dy+\frac{1}{2}[M(a,b)+N(a,b)],
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 3.

Proof. Since f and g are convex on $[a,b]$ we have

$$
\begin{aligned}
f\left(t\frac{a+b}{2}+(1-t)y\right) &\leq tf\left(\frac{a+b}{2}\right)+(1-t)f(y),\\
g\left(t\frac{a+b}{2}+(1-t)y\right) &\leq tg\left(\frac{a+b}{2}\right)+(1-t)g(y),
\end{aligned}
$$

for x in $[a,b]$ and all $t\in[0,1]$. Because f and g are nonnegative, these above inequalities can be multiplied by side

$$
\begin{aligned}
&f\left(t\frac{a+b}{2}+(1-t)y\right)g\left(t\frac{a+b}{2}+(1-t)y\right)\\
\leq & t^2 f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+(1-t)^2 f(y)g(y)\\
&+t(1-t)f\left(\frac{a+b}{2}\right)g(y)+t(1-t)f(y)g\left(\frac{a+b}{2}\right).
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$ we obtain

$$
\begin{aligned}
&\int_0^1 f\left(t\frac{a+b}{2}+(1-t)y\right)g\left(t\frac{a+b}{2}+(1-t)y\right)dt\\
\leq & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 t^2dt+f(y)g(y)\int_0^1(1-t)^2dt\\
&+f\left(\frac{a+b}{2}\right)g(y)\int_0^1 t(1-t)\,dt+f(y)g\left(\frac{a+b}{2}\right)\int_0^1 t(1-t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{3}\left[f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+f(y)\,g(y)\right] \\
&\quad +\frac{1}{6}\left[f\left(\frac{a+b}{2}\right)g(y)+f(y)\,g\left(\frac{a+b}{2}\right)\right].
\end{aligned}
$$

Now integrating both sides of the above inequality on $[a,b]$, using the right half of the Hadamard's inequality and the convexity of f and g we observe that

$$
\begin{aligned}
&\int_a^b\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,y\right)g\left(t\frac{a+b}{2}+(1-t)\,y\right)dtdy \\
&\le \frac{1}{3}\int_a^b\left[f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+f(y)\,g(y)\right]dy \\
&\quad +\frac{1}{6}\int_a^b\left[f\left(\frac{a+b}{2}\right)g(y)+f(y)\,g\left(\frac{a+b}{2}\right)\right]dy \\
&= \frac{1}{3}\int_a^b\left[f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)+f(y)\,g(y)\right]dy \\
&\quad +\frac{1}{6}f\left(\frac{a+b}{2}\right)\int_a^b g(y)\,dy+\frac{1}{6}g\left(\frac{a+b}{2}\right)\int_a^b f(y)\,dy \\
&\le \frac{1}{3}(b-a)\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}+\frac{1}{3}\int_a^b f(y)\,g(y)\,dy \\
&\quad +\frac{1}{6}\frac{f(a)+f(b)}{2}(b-a)\frac{g(a)+g(b)}{2}+\frac{1}{6}\frac{g(a)+g(b)}{2}(b-a)\frac{f(a)+f(b)}{2} \\
&= \frac{(b-a)}{12}\left[M(a,b)+N(a,b)\right]+\frac{1}{3}\int_a^b f(y)\,g(y)\,dy \\
&\quad +\frac{(b-a)}{12}\left[M(a,b)+N(a,b)\right] \\
&= \frac{1}{3}\int_a^b f(y)\,g(y)\,dy+\frac{(b-a)}{6}\left[M(a,b)+N(a,b)\right].
\end{aligned}
$$

Now multiplying both sides of the above inequalities by $\frac{3}{b-a}$, we get required inequality in (9).

$$
\begin{aligned}
&\frac{3}{b-a}\int_a^b\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,y\right)g\left(t\frac{a+b}{2}+(1-t)\,y\right)dtdy \\
&\le \frac{1}{b-a}\int_a^b f(y)\,g(y)\,dy+\left[\frac{M(a,b)+N(a,b)}{2}\right].
\end{aligned}
$$

The proof is complete. □

Theorem 7. *Let $f,g:[a,b]\to\mathbb{R}, a,b\in[0,\infty)$, $a<b$ be nonnegative and s-convex function in the second sense. If f is s_1-convex in the second sense and*

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g is s_2-convex in the second sense on $[a,b]$ for some $t\in[0,1]$ and $s_1,s_2\in(0,1]$, then;

$$\begin{aligned} &\frac{1}{b-a}\int_a^b\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\,y\right)g\left(t\frac{a+b}{2}+(1-t)\,y\right)dtdy\\ \le\ &\frac{1}{s_1+s_2+1}\frac{1}{b-a}\int_a^b f(y)\,g(y)\,dy \qquad (10)\\ &+\left[\frac{1}{4(s_1+s_2+1)}+\frac{\Gamma(s_1+1)\,\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}\right][M(a,b)+N(a,b)], \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 3.

Proof. The proof of this theorem can easily be made like Theorem 5. □

3. Applications to Some Special Means

We now consider the applications of our theorems to the following special means:

The power mean: $M_p=M_p(x_1,...,x_n):=\left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p}$, $a,b\ge 0$.

The arithmetic mean: $A=A(a,b):=\frac{a+b}{2}$, $a,b\ge 0$.

The geometric mean: $G=G(a,b):=\sqrt{ab}$, $a,b\ge 0$.

The harmonic mean: $H=H(a,b):=\frac{2ab}{a+b}$, $a,b\ge 0$.

The quadratic mean: $K=K(a,b):=\sqrt{\frac{a^2+b^2}{2}}$ $a,b\ge 0$.

The logarithmic mean: $L=L(a,b):=\begin{cases} a, & \text{if } a=b,\\ \frac{b-a}{\ln b-\ln a}, & \text{if } a\ne b,\end{cases}$ $a,b\ge 0$.

The Identric mean: $I=I(a,b):=\begin{cases} a, & \text{if } a=b,\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a\ne b,\end{cases}$ $a,b\ge 0$,

The p-logarithmic mean: $L_p=L_p(a,b):=\begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & \text{if } a\ne b,\\ a, & \text{if } a=b,\end{cases}$ $p\in\mathbb{R}\backslash\{-1,0\}$; $a,b>0$.

The following inequality is well known in the resources:

$$H\le G\le L\le I\le A\le K.$$

It is also known that L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0=I$ and $L_{-1}=L$.

In [3] the above Example 1 is given:

Let $s\in(0,1)$ and $a,b,c\in\mathbb{R}$.We define function $f:[0,\infty)\to\mathbb{R}$ as

$$f(t)=\begin{cases} a, & t=0,\\ bt^s+c, & t>0.\end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. Consequently, for $a = c = 0, b = 1, s = 1/2$, we have $f : [0,1] \to [0,1]$, $f(t) = t^{\frac{1}{2}}$, $f \in K_s^2$.

Proposition 8. *Let $a, b \in [0, \infty)$, $a < b$ and $b - a \leq 1$. Then one has the inequality:*

$$11K^2(a,b) + 7G^2(a,b) \leq \frac{4\left(b^3 - a^3\right)}{b-a} + \frac{6A^2(a,b)}{(b-a)^2}. \tag{11}$$

Proof. If f is s_1-convex function in the second sense and g is s_2-convex function in the second sense on $[a,b]$ for some $t \in [0,1]$ and $s_1, s_2 \in (0,1]$, then in Theorem 3, if we choose $f, g : [0,1] \to [0,1]$ $f(x) = x^{s_1}$, $g(x) = x^{s_2}$, $x \in [a,b]$ and $s_1 = s_2 = 1$ and $b - a \leq 1$, so

$$\begin{aligned}
& \int_a^b \int_a^b \int_0^1 (tx + (1-t)y)^2 \, dtdydx \\
\leq \; & \frac{2(b-a)}{3} \int_a^b x^2 dx + \frac{\Gamma(2)\Gamma(2)}{2\Gamma(4)} \left[a^2 + b^2 + 2ab\right] \\
\Longrightarrow \; & \frac{1}{36}(b-a)^2\left(11a^2 + 14ab + 11b^2\right) \\
\leq \; & \frac{2(b-a)}{9}\left(b^3 - a^3\right) + \frac{1}{12}(a+b)^2 \\
\Longrightarrow \; & (b-a)^2\left(11\left(a^2+b^2\right) + 14ab\right) \\
\leq \; & 8(b-a)\left(b^3 - a^3\right) + 3(a+b)^2 \\
\Longrightarrow \; & (b-a)^2\left(22K^2(a,b) + 14G^2(a,b)\right) \\
\leq \; & 8(b-a)\left(b^3 - a^3\right) + 12A^2(a,b).
\end{aligned}$$

Now multiplying both sides of the above inequalities by $\frac{1}{(b-a)^2}$, so

$$11K^2(a,b) + 7G^2(a,b) \leq \frac{4\left(b^3 - a^3\right)}{b-a} + \frac{6A^2(a,b)}{(b-a)^2}.$$

The proof is complete. □

Proposition 9. *Let $a, b \in [0, \infty)$, $a < b$ and $b - a \geq 1$. Then one has the inequality:*

$$11K^2(a,b) + 7G^2(a,b) \geq \frac{4\left(b^3 - a^3\right)}{b-a} + \frac{6A^2(a,b)}{(b-a)^2}. \tag{12}$$

Proof. The proof of this proposition is similar to the proof of Proposition 8. □

Proposition 10. *Let $a, b \in [0, \infty), a < b$. Then one has the inequality:*

$$36G^2(a,b)\,A\left(a^2, b^2\right) \leq \frac{5\left(b^5 - a^5\right)}{b-a} + 9A\left(a^4, b^4\right) + 2G^4(a,b) \tag{13}$$

Proof. In Theorem 5, let us assume that $f, g : [a, b] \to \mathbb{R}, a, b \in [0, \infty)$, $a < b$; Then we have for convex mapping $f(x) = x^2$ and $g(x) = x^2$, $x > 0$. We have

$$\begin{aligned}
& \frac{3}{b-a}\int_a^b\int_0^1\left(t\frac{a+b}{2}+(1-t)\,y\right)^4 dtdy \\
\leq\ & \frac{1}{b-a}\int_a^b x^4dx+\frac{1}{2}\left[a^4+b^4+2a^2b^2\right] \\
\Longrightarrow\ & \frac{8}{25}a^4+\frac{18}{25}a^3b+\frac{23}{25}a^2b^2+\frac{18}{25}ab^3+\frac{8}{25}b^4 \\
\leq\ & \frac{b^5-a^5}{5\,(b-a)}+\frac{a^4+b^4}{2}+a^2b^2 \\
\Longrightarrow\ & \frac{18}{25}\left(ab\left(a^2+b^2\right)\right) \\
\leq\ & \frac{b^5-a^5}{5\,(b-a)}+\frac{9}{50}\left(a^4+b^4\right)+\frac{2}{25}a^2b^2 \\
\Longrightarrow\ & 18ab\left(a^2+b^2\right) \\
\leq\ & \frac{5\left(b^5-a^5\right)}{b-a}+9\left(\frac{a^4+b^4}{2}\right)+2a^2b^2.
\end{aligned}$$

We get required inequality in (13). The proof is complete. □

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