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# OSCILLATION PROPERTIES OF SOLUTIONS OF $n$-TH ORDER DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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**Abstract:** We investigate some qualitative properties of differential equations with "maxima". The attention is paid to the existence of oscillatory solutions of a class of such equations. We prove some conditions under which that class of differential equations is almost oscillatory.

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**Key Words:** oscillation, almostoscillation, $n$-th order differential equation, maxima

## 1. Introduction

We study the $n$-th order differential equation with "maxima"

$$L_n x(t) + \delta f(t, M_t^1(x), \dots, M_t^m x) = 0, \tag{1}$$

where $M_t^j x = \max_{\sigma_j(t) \le s \le \tau_j(t)} x(s)$, $j = 1, \dots, m$. Here $n \ge 2$ is an integer, $\delta = \pm 1$, $t \in J = [\alpha, +\infty) \le R_+ = [0, +\infty)$ and

$$L_0 x(t) = x(t), \quad L_k x(t) = r_k(t)(L_{k-1}x(t))', \quad k = 1, \dots, n.$$

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The domain $D(L_n)$ of $L_n$ is defined to be the set of all functions $x : [t_0, +\infty) \to R$ such that $L_k x(t), k = 1, \dots, n$ exist and are continuous on an interval $[t_0, +\infty) \leq J$. By *proper* solution of inequality (1) is meant a function $x \in D(L_n)$ which satisfies (1) for all sufficiently large $t$ and $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq t_0$. We assume that equation (1) possess proper solutions. A proper solution of inequality (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *non-oscillatory*. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

Equation (1) is said to be *almost oscillatory* if:

(i) For $\delta = 1$ and $n$ even, equation (1) is oscillatory.

(ii) For $\delta = 1$ and $n$ odd, every proper solution $x$ of equation (1) is either oscillatory or $|L_i x(t)| \to 0$ monotonically as $t \to +\infty$, $i = 0, 1, \dots, n-1$.

(iii) For $\delta = -1$ and $n$ even, every proper solution $x$ of equation (1) is oscillatory, $|L_i x(t)| \to 0$ monotonically as $t \to +\infty$, $i = 0, 1, \dots, n-1$ or $|L_i x(t)| \to +\infty$ monotonically as $t \to +\infty$, $i = 0, 1, \dots, n-1$.

(iv) For $\delta = -1$ and $n$ odd, every proper solution $x$ of equation (1) is either oscillatory or $|L_i x(t)| \to +\infty$ monotonically as $t \to +\infty$, $i = 0, 1, \dots, n-1$.

We assume that

$$\int_0^\infty \frac{ds}{r_i(s)} = +\infty \quad \text{and} \quad f(t, x_1, \dots, x_m) \operatorname{sgn} x_1 \geq a(t) \prod_{j=1}^m |x_j|^{\lambda_j}$$

for $i = 1, \dots, n-1$, $t \in J$ and $x_1 x_j > 0$, $j = 1, \dots, m$, where $a \in C(J, R_+)$, $\lambda_j \geq 0$, $j = 1, \dots, m$ and $\lambda = \sum_{j=1}^m \lambda_j > 0$

In this paper sufficient conditions are found under which equation (1) is almost oscillatory. In the main Theorems 1, 2 and 3 the cases $\lambda > 0$, $0 < \lambda < 1$ and $\lambda = 1$ are considered, respectively.

The oscillatory and asymptotic behavior of equation with "maxima" have been considered in Bainov et al [1]-[3], Binggen Zang, Petrov [5], Petrov [8], [9].

Differential equations with "maxima" are often met in the applications, for instance in the theory of automatic control, see [7], [10].

## 2. Preliminary Notes

Introduce the following conditions:

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H1. $r_i \in C(J,(0,+\infty))$, $i = 1,\dots,n-1$, $r_n \equiv 1$ and

$$\int_{\alpha}^{\infty} \frac{ds}{r_i(s)} = +\infty, \quad i = 1,\dots,n-1. \tag{2}$$

H2. $f \in C(J \times R^m, R)$ and there exist a function $a \in C(J, R_+)$ and constants $\lambda_j \geq 0$, $j = 1,\dots,m$ such that $\sum_{j=1}^{m} \lambda_j = \lambda > 0$ and

$$f(t, x_1, \dots, x_m) \operatorname{sgn} x_1 \geq a(t) \prod_{j=1}^{m} |x_j|^{\lambda_j} \tag{3}$$

for $t \in J$ and $x_1 x_j > 0$, $j = 1,\dots,m$.

H3. $\sigma_j, \tau_j \in C(J,R)$, $j = 1,\dots,m$; $\lim_{t\to+\infty} \sigma_j(t) = +\infty$; $\sigma_j(t) \leq \tau_j(t) \leq t$, $j = 1,\dots,m$, $t \in J$. Moreover, we suppose that there exists a number $H \in R$ such that $\tau_j(t) - \sigma_j(t) \leq H$, $j = 1,\dots,m$, $t \in J$.

To formulate our results we use the following notations:

$$I_0 = 1, I_j(t,s;p_j,\dots,p_1) = \int_s^t \frac{1}{p_j(u)} I_{j-1}(u,s;p_{j-1},\dots,p_1)du, \quad j = 1,2,\dots,$$

where $p_j \in C(J,(0,+\infty)), j = 1,2,\dots$.

It is easy to verify that for $j = 1,2,\dots,n-1$

$$I_j(t,s;p_j,\dots,p_1) = (-1)^j I_j(s,t;p_1,\dots,p_j), \tag{4}$$

$$I_j(t,s;p_j,\dots,p_1) = \int_s^t \frac{1}{p_1(u)} I_{j-1}(t,u;p_j,\dots,p_2)du, \tag{5}$$

$$\frac{\partial I_j}{\partial t}(t,s;p_j,\dots,p_1) = \frac{1}{p_j(u)} I_{j-1}(t,s;p_{j-1},\dots,p_1), \tag{6}$$

$$\frac{\partial I_j}{\partial s}(t,s;p_j,\dots,p_1) = -\frac{1}{p_1(u)} I_{j-1}(t,s;p_j,\dots,p_2). \tag{7}$$

We will need the following lemmas:

**Lemma 1.** *Let $p_j \in C(J,(0,+\infty))$, $j = 1,2,\dots$ and $\alpha \leq T < u$. Then:*

*(i) $I_j(u,t;p_1,\dots,p_j) \leq I_1(u,t;p_j) I_{j-1}(u,t;p_1,\dots,p_{j-1})$, $T \leq t < u$;*

*(ii) The function* $\frac{I_j(u,t;p_1,\dots,p_j)}{I_1(u,t;p_j)}$ *is non-increasing in* $t \in [T,u)$;

*(iii)* $\dfrac{I_j(u,T;p_1,\dots,p_j)}{I_1(u,T;p_j)} \geq \dfrac{I_j(u,t;p_1,\dots,p_j)}{I_1(u,t;p_j)}$, $T \leq t < u$;

*(iv)* $I_j(t,T;p_1,\dots,p_j) \geq \dfrac{I_j(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j)$, $T \leq t < u$;

*(v)* $\displaystyle\int_T^t \frac{1}{p_j(s)} I_{j-1}(u,s;p_1,\dots,p_{j-1})ds \geq \frac{I_1(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j)$, $T \leq t < u$.

*Proof.* Using that $I_{j-1}(u,s;p_1,\dots,p_{j-1})$ is non-increasing in $s \in [t,u]$, we obtain (i):

$$I_j(u,t;p_1,\dots,p_j) = \int_t^u \frac{1}{p_j(s)} I_{j-1}(u,s;p_1,\dots,p_{j-1})ds$$
$$\leq I_{j-1}(u,t;p_1,\dots,p_{j-1}) \int_t^u \frac{ds}{p_j(s)} = I_{j-1}(u,t;p_1,\dots,p_{j-1}) I_1(u,t;p_j).$$

Using equality (7) and (i) we obtain the inequality

$$\frac{\partial}{\partial t}\left(\frac{I_j(u,t;p_1,\dots,p_j)}{I_1(u,t;p_j)}\right) \leq 0, \quad T \leq t < u,$$

which proves (ii) and (iii)

It follows from (iii) that

$$I_1(u,t;p_j) I_j(u,T;p_1,\dots,p_j) \geq I_1(u,T;p_j) I_j(u,t;p_1,\dots,p_j), \quad T \leq t < u.$$

Using this we obtain successively

$$[I_1(u,T;p_j) - I_1(t,T;p_j)] I_j(u,T;p_1,\dots,p_j)$$
$$\geq I_1(u,T;p_j) I_j(u,t;p_1,\dots,p_j), \quad T \leq t < u,$$

$$I_1(u,T;p_j)[I_j(u,T;p_1,\dots,p_j) - I_j(u,t;p_1,\dots,p_j)]$$
$$\geq I_1(t,T;p_j) I_j(u,t;p_1,\dots,p_j), \quad T \leq t < u,$$

$$I_j(t,T;p_1,\dots,p_j) \geq \frac{I_1(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j), \quad T \leq t < u,$$

$$\int_T^t \frac{1}{p_j(s)} I_{j-1}(u,s;p_1,\dots,p_{j-1})ds \ge \int_T^t \frac{1}{p_j(s)} I_{j-1}(t,s;p_1,\dots,p_{j-1})ds$$
$$= I_j(t,T;p_1,\dots,p_j) \ge \frac{I_1(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j), \quad T \le t < u,$$

which proves (iv) and (v). □

**Lemma 2.** *If $x \in D(L_n)$, then for $t, s \in J$ and $0 \le i < \nu \le n$:*

*(i)* $$L_i x(t) = \sum_{j=i}^{\nu-1} I_{j-i}(t,s;r_{i+1},\dots,r_j) L_j x(s) + \int_s^t I_{\nu-i-1}(t,u;r_{i+1},\dots,r_{\nu-1}) \frac{L_\nu x(u)}{r_\nu(u)} du;$$

*(ii)* $$L_i x(t) = \sum_{j=i}^{\nu-1} (-1)^{j-i} I_{j-i}(s,t;r_j,\dots,r_{i+1}) L_j x(s) + (-1)^{\nu-i} \int_t^s I_{\nu-i-1}(u,t;r_{\nu-1},\dots,r_{i+1}) \frac{L_\nu x(u)}{r_\nu(u)} du.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus.

**Lemma 3.** *Suppose condition H2 holds and the functions $L_n x$ and $x \in D(L_n)$ are of constant sign and not identically zero for $t \ge t_* \ge \alpha$. Then:*

*(i) There exist a $t_k \ge t_*$ and an integer $k$, $0 \le k \le n$ with $n+k$ even for $x(t)L_n x(t)$ nonnegative or $n+k$ odd for $x(t)L_n x(t)$ non-positive and such that for every $t \ge t_k$*

$$x(t)L_i x(t) > 0, \quad i = 0,1,\dots,k,$$
$$(-1)^{k-i} x(t) L_i x(t) > 0, \quad i = k, k+1, \dots, n.$$

*(ii) The following inequality is valid*

$$\frac{I_\nu(t,t_0;r_{k-\nu+1},\dots,r_k)}{|L_{k-\nu}x(t)|} \ge \frac{I_{\nu+1}(t,t_0;r_{k-\nu},\dots,r_k)}{|L_{k-\nu-1}x(t)|}, \tag{8}$$

*for $t \ge t_0 \ge t_k$ and $\nu = 0,1,\dots,k-1$.*

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*Proof.* This lemma generalizes the well-known lemma of Kiguradze [6] and can be proved similarly. We omit the proof of part (i) and prove part (ii).

Suppose without loss of generality that $x(t) > 0$, $t \geq t_*$. Since $L_k x(t)$ is non-increasing for $t \geq t_k$ and $L_{k-1}x(t_0) > 0$, then

$$L_{k-1}x(t) = L_{k-1}x(t_0) + \int_{t_0}^{t} (L_{k-1}x(s))' ds = \int_{t_0}^{t} \frac{L_k x(s)}{r_k(s)} ds$$

$$\geq \int_{t_0}^{t} \frac{ds}{r_k(s)} L_k x(t) = I_1(t, t_0; r_k) L_k x(t), \quad t \geq t_0.$$

We proved that (8) is true for $\nu = 0$

Suppose that (8) is true for $\nu$, $0 \leq \nu \leq j < k-1$. Then

$$\frac{I_j(t, t_0; r_{k-j+1}, \ldots, r_k)}{L_{k-j}x(t)} \geq \frac{I_{j+1}(t, t_0; r_{k-j}, \ldots, r_k)}{L_{k-j-1}x(t)}, \quad t \geq t_0. \tag{9}$$

We prove that (8) is true for $\nu = j+1$, that is,

$$f(t) = \frac{I_{j+1}(t, t_0; r_{k-j}, \ldots, r_k)}{L_{k-j-1}x(t)} \geq \frac{I_{j+2}(t, t_0; r_{k-j-1}, \ldots, r_k)}{L_{k-j-2}x(t)} = g(t) \tag{10}$$

for $t \geq t_0$

We have that $f(t_0) = g(t_0)$. We prove that (10) holds for each $t \geq t_0$. Assume the opposite, that $f(t_1) < g(t_1)$ for $t_1 > t_0$. Then there exists $T \in [t_0, t_1)$ such that

$$f(T) = g(T) \text{ and } f(t) < g(t), \quad T < t \leq t_1. \tag{11}$$

Since

$$f'(t) = \frac{I_j(t, t_0; r_{k-j+1}, \ldots, r_k) L_{k-j-1}x(t)}{r_{k-j}(t)(L_{k-j-1}x(t))^2} - \frac{I_{j+1}(t, t_0; r_{k-j}, \ldots, r_k) L_{k-j}x(t)}{r_{k-j}(t)(L_{k-j-1}x(t))^2},$$

$$g'(t) = \frac{I_{j+1}(t, t_0; r_{k-j}, \ldots, r_k) L_{k-j-2}x(t)}{r_{k-j-1}(t)(L_{k-j-2}x(t))^2} - \frac{I_{j+2}(t, t_0; r_{k-j-1}, \ldots, r_k) L_{k-j-1}x(t)}{r_{k-j-1}(t)(L_{k-j-2}x(t))^2},$$

then it follows from (9) and (11) that

$$f'(t) \geq 0 > g'(t), \quad T < t \leq t_1.$$

This implies $f(t) > g(t)$, $T < t \leq t_1$ which contradicts (11). $\square$

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**Remark 1.** From inequality (8) with $\nu = 0, 1, \ldots, k-1$ it follows

$$|L_{k-1}x(t)| \geq I_1(t, t_0; r_k)|L_k x(t)|, \quad t \geq t_0 \geq t_k, \tag{12}$$

$$|x(t)| \geq I_k(t, t_0; r_1, \ldots, r_k)|L_k x(t)|, \quad t \geq t_0 \geq t_k, \tag{13}$$

$$|x(t)| \geq \frac{I_k(t, t_0; r_1, \ldots, r_k)}{I_1(t, t_0; r_k)}|L_{k-1}x(t)|, \quad t > t_0 \geq t_k. \tag{14}$$

For every $T \geq \alpha$ and $t \geq s \geq T$ we set

$$R_j(t, T) = I_1(t, T; r_j) = \int_T^t \frac{ds}{r_j(s)}, \quad j = 1, 2, \ldots, n-1,$$

$$\alpha_j(t, s) = I_j(t, s; r_1, \ldots, r_j), \quad j = 1, 2, \ldots, n-1,$$

$$\beta_j(t, s) = I_{n-j-1}(t, s; r_{n-1}, \ldots, r_{j+1}), \quad j = 0, 1, \ldots, n-1,$$

$$\gamma_i(\sigma_j(t), T) = \int_T^{\sigma_j(t)} \alpha_{i-2}(\sigma_j(t), s)\frac{R_i(s, T)}{r_{i-1}(s)}ds, i = 2, \ldots, n-1, j = 1, \ldots, m$$

and

$$\gamma_1(\sigma_j(t), T) = R_1(\sigma_j(t), T), \quad j = 1, 2, \ldots, m.$$

## 3. Main Results

### 3.1. Case $\lambda > 1$

**Theorem 1.** *Let conditions H1-H3 hold and $\lambda > 1$. Then a sufficient condition for equation (1) to be almost oscillatory is that:*

*(i) When $\delta = 1$ and $n$ is even, for every $k \in \{1, 3, \ldots, n-1\}$ and all sufficiently large $T$*

$$\int_T^\infty \frac{1}{r_k(t)} \int_t^\infty \beta_k(u, t)a(u) \prod_{j=1}^m \left(\frac{\gamma_k(\sigma_j(u), T)}{R_k(u, T)}\right)^{\lambda_j} dudt = +\infty. \tag{15; k}$$

*(ii) When $\delta = 1$ and $n$ is odd, condition (15, k), $k \in \{2, 4, \ldots, n-1\}$ hold and for all sufficiently large $T$*

$$\int^\infty \beta_0(s, T)a(s)ds = +\infty. \tag{16}$$

*(iii) When $\delta = -1$ and $n$ is even, conditions (15;k), $k \in \{2, 4, \dots, n-2\}$ and (16) hold and for all sufficiently large $T$*

$$\int^{\infty} a(s) \prod_{j=1}^{m} (\alpha_{n-1}(\sigma_j(s), T))^{\lambda_i} ds = +\infty. \tag{17}$$

*(iv) When $\delta = -1$ and $n$ is odd, conditions (15;k), $k \in \{1, 3, \dots, n-2\}$ and (17) hold.*

*Proof.* Let $x(t)$ be a non-oscillatory solution of equation (1). Without loss of generality we assume that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists $t_1 \geq t_0$ : $x(\sigma_j(t)) > 0$ for $t \geq t_1$, $j = 1, 2. \dots .m$.

By Lemma 3 (i) there exist $t_k \geq t_1$ and $k \in \{0, 1, \dots, n-1\}$ with $n + k$ odd if $\delta = 1$ and $n + k$ even if $\delta = -1$ such that

$$\begin{gathered} L_i x(t) > 0, \quad i = 0, \dots, k, \quad t \geq t_k, \\ (-1)^{i-k} L_i x(t) > 0, \quad i = k, k+1 \dots, n, \quad t \geq t_k. \end{gathered} \tag{18}$$

From Lemma 2 (ii) we have

$$\begin{aligned} L_k x(t) = \sum_{j=k}^{n-1} & (-1)^{j-k} I_{j-k}(s, t; r_j, \dots, r_{k+1}) L_j x(s) \\ & + (-1)^{n-k} \int_t^s I_{n-k-1}(u, t; r_{n-1}, \dots, r_{k+1}) L_n x(u) du, \quad t \leq s. \end{aligned}$$

Using (3), (18) and letting $s \to +\infty$ we obtain

$$L_k x(t) \geq \int_t^{\infty} \beta_k(u, t) a(u) \prod_{j=1}^{m} |M_u^j x|^{\lambda_j} du, \quad t \geq t_k. \tag{19}$$

Notice that if $k \geq 1$, then $x(t)$ is increasing for $t \geq t_k$ and $M_t^j x \geq x(\sigma_j(t))$ for $t \geq t_2$, where $t_2$ is such that $\sigma_j(t) \geq t_k$, $t \geq t_2$, $j = 1, \dots, m$. In this case

$$L_k x(t) \geq \int_t^{\infty} \beta_k(u, t) a(u) \prod_{j=1}^{m} (x(\sigma_j(u)))^{\lambda_j} du, \quad t \geq t_2. \tag{20}$$

*Case 1.* $k \geq 2$. From Lemma 2 (i) we obtain

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$$x(t) = \sum_{j=0}^{k-2} Ij(t, t_k; r_1, \dots, r_j) L_j x(t_k)$$
$$+ \int_{t_k}^{t} I_{k-2}(t, u; r_1, \dots, r_{k-2}) \frac{L_{k-1}x(u)}{r_{k-1}(u)} du \geq \int_{t_k}^{t} \alpha_{k-2}(t, u) \frac{L_{k-1}x(u)}{r_{k-1}(u)} du,$$
$$t \geq t_k.$$

Then

$$x(\sigma_j(t)) \quad \geq \quad \int_{t_k}^{\sigma_j(t)} \alpha_{k-2}(\sigma_j(t), u) \frac{L_{k-1}x(u)}{r_{k-1}(u)} du, \quad j \quad = \quad 1, \dots, m, \quad t \quad \geq \quad t_2. \quad (21)$$

Since (12) implies that $\frac{L_{k-1}x(u)}{R_k(u,t_k)}$ is a non-increasing for $u \geq t_k$, then it follows from (21) that

$$x(\sigma_j(t)) \geq \frac{L_{k-1}x(t)}{R_k(t, t_k)} \int_{t_k}^{\sigma_j(t)} \alpha_{k-2}(\sigma_j(t), u) \frac{R_k(u, t_k)}{r_{k-1}(u)} du, \quad t \geq t_2,$$

that is

$$x(\sigma_j(t)) \geq \gamma_k(\sigma_j(t), t_k) \frac{L_{k-1}x(t)}{R_k(t, t_k)}, \quad j = 1, \dots, m, \quad t \geq t_2. \quad (22)$$

From (20) and (22) we conclude that

$$\frac{(L_{k-1}x(t))'}{(L_{k-1}x(t))^{\lambda}} \geq \frac{1}{r_k(t)} \int_{t}^{\infty} \beta_k(u, t) a(u) \prod_{j=1}^{m} \left( \frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} du, \quad t \geq t_2.$$

Integrating this inequality from $t_2$ to $\tau \geq t_2$ we obtain

$$\int_{t_2}^{\tau} \frac{1}{r_k(t)} \int_{t}^{\infty} \beta_k(u, t) a(u) \prod_{j=1}^{m} \left( \frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} dudt \;\leq\; \int_{L_{k-1}x(t_2)}^{L_{k-1}x(\tau)} w^{-\lambda} dw \;<\; +\infty,$$

which contradicts (15).

*Case 2.* $k = 1$. Then $\delta = 1$ and $n$ is even or $\delta = -1$ and $n$ is odd. It follows from (20) that

$$x'(t) \geq \frac{1}{r_1(t)} \int_{t}^{\infty} \beta_1(u, t) a(u) \prod_{j=1}^{m} (x(\sigma_j(u)))^{\lambda_j} du, \quad t \geq t_2. \quad (23)$$

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Since $\frac{x(t)}{R_1(t,t_k)}$ is non-increasing for $t \geq t_k$ and $\sigma_j(u) \leq u$ we have

$$x(\sigma_j(u)) \geq \frac{R_1(\sigma_j(u), t_k)}{R_1(u, t_k)} x(t), \quad u \geq t \geq t_2$$

and then we conclude from (23) that

$$\frac{x'(t)}{(x(t))^\lambda} \geq \frac{1}{r_1(t)} \int_t^\infty \beta_1(u,t)a(u) \prod_{j=1}^m \left( \frac{\gamma_1(\sigma_j(u), t_k)}{R_1(u, t_k)} \right)^{\lambda_j} du, \quad t \geq t_2.$$

This implies the inequality

$$\int_{t_2}^\infty \frac{1}{r_1(t)} \int_t^\infty \beta_1(u,t)a(u) \prod_{j=1}^m \left( \frac{\gamma_1(\sigma_j(u), t_k)}{R_1(u, t_k)} \right)^{\lambda_j} dudt < +\infty$$

which contradicts condition (15;$k$) for $k = 1$.

*Case 3.* $k = 0$. Then $\delta = 1$ and $n$ is odd or $\delta = -1$ and $n$ is even. It follows from (19) with $k = 0$

$$x(t) \geq \int_t^\infty \beta_0(u,t)a(u) \prod_{j=1}^m (M_u^j x)^{\lambda_j} du, \quad t \geq t_k. \tag{24}$$

Since $x(t)$ is decreasing and positive for $t \geq t_k$, there exists $\lim_{t\to+\infty} x(t) = c \geq 0$. If $c > 0$, then there exists $t_2 \geq t_k$ such that $2c \geq M_u^j x \geq c$, $t \geq t_2$, $j = 1, \ldots, m$. Then (24) implies the inequality

$$2c \geq x(t_2) \geq \int_{t_2}^\infty \beta_0(u, t_2)a(u)du.c^\lambda$$

which contradicts (16). Hence $c = 0$.

*Case 4.* $k = n$. Then $\delta = -1$ and $n$ is either even or odd. From (18) we have

$$L_i x(t) > 0, \quad i = 0, 1, \ldots, n, \quad t \geq t_k.$$

Furthermore, by l'Hôpital's rule,

$$\lim_{t\to+\infty} \frac{x(t)}{\alpha_{n-1}(t, t_k)} = \lim_{t\to+\infty} L_{n-1}x(t) > 0.$$

Since $\sigma_j(t) \to +\infty$ as $t \to +\infty$ $(j = 1, \ldots, m)$, there exist a constant $c > 0$ and $t_2 \geq t_k$ such that

$$x(\sigma_j(t)) \geq c\alpha_{n-1}(\sigma_j(t), t_k), \quad j = 1, \ldots, m, \quad t \geq t_2. \tag{25}$$

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Integrating equation (1) for $\delta = -1$ from $t_2$ to $t$ and using (3) and (25) we obtain

$$L_{n-1}x(t) \geq L_{n-1}x(t_2) + \int_{t_2}^{t} f(s, M_s^1 x, \ldots, M_s^m x)ds$$
$$\geq \int_{t_2}^{t} a(s) \prod_{j=1}^{m} (x(\sigma_j(s)))^{\lambda_j} ds \geq c^{\lambda} \int_{t_2}^{t} a(s) \prod_{j=1}^{m} (\alpha_{n-1}(\sigma_j(s), t_k))^{\lambda_j} ds.$$

Therefore (17) implies

$$\lim_{t\to+\infty} L_{n-1}x(t) = +\infty$$

and hence $\lim_{t\to+\infty} L_i x(t) = +\infty$, $i = 0, 1, \ldots, n-1$ monotonically. □

Consider the equation

$$x^{(n)}(t) + \delta f(t, \max_{\sigma_1(t)\leq s\leq \tau_1(t)} x(s), \ldots, \max_{\sigma_m(t)\leq s\leq \tau_m(t)} x(s)) = 0, \tag{26}$$

which is a particular case of equation (1) with $r_1 = r_2 = \cdots = r_n = 1$.

**Corollary 1.** *Let conditions H2 and H3 hold and $\lambda > 1$. Then a sufficient condition for equation (26) to be almost oscillatory is that:*

*(i) When $\delta = 1$ and $n$ is even, for every $k \in \{1, 3, \ldots, n-1\}$*

$$\int^{\infty} u^{n-k-\lambda} a(u) \prod_{j=1}^{m} (\sigma_j(u))^{k\lambda_j} du = +\infty. \tag{27;$k$}$$

*(ii) When $\delta = 1$ and $n$ is odd, conditions (27;k) for $k \in \{2, 4, \ldots, n-1\}$ hold and*

$$\int^{\infty} s^{n-1} a(s) ds = +\infty. \tag{28}$$

*(iii) When $\delta = -1$ and $n$ is even, conditions (27;k), $k \in \{2, 4, \ldots, n-2\}$ and (28) hold and*

$$\int^{\infty} a(s) \prod_{j=1}^{m} (\sigma_j(s))^{(n-1)\lambda_j} ds = +\infty. \tag{29}$$

*(iv) When $\delta = -1$ and $n$ is odd, conditions (27;k), $k \in \{1, 3, \ldots, n-2\}$ and (29) hold.*

### 3.2. Case $0 < \lambda < 1$

**Theorem 2.** *Let conditions H1-H3 hold and $0 < \lambda < 1$. Then a sufficient condition for equation (1) to be almost oscillatory is that*

*(i) When $\delta = 1$ and $n$ is even*

$$\int_{\alpha}^{\infty} a(u) \prod_{j=1}^{m} (\gamma_{n-1}(\sigma_j(u), T))^{\lambda_j} du = +\infty \tag{30}$$

*and*

$$\int_{T}^{\infty} \frac{(R_k(t,T))^{\lambda}}{r_{k+1}(t)} \int_{t}^{\infty} \beta_{k+1}(u,t) a(u) \prod_{j=1}^{m} \left( \frac{\gamma_k(\sigma_j(u), T)}{R_k(u,T)} \right)^{\lambda_j} du dt = +\infty, \tag{31;k}$$

*for all sufficiently large $T$ and $k \in \{1, 3, \dots, n-3\}$.*

*(ii) When $\delta = 1$ and $n$ is odd, conditions (31;k), $k \in \{2, 4, \dots, n-3\}$, (16) and (30) hold.*

*(iii) When $\delta = -1$ and $n$ is even, condition (31;k), $k \in \{2, 4, \dots, n-2\}$, (16) and (17) hold.*

*(iv) When $\delta = -1$ and $n$ is odd, conditions (31;k), $k \in \{1, 3, \dots, n-2\}$ and (17) hold.*

*Proof.* Let $x(t)$ be a non-oscillatory solution of equation (1). Assume without loss of generality that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists $t_1 \geq t_0$ such that $x(\sigma_j(t)) > 0$ for $t \geq t_1$, $j = 1, \dots, m$. As in the proof of Theorem 1, we conclude that there exist $t_k \geq t_1$ and $k \in \{0, 1, \dots, n\}$ with $n + k$ odd if $\delta = 1$ or $n + k$ even if $\delta = -1$ such that (18) holds.

*Case 1.* $k \in \{1, \dots, n-2\}$. From Lemma 2 (ii) we have

$$L_{k+1}x(t) = \sum_{j=k+1}^{n-1} (-1)^{j-k-1} I_{j-k-1}(s,t; r_j, \dots, r_{k+2}) L_j x(s) + (-1)^{n-k-1} \int_{t}^{s} I_{n-k-2}(u,t; r_{n-1}, \dots, r_{k+2}) L_n x(u) du$$

for $s \geq t \geq t_k$. Using (1), (18), (3) and letting $s \to +\infty$ we get

$$-L_{k+1}x(t) \geq \int_{t}^{\infty} \beta_{k+1}(u,t) a(u) \prod_{j=1}^{m} (x(\sigma_j(u)))^{\lambda_j} du, \quad t \geq t_k.$$

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Taking in view (22) we obtain

$$-L_{k+1}x(t) \geq (L_{k-1}x(t))^{\lambda} \int_t^{\infty} \beta_{k+1}(u,t)a(u) \prod_{j=1}^{m} \left( \frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u,t_k)} \right)^{\lambda_j} du$$

for $t \geq t_2 \geq t_k$. Then using (12) we conclude

$$-\frac{(L_k x(t))'}{(L_k x(t))^{\lambda}} \geq \frac{(R_k(t,t_k))^{\lambda}}{r_{k+1}(t)} \int_t^{\infty} \beta_{k+1}(u,t)a(u) \prod_{j=1}^{m} \left( \frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u,t_k)} \right)^{\lambda_j} du, \quad t \geq t_2$$

Integrating this inequality from $t_2$ to $\tau \geq t_2$ we obtain

$$\int_{t_2}^{\tau} \frac{(R_k(t,t_k))^{\lambda}}{r_{k+1}(t)} \int_t^{\infty} \beta_{k+1}(u,t)a(u) \prod_{j=1}^{m} \left( \frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u,t_k)} \right)^{\lambda_j} dudt$$

$$\leq \int_{L_k x(\tau)}^{L_k x(t_2)} w^{-\lambda} dw < +\infty,$$

which contradicts (31).

*Case 2.* $k = n-1$. Then $\delta = 1$ and either $n$ is odd or even. From (1) and (22) we have

$$-L_n x(t) \geq a(t) \prod_{j=1}^{m} \left( \frac{\gamma_{n-1}(\sigma_j(t), t_k)}{R_{n-1}(t,t_k)} L_{n-2}x(t) \right)^{\lambda_j}, \quad t \geq t_2,$$

and applying (12) with $k = n-1$ we get

$$-\frac{L_n x(t)}{(L_{n-1}x(t))^{\lambda}} \geq a(t) \prod_{j=1}^{m} (\gamma_{n-1}(\sigma_j(t), t_k))^{\lambda_j}, \quad t \geq t_2.$$

Integrating the above inequality from $t_2$ to $t \geq t_2$ we obtain

$$\int_{t_2}^{t} a(u) \prod_{j=1}^{m} (\gamma_{n-1}(\sigma_j(u), t_k))^{\lambda_j} du \leq \int_{L_{n-1}x(t)}^{L_{n-1}x(t_2)} w^{-\lambda} dw < +\infty$$

which contradicts (30).

In the cases $k = 0$ and $k = n$ the proof is the same as the proof for these cases in Theorem 1 and we omit it here. □

**Corollary 2.** *Let conditions H2 and H3 hold and* $0 < \lambda < 1$. *Then the sufficient condition for equation (26) to be almost oscillatory is that:*

*(i) When $\delta = 1$ and $n$ is even, for every $k \in \{1, 3, \dots, n-1\}$*

$$\int^{\infty} a(u) u^{n-k-1} \prod_{j=1}^{m} (\sigma_j(u))^{k\lambda_j} du = +\infty. \tag{32; k}$$

*(ii) When $\delta = 1$ and $n$ is odd, conditions (32;k), $k \in \{2, 4, \dots, n-1\}$ and (28) hold.*

*(iii) When $\delta = -1$ and $n$ is even, conditions (32;k), $k \in \{2, 4, \dots, n-2\}$, (28) and (29) hold.*

*(iv) When $\delta = -1$ and $n$ is odd, conditions (32;k), $k \in \{1, 3, \dots, n-2\}$ and (29) hold.*

### 3.3. Case $\lambda = 1$

**Theorem 3.** *Let conditions H1-H3 hold and $\lambda = 1$. Then a sufficient condition for equation (1) to be almost oscillatory is that:*

*(i) When $\delta = 1$ and $n$ is even, for every $k \in \{1, 3, \dots, n-1\}$ and some $T_0 \geq \alpha$*

$$\limsup_{t \to +\infty} \Big\{ \frac{1}{R_k(t, \alpha)} \int_{T_0}^{t} \beta_{k-1}(u, \alpha) a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u), \alpha))^{\lambda_j} du + R_k(t, \alpha) \int_{t}^{\infty} \frac{\beta_{k-1}(t, \alpha)}{R_k(u, \alpha)} a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u), \alpha))^{\lambda_j} du \Big\} > 1. \tag{33;k}$$

*(ii) When $\delta = 1$ and $n$ is odd, conditions (33;k), $k \in \{2, 4, \dots, n-1\}$ hold and*

$$\int^{\infty} \beta_0(s, \alpha) a(s) ds = +\infty. \tag{34}$$

*(iii) When $\delta = -1$ and $n$ is even , conditions (33;k), $k \in \{2, 4, \dots, n-2\}$ and (34) hold and*

$$\int^{\infty} a(s) \prod_{j=1}^{m} (\alpha_{n-1}(\sigma_j(s), \alpha))^{\lambda_j} ds = +\infty. \tag{35}$$

*(iv) When $\delta = -1$ and $n$ is odd, conditions (33;k), $k \in \{1, 3, \dots, n-2\}$ and (35) hold.*

*Proof.* Let $x(t)$ be a non-oscillatory solution of equation (1). Assume without loss of generality that $x(t)$ is eventually positive: $x(t) > 0$, $M_t^j x > 0$, $x(\sigma_j(t)) > 0$,

$t \geq t_1 \geq \alpha$, $j = 1, \dots, m$. By Lemma 3 (i) there exist $t_k \geq t_1$ and $k \in \{0, 1, \dots, n\}$ with $n + k$ odd if $\delta = 1$ and $n + k$ even if $\delta = -1$ such that (18) holds. Applying Lemma 2 (ii) with $i = k$ taking into account (18), (1) and (3) we get

$$L_k x(t) \geq \int_t^\infty \beta_k(u,t) a(u) \prod_{j=1}^m (M_u^j x)^{\lambda_j} du, \quad t \geq t_k.$$

*Case 1.* $k \in \{1, 2, \dots, n-1\}$. Then $x(t)$ is non-decreasing and $M_u^j x \geq x(\sigma_j(u))$, $u \geq t_k$. Therefore

$$L_k x(t) \geq \int_t^\infty \beta_k(u,t) a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} du, \quad t \geq t_k.$$

Integrating this inequality from $t_k$ to $t \geq t_k$ we have

$$L_{k-1} x(t) \geq L_{k-1} x(t_k) + \int_{t_k}^t \frac{1}{r_k(s)} \int_s^\infty \beta_k(u,s) a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} du ds, \quad t \geq t_k.$$

Keeping in mind (18) and changing the order of integration we obtain

$$L_{k-1} x(t) \geq \int_{t_k}^t a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} \int_{t_k}^u \frac{\beta_k(u,s)}{r_k(s)} ds du + \int_t^\infty a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} \int_{t_k}^t \frac{\beta_k(u,s)}{r_k(s)} ds du, \quad t \geq t_k. \quad (36)$$

From (5) and Lemma 1 (v) it follows

$$\int_{t_k}^u \frac{\beta_k(u,s)}{r_k(s)} ds = \beta_{k-1}(u, t_k), \quad t \geq u \geq t_k,$$

and

$$\int_{t_k}^t \frac{\beta_k(u,s)}{r_k(s)} ds \geq \frac{R_k(t,t_k)}{R_k(u,t_k)} \beta_{k-1}(u, t_k), \quad t_k \leq t < u.$$

This together with (36) implies

$$L_{k-1} x(t) \geq \int_{t_k}^t \beta_{k-1}(u, t_k) a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} du$$

$$+ R_k(t,t_k)\int_{t_k}^{\infty}\frac{\beta_{k-1}(u,t_k)}{R_k(u,t_k)}a(u)\prod_{j=1}^{m}(x(\sigma_j(u)))^{\lambda_j}du, \quad t \geq t_k. \quad (37)$$

Due to condition H3 there exists $t_2 \geq t_k$ such that $\sigma_j(u) \geq t_k$ for $t \geq t_2$, $j = 1,\dots,m$. Therefore from (14) we have

$$x(\sigma_j(u)) \geq \frac{\alpha_k(\sigma_j(u),t_k)}{R_k(\sigma_j(u),t_k)}L_{k-1}x(\sigma_j(u)), \quad u \geq t_2. \quad (38)$$

Since $\frac{L_{k-1}x(t)}{R_k(t,t_k)}$ is non-increasing and $L_{k-1}x(t)$ is nondecreasing for $t \geq t_k$, we have

$$L_{k-1}x(\sigma_j(u)) \geq \frac{R_k(\sigma_j(u),t_k)}{R_k(t,t_k)}L_{k-1}x(t), \quad t > u \geq t_2, \quad (39)$$

and

$$L_{k-1}x(\sigma_j(u)) \geq \frac{R_k(\sigma_j(u),t_k)}{R_k(u,t_k)}L_{k-1}x(t), \quad u > t \geq t_2.$$

Now we conclude from (37)-(39) that

$$1 \geq \frac{1}{(R_k(t,t_k))^2}\int_{T_1}^{t}\beta_{k-1}(u,t_k)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),t_k))^{\lambda_j}du$$
$$+ R_k(t,t_k)\int_{t}^{\infty}\frac{\beta_{k-1}(u,t_k)}{(R_k(u,t_k))^2}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),t_k))^{\lambda_j}du \quad (40)$$

for $t \geq T_1 > \max(t_2,T_0)$.

From (33) it follows that there exists $\eta \in (0,1)$ such that

$$\eta \limsup_{t\to+\infty}\{\frac{1}{R_k(t,\alpha)}\int_{T_0}^{t}\beta_{k-1}(u,\alpha)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}du$$
$$+ R_k(t,\alpha)\int_{t}^{\infty}\frac{\beta_{k-1}(u,\alpha)}{(R_k(u,\alpha))^2}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}du\} > 1. \quad (41)$$

Set $\mu = \eta^{\frac{1}{3}}$. Since $\mu \in (0,1)$ and the functions $\sigma_j(t)$, $R_k(t,t_k)$, $\alpha_k(t,t_k)$ and $\beta_{k-1}(t,t_k)$ tend to infinity as $t \to +\infty$, there exists $t_\mu \geq t_2$ such that

$$R_k(u,\alpha) \geq R_k(u,t_k) \geq \mu R_k(t,\alpha), \quad u \geq t_\mu,$$

$$\alpha_k(\sigma_j(u),t_k) \geq \mu\alpha_k(\sigma_j(u),\alpha), \quad j = 1,\dots,m, \quad u \geq t_\mu, \quad (42)$$

$$\beta_{k-1}(u,t_k) \geq \mu\beta_{k-1}(u,\alpha), \quad u \geq t_\mu$$

Keeping in mind (42) and (40) we obtain

$$1 \geq \eta\Big\{\frac{1}{R_k(t,\alpha)}\int_{T_1}^{t}\beta_{k-1}(u,\alpha)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du + R_k(t,\alpha)\int_{t}^{\infty}\frac{\beta_{k-1}(u,\alpha)}{(R_k(u,\alpha))^2}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du\Big\}, \quad t \geq T_1. \quad (43)$$

Taking into account (43) and the equality

$$\lim_{t\to+\infty}\frac{1}{R_k(t,\alpha)}\int_{T_0}^{T_1}\beta_{k-1}(u,\alpha)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du = 0,$$

we obtain the inequality

$$1 \geq \eta\limsup_{t\to+\infty}\Big\{\frac{1}{R_k(t,\alpha)}\int_{T_0}^{t}\beta_{k-1}(u,\alpha)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du + R_k(t,\alpha)\int_{t}^{\infty}\frac{\beta_{k-1}(u,\alpha)}{(R_k(u,\alpha))^2}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du\Big\},$$

which contradicts (41).

In the cases $k = 0$ and $k = n$ the proof is the same as the proof of Theorem 1. □

**Corollary 3.** *Let conditions H2 and H3 hold and $\lambda = 1$. Then a sufficient condition for equation (26) to be almost oscillatory is that:*

*(i) When $\delta = 1$ and $n$ is even, for every $k \in \{1,3,\ldots,n-1\}$ and some $T_0 \geq \alpha$*

$$\limsup_{t\to+\infty}\Big\{\frac{1}{t}\int_{T_0}^{t}\frac{u^{n-k}}{(n-k)!k!}a(u)\prod_{j=1}^{m}(\sigma_j(u))^{k\lambda_j}\,du + t\int_{t}^{\infty}\frac{u^{n-k-2}}{(n-k)!k!}a(u)\prod_{j=1}^{m}(\sigma_j(u))^{k\lambda_j}\,du\Big\} > 1. \quad (44;k)$$

*(ii) When $\delta = 1$ and $n$ is odd, conditions (44;k), $k \in \{2,4,\ldots,n-1\}$ and (28) hold.*

*(iii) When $\delta = -1$ and $n$ is even, conditions (44;k), $k \in \{2, 4, \ldots, n-2\}$, (28) and (29) hold.*

*(iv) When $\delta = -1$ and $n$ is odd, conditions (44;k), $k \in \{1, 3, \ldots, n-2\}$ and (29) hold.*

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