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IDENTIFY FULLY-UNCERTAIN PARAMETERS AND DESIGN CONTROLLERS BASED ON SYNCHRONIZATION FOR CHAOTIC FRACTIONAL-ORDER UNIFIED SYSTEMS

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Abstract: Synchronization in chaotic fractional-order differential systems is studied both theoretically and numerically. The controller is designed to achieve chaos synchronization of so-called unified chaotic systems. Some sufficient conditions on synchronization are also derived based on the Laplace transformation theory. Computer simulations are used for demonstration.

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Key Words: chaos, parameter identification, chaos synchronization, linear feedback, fractional-order

1. Introduction

It has been found that many systems in interdisciplinary fields can be described by fractional differential equations, for example, viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves, see [11], [6]. Studying fractional-order differential systems is becoming an active research field.

There are material differences in many aspects, e.g. qualitative properties, between ordinary differential equation (ODE) systems and fractional-order differential systems. That most of properties or conclusions of the ODE systems cannot be simply extend to the case of the fractional-order differential systems is a known fact, see [2], [7]. Now, a problem we ask is when an ODE system is chaotic, under what condition the corresponding fractional-order system is also chaotic. More exactly,

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for what orders, the fractional-order system is chaotic. Furthermore, given a chaotic fractional-order differential system, how to design a scheme such that synchronization of many such systems is achieved.

The latter problem even in the ODE case has not yet been thoroughly solved since feasibility of a designed synchronization scheme depends heavily on the considered single system. There is no doubt that studying a fractional-order system is much more difficult than doing an ODE system whatever in either theoretical analysis or numerical simulation. Therefore, designing synchronization schemes to fractional-order differential systems is a more challenging task, which is the main concern of this paper.

It is known that some fractional-order differential systems behave chaotically, e.g., the fractional Duffing system [1], the fractional Chua system [5], the fractional Chen system [8], the fractional Lu system [4]. These fractional-order systems can be also synchronized [1]-[4]. In this paper, we further investigate the synchronization problem of chaotic fractional-order differential systems both analytically and numerically. By taking the unified chaotic system (see [9]) as an example, we design controller to realize synchronization of two identical unified systems. In fact, our design scheme and numerical simulation can been easily extended to other similar fractional-order differential systems.

2. Definitions and Systems

There are several definitions of a fractional-order differential system. In the following, we introduce the most common one of them:

$$D_*^\alpha x(t) = J^{m-\alpha} x(t)^{(m)}, \quad \text{with } \alpha > 0, \tag{1}$$

with $m = [\alpha]$, i.e., m is the first integer which is not less than α, x^m is the m-order derivative in the usual sense, and $J^\beta (\beta > 0)$ is the β-order Reimann-Liouville integral operator with expression

$$J^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{(\beta-1)} y(\tau) d\tau. \tag{2}$$

Here τ stands for Gamma function, and the operator D_*^α is generally called "α-order Caputo differential operator", see [3].

Lü and Chen [9] introduced a chaotic system of three-dimensional quadratic autonomous ordinary differential equations, which can simultaneously display two 1-scroll chaotic attractor with only three equilibria, and two 2-scroll chaotic attractors with five equilibria. This system is described as:

$$\begin{cases} \dot{x} = -\dfrac{ab}{a+b} x - yz + c, \\ \dot{y} = ay + xz, \\ \dot{z} = bz + xy, \end{cases} \tag{3}$$

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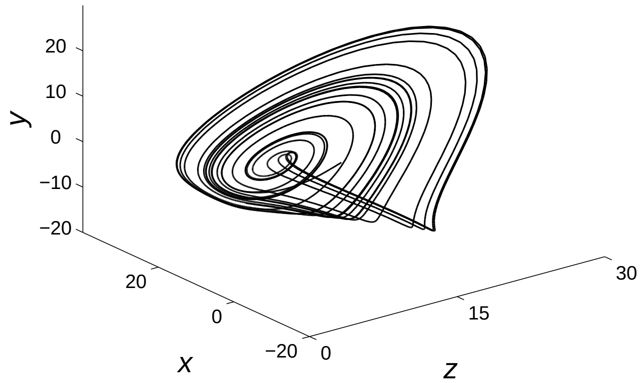

Figure 1: The attractor for $q = (0.99, 0.98, 0.995)$ and $a = -10$, $b = -4$ and $c = 29.9$

where a, b, c are real constants and x, y, z are state variables.

Now, let us introduce its fractional version as follows:

$$\begin{cases} \dfrac{d^{q_1}x}{dt^{q_1}} = -\dfrac{ab}{a+b}x - yz + c, \\ \dfrac{d^{q_2}y}{dt^{q_2}} = ay + xz, \\ \dfrac{d^{q_3}z}{dt^{q_3}} = bz + xy, \end{cases} \tag{4}$$

where $\frac{d^{q_i}}{dt^{q_i}} = D_*^{q_i}$ $(i = 1, 2, 3)$, its order denoted by $q = (q_1, q_2, q_3)$ is subject to $0 < q_1, q_2, q_3 \leq 1$ and $a, b, c,$ and in particular we have found that for a set of parameter values: $a = -10$, $b = -4$ and $|c| < 40$ and $q = (0.99, 0.98, 0.995)$ the fractional-order unified system can display chaotic attractors as shown in Figures 1-4.

3. Update Law of Parameters Identification and Design of Controllers

In this section, we will design controllers and propose an update law for the time evolution of unknown parameters by taking the system (5) as an example.

In order to observe the synchronization behavior in two identical fractional-order unified chaotic systems, we build a PC drive-response configuration with a

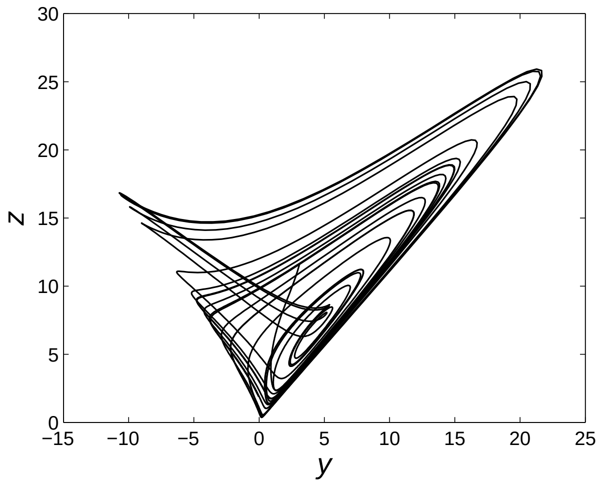

Figure 2: The $z-y$ projection of the attractor for $q=(0.99, 0.98, 0.995)$ and $a=-10$, $b=-4$ and $c=29.9$

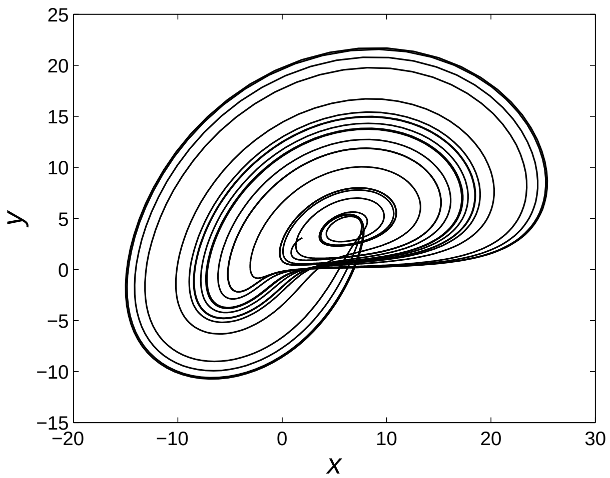

Figure 3: The $x-y$ projection of the attractor for $q=(0.99, 0.98, 0.995)$ and $a=-10$, $b=-4$ and $c=29.9$

drive system given by the fractional-order unified system (with three state variables denoted by the subscript 1) and with a response system given by the identify system.

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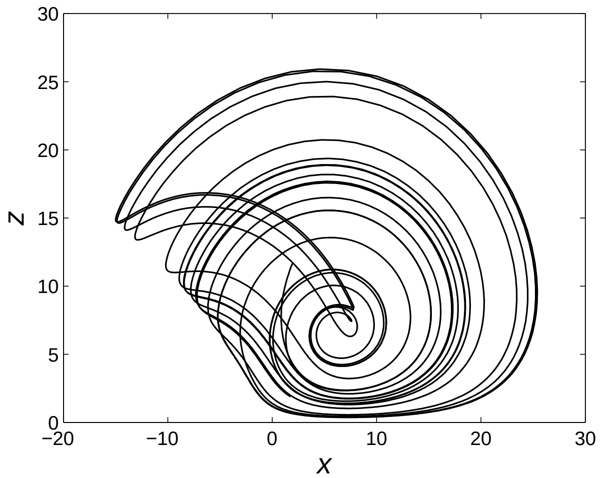

Figure 4: The $x - z$ projection of the attractor for $q = (0.99, 0.98, 0.995)$ and $a = -10$, $b = -4$ and $c = 29.9$

We will use the chaotic signal to drive the response system whose variables are denoted by subscript 2. The drive and response system are described by the following differential equations, respectively:

$$\begin{cases} \dfrac{d^{q_1}x_1}{dt^{q_1}} = -\dfrac{ab}{a+b}x_1 - y_1z_1 + c, \\ \dfrac{d^{q_2}y_1}{dt^{q_2}} = ay_1 + x_1z_1, \\ \dfrac{d^{q_3}z_1}{dt^{q_3}} = bz_1 + x_1y_1, \end{cases} \tag{5}$$

where the parameters a, b, and c are unknown in advance, and need to be estimated. We assume that time series for all variables of equations (4) or (5), as the experimental output of the system, are available, that is, solutions $x_1(t)$, $y_1(t)$ and $z_1(t)$ of system (5) with certain initial conditions are known in the case of

$$\begin{cases} \dfrac{d^{q_4}a}{dt^{q_4}} = 0, \\ \dfrac{d^{q_5}b}{dt^{q_5}} = 0, \\ \dfrac{d^{q_6}c}{dt^{q_6}} = 0. \end{cases} \tag{6}$$

To identify fully uncertain parameters from these time series, we introduce a new system with identical structure to that of equations (5), namely the receiver system (correspondingly, the system (5) is called the drive system), and with linear feedback controllers as follows:

$$\begin{cases} \dfrac{d^{q_1}x_2}{dt^{q_1}} = -\dfrac{a_1b_1}{a_1+b_1}x_2 - y_2z_2 + c_1 + u_1(t), \\ \dfrac{d^{q_1}y_2}{dt^{q_1}} = a_1y_2 + x_2z_2 + u_2(t), \\ \dfrac{d^{q_1}z_2}{dt^{q_1}} = b_1z_2 + x_2y_2 + u_3(t), \end{cases} \tag{7}$$

where the parameters a_1, b_1 and c_1 are not the same as a, b and c respectively, and u_1, u_2 and u_3 in equations (7) are controllers to be designed. The equations governing the evolution of the parameters a_1, b_1 and c_1 are chosen similar to the adaptive control used in [12], with the following simple form:

$$\begin{cases} \dfrac{d^{q_4}a_1}{dt^{q_4}} = -e_2y_2, \\ \dfrac{d^{q_5}b_1}{dt^{q_5}} = -e_3z_2, \\ \dfrac{d^{q_6}c_1}{dt^{q_6}} = -e_1, \end{cases} \tag{8}$$

where

$$\begin{cases} e_1(t) = x_2(t) - x_1(t), \\ e_2(t) = y_2(t) - y_1(t), \\ e_3(t) = z_2(t) - z_1(t), \end{cases} \tag{9}$$

which are practically synchronization error functions between the drive system and the receiver system.

In addition, let $p_1 = -\frac{a_1b_1}{a_1+b_1}$, $p = -\frac{ab}{a+b}$. To this end, we design simple feedback controllers by using the known information from the drive system:

$$\begin{cases} u_1 = -[k_1e_1 + (p_1 - p)x_2], \\ u_2 = -k_2e_2, \\ u_3 = -k_3e_3, \end{cases} \tag{10}$$

where k_1, k_2 and k_3 are real constants.

The remained question is whether or not all the above designs can reach the goal chaos synchronization mentioned in the introduction. The next section will then give a positive answer to it.

4. A Sufficient Condition

Note that systems (5)-(9) with equations (10) can be viewed as a nonlinear coupled system. The main analysis result of the paper is concluded in the following theorem.

Theorem 4.1. *It is easy to verify that the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$ satisfy the following ordinary differential equations:*

$$\begin{cases} p - k_1 < 0, \\ (p - k_1)(k_2 - a) < 0, \\ (p - k_1)(a + k_2)(k_3 - b) > \dfrac{1}{4}(p - k_1)(M_1 + M_2)^2, \end{cases} \tag{11}$$

where $M_1 > |x_1|$ and $M_2 > |x_2|$ are positive constants, then the coupled system will achieve chaos synchronization.

Before proving the theorem, we give the following comments:

— Constants M_1 and M_2 mean that a solution of the drive system is bounded. Such an assumption is reasonable since the experimental output of the system is available;

— The synchronization result leads that all unknown parameters can be identified at the same time.

Proof. It is easy to verify that the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$ satisfy the following fractal-order ordinary differential equations:

$$\begin{cases} \dfrac{d^{q_1}e_1}{dt^{q_1}} = pe_1 + (p_1 - p)x_2 - y_2z_2 + y_1z_1 + c_1 - c - [k_1e_1 + (p_1 - p)x_2], \\ \dfrac{d^{q_2}e_2}{dt^{q_2}} = ae_2 + (a_1 - a)y_2 + x_2z_2 - x_1z_1 - k_2e_2, \\ \dfrac{d^{q_3}e_3}{dt^{q_3}} = be_3 + (b_1 - b)z_2 + x_2y_2 - x_1y_1 - k_3e_3. \end{cases} \tag{12}$$

First we consider the following lemma.

Lemma 4.1. *For the fractional differential equations (5), when $q = (q_1, q_2, ..., q_n) < 1$ and if there exists a positive matrix P, such that for the every vector $x = (x_1, x_2, ..., x_n)^T$, the equation*

$$J = x^T P \frac{d^{q_i}x}{dt^{q_i}} \le 0 \tag{13}$$

comes to existence.

According to Lemma 4.1, we know that we could design the function J similar to the Lyapunov function to proof the above theorem.

Now we select the following J function:

$$J(e_1, e_2, e_3) = \frac{1}{2}(e_1\frac{d^{q_1}e_1}{dt^{q_1}} + e_2\frac{d^{q_2}e_2}{dt^{q_2}} + e_3\frac{d^{q_3}e_3}{dt^{q_3}} + \tilde{a}\frac{d^{q_4}\tilde{a}}{dt^{q_4}} + \tilde{b}\frac{d^{q_5}\tilde{b}}{dt^{q_5}} + \tilde{c}\frac{d^{q_6}\tilde{c}}{dt^{q_6}}). \tag{14}$$

It follows from (8), (10), (12), and (14) that:

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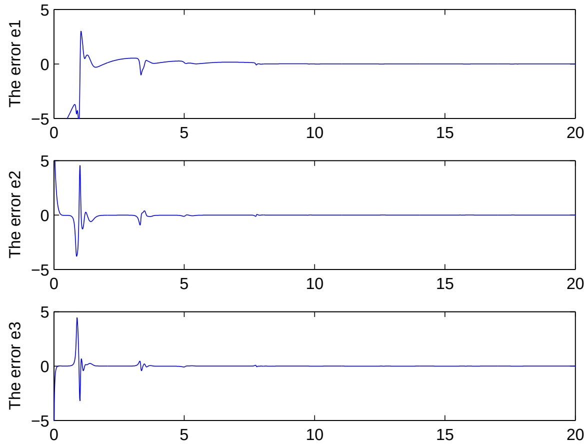

Figure 5: The time evolution of the error functions between systems (5) and (7): $e_1(t) = x_2(t) - x_1(t)$, $e_2(t) = y_2(t) - y_1(t)$, $e_3(t) = z_2(t) - z_1(t)$, where fractional order is $q = (0,99, 0.98, 0.995, 0,99, 0.98, 0.995)$

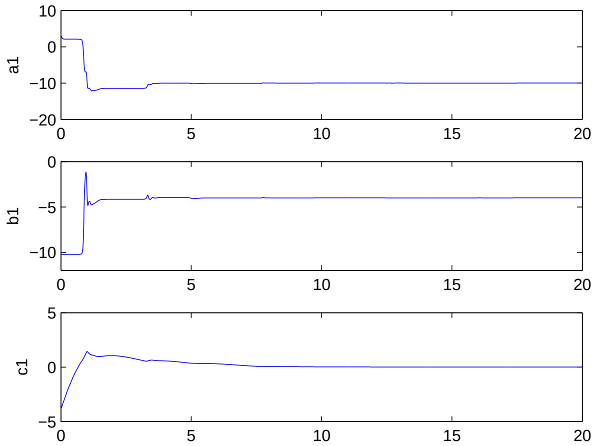

Figure 6: Time evolution of a1, b1 and c1

$$J = e_1\left[pe_1 + (p_1 - p)x_2 - y_2z_2 + y_1z_1 + c_1 - c - [k_1e_1 + (p_1 - p)x_2]\right] \\ + e_2\left[ae_2 + (a_1 - a)y_2 + x_2z_2 - x_1z_1 - k_2e_2\right]$$

$$
\begin{aligned}
&+ e_3\left[be_3 + (b_1 - b)z_2 + x_2y_2 - x_1y_1 - k_3e_3\right] \\
&+ \tilde{a}\frac{d^{q_4}a_1}{dt^{q_4}} + \tilde{b}\frac{d^{q_5}b_1}{dt^{q_5}} + \tilde{c}\frac{d^{q_6}c_1}{dt^{q_6}} \\
&= (p - k_1)e_1^2 + (a - k_2)e_2^2 + (b - k_3)e_3^2 + (x_1 + x_2)e_2e_3 \\
&\le (p - k_1)e_1^2 + (a - k_2)e_2^2 + (b - k_3)e_3^2 + (M_1 + M_2)||e_2||||e_3|| = -ePe^T.
\end{aligned}
$$

Then we get:

$$
P = \begin{pmatrix} k_1 - p & 0 & 0 \\ 0 & k_2 - a & -\frac{1}{2}(M_1 + M_2) \\ 0 & -\frac{1}{2}(M_1 + M_2) & k_3 - b \end{pmatrix} \tag{15}
$$

to achieve the desired synchronization, the matrix P should be positive definite. This will be ensured if conditions (11) are satisfied, and the theorem is then proven. □

5. Numerical Simulation

Figure 5 shows the time series of the synchronization errors between state variables of the drive system and the receiver system. Obviously, they tend to zero after a sufficiently long time.

Figure 6 shows the evolution of parameters a1,b1,c1 in the receiver system. Obviously, they differently tend to values $a = -10$, $b = -5$, $c = 0$.

6. Conclusion

The paper has designed controller for synchronization of chaotic fractional-order unified systems. Numerical simulations have also verified reasonability and feasibility of this. We emphasize that this schemes can straightforwardly be extended to cases of synchronization of other similar chaotic fractional-order differential systems. However, the relationship between orders of fractional-order systems and coupling strength in one-way coupling case and between orders of the fraction-order systems and driving signals in PC method is worth further studying.

Acknowledgments

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